Classifying unary quantifiers

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1 Introduction

Andrzej Mostowski presented a way to generalize the notion of a quantifier in 1957 [Mos57]. Preserving the syntactical form of first order quantifiers, he enhanced Tarski's truth definition by the clause

$$\mathfrak{M} \models Qx\psi(x) \iff \big(|\psi^{\mathfrak{M}}|, |\operatorname{Dom}(\mathfrak{M}) \smallsetminus \psi^{\mathfrak{M}}|\big) \in \mathcal{R}(Q)$$

where $\mathcal{R}(Q)$ is the binary relation between cardinals which determines the meaning of a generalized quantifier Q. Model theory based on first order logic was young, but already showing some promise at that time. However, it was all too clear that first order logic was unable to express some interesting properties, especially infinite sizes of predicates. Apparently, Mostowski's paper was a research initiative for developing stronger logics with good logical properties. In accordance to that, much of the subsequent research on generalized quantifiers was done on cardinality quantifiers Q_α expressing that there are at least $\aleph_\alpha$ elements satisfying a formula. This culminated in Keisler's landmark paper (see, e.g., [Fuh65, Kei70]).

Mostowski's work inspired some further generalizations of quantifiers by Härtig and Rescher [Här65, Res62], which did not follow the first order syntactical form any more. Finally, Lindström presented the most general form of a generalized quantifier in 1966 [Lin66] where a quantifier is basically treated as a logical oracle. A bit paradoxically, Lindström soon afterwards proved [Lin69] his famous characterization theorem which showed that first order logic FO is the only regular logic satisfying Löwenheim–Skolem and compactness theorems. So in one rigorously defined sense, the quest for better logics than FO was futile. Though there were many interesting results in abstract model theory in 1970's, e.g., Shelah found the compact quantifiers $\mathsf{Q}_\alpha^{\mathrm{cf}}$ [She75], the research interest in this area started to decline in 1980's. Probably the general feeling was that abstract model theory was bearing too little fruit against the efforts that were spent on it. Currently abstract elementary classes (for a survey, see [Gro02]) seem to have replaced abstract model theory as a progressive way of doing model theory.

However, there are other reasons to study generalized quantifiers than model-theoretic, such as set-theoretical and combinatorial. Indeed, generalized quantifiers re-emerged in

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finite model theory in the early 1990's [KV95, Hel96]. The connections of the field to complexity theory provide very interesting research problems concerning generalized quantifiers.

This paper is a survey on a seemingly limited aspect of generalized quantifiers, namely about mutual definability problems between unary quantifiers. We try to develop this subject matter as a coherent theory, sketching some know results but filling some gaps as well. The main references are [KV95, Luo00, Luo99, NV96]. It will become clear that the unary quantifier definability problems are very combinatorial in nature, even if we skip the Ramsey-theoretic arguments needed.

In Section 2 we introduce the tools needed in the rest of the paper. In particular, we prove the known result that definability problems between unary quantifiers reduce to combinatorial questions on relations. Sections 3 and 4 work around the question which unary quantifiers are definable by (certain types of) Mostowski quantifiers, and how regularity affects this question. Then in Section 5, we observe that the collection of unary quantifiers has a hierarchical structure and the steps of this hierarchy are determined by the unary dimension of a quantifier. Section 6, Regularity gap, is inspired by the fact that in relation to descriptive complexity theory, irregular quantifiers, such as Maj, are often used. In the context where they are most frequent, on ordered structures, they are usually equivalent to regular ones, but on all finite structures this is no longer true. The main point of the section is that from the logical point of view, this seems to be a defect.

2 Relations and games

This section consists of some preliminary material intensively used in subsequent sections. To fix some notation, we start with generalized quantifiers as they are treated in modern expositions. After a quick glance at Ehrenfeucht–Fraïssé-games, it is explained how definability problem about unary quantifiers are reduced to combinatorial problems between relations. In a sense, this reduction means a return to the origins of generalized quantifiers as introduced by Mostowski, as the relations of quantifiers enable a definition of semantics of the quantifier which is very much in the spirit of Mostowski's definition.

The model-theoretic notation here is quite standard. We write $\mathrm{Dom}(\mathfrak{M})$ for the universe of a structure $\mathfrak{M}$ and $\mathrm{card}(\mathfrak{M})$ for its size.

In this paper, the numbers most frequently used are cardinals and integers. We write Card for the class of cardinals. Addition of cardinals and that of integers is extended to addition on $\mathrm{Card} \cup \mathbb{Z}$ by setting $\kappa + n = n + \kappa = \kappa$ for k an infinite cardinal and $n \in \mathbb{Z}$. Occasionally, real numbers are needed, too, and we write lb for the binary logarithm, i.e., $\mathrm{lb}\, x = \log_2 x$, for $x > 0$.

Concepts related to abstract logics are only needed for discussion. In brief, we assume that the definition presented in [Ebb85] with the strengthening that for every sentence φ there exists the vocabulary of φ, τ_φ, *the symbols occurring in* φ. *A logic* $\mathcal{L}$ is then a collection of sentences together with a truth relation $\models$ such that $\models$ is invariant under isomorphisms and renamings of symbols. In addition, $\mathfrak{A} \models \varphi$ is only meaningful for a

σ-structure $\mathfrak{A}$ such that $\sigma \supseteq \tau_\varphi$, and $\mathfrak{A} \models \varphi$ iff $\mathfrak{A}{\restriction}\tau_\varphi \models \varphi$.

A logic $\mathcal{L}$ is *semiregular* if it is FO-closed (contains atomic sentences, and is closed under Boolean connectives and existential quantification) and closed under substitution of a formula for a predicate. It is well-known that a logic $\mathcal{L}$ with finite occurrence property (τ_φ always finite) is semiregular iff it can be presented in the form $\mathcal{L} \equiv \mathrm{FO}(\mathcal{Q})$ where $\mathcal{Q}$ is a collection of quantifiers of finite width (more on quantifiers later in this section). A logic is *regular* if it is semiregular and closed under relativization.

We often compare logics $\mathcal{L}$ and $\mathcal{L}'$ as in [Ebb85], using the ordinary notation. Also a restricted version is used: If $\mathcal{S}$ is a class of structures, $\mathcal{L} \leq \mathcal{L}' \,/\mathcal{S}$ means that for every $\varphi \in \mathcal{L}$ there is $\varphi' \in \mathcal{L}'$ with $\tau_\varphi = \tau_{\varphi'}$ and such that for every $\mathfrak{M} \in \mathrm{Str}(\tau) \cap \mathcal{S}$ with $\tau \supseteq \tau_\varphi$, we have $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi'$. $\mathcal{L} \equiv \mathcal{L}' \,/\mathcal{S}$ stands for $\mathcal{L} \leq \mathcal{L}' \,/\mathcal{S}$ and $\mathcal{L}' \leq \mathcal{L} \,/\mathcal{S}$. We deal with the following classes: For $C \subseteq \mathrm{Card}$, $\mathcal{S}_C$ is the class of structures $\mathfrak{M}$ with $\mathrm{card}(\mathfrak{M}) \in S$. We write $\mathcal{F} = \mathcal{S}_\omega$ and $\mathcal{O}$ for the class of finite ordered structures.

A *(generalized) quantifier* Q is a symbol for its *defining class* $K_Q \subseteq \mathrm{Str}(\tau_Q)$ where τ_Q is *the vocabulary of the quantifier* Q. K_Q is always assumed to be closed under isomorphisms and $\mathcal{Q}$ to be relational. A logic $\mathcal{L}$ is *closed under the Q-introduction rule*, if for every vocabulary σ and sequence $\big(\psi_R(\boldsymbol{x}_R)\big)_{R\in\tau_Q}$ of σ-formulas of $\mathcal{L}$ such that $n_R = |\boldsymbol{x}_R|$, for every $R \in \tau_Q$, there is a sentence

$$\varphi = Q(\boldsymbol{x}_R\psi_R(\boldsymbol{x}_R))_{R\in\tau_Q}$$

such that for every $\mathfrak{A} \in \mathrm{Str}(\sigma)$, we have

$$\mathfrak{A} \models \varphi \text{ iff } F(\mathfrak{A}) \in K_Q$$

where the interpreted structure $F(\mathfrak{A})$ has the universe $\mathrm{Dom}(F(\mathfrak{A})) = \mathrm{Dom}(\mathfrak{A})$ and for every $R \in \tau_Q$, it holds that $R^{F(\mathfrak{A})} = \psi_R^{\mathfrak{A}} = \{\, \boldsymbol{a} \in \mathrm{Dom}(\mathfrak{A})^{n_R} \mid \mathfrak{A} \models \psi_R[\boldsymbol{a}] \,\}$ where n_R is the arity of R.

We are mainly interested in logics $\mathrm{FO}(\mathcal{Q})$, i.e., quantifier logics with the base logic FO and set of quantifiers $\mathcal{Q}$. By definition, $\mathrm{FO}(\mathcal{Q})$ is the least logic which is closed under first order formation rules and all Q-introduction rules for $Q \in \mathcal{Q}$. Other base logics make sense and have been studied, in particular, many of the results in this paper extend to $\mathrm{FVL}(\mathcal{Q})$ where $\mathrm{FVL} = \mathcal{L}^\omega_{\infty\omega}$ is *the finite variable logic* (see, e.g., [KV95]).

Example 2.1. a) The Härtig quantifier I is a quantifier of vocabulary $\tau_\mathsf{I} = \{U, V\}$ and defining class

$$K_\mathsf{I} = \{\, \mathfrak{M} \in \mathrm{Str}(\tau_\mathsf{I}) \mid |U^\mathfrak{M}| = |V^\mathfrak{M}| \,\}.$$

Consider the simplest kind of I-quantified sentence, namely, $\mathsf{I}(xU(x), yV(y))$ of $\mathrm{FO}(\mathsf{I})$, which is usually written in the form $\mathsf{I}\,xy(U(x), V(y))$. Clearly, the definition of K_I is equivalent to the statement that for every τ_I-structure $\mathfrak{M}$, we have

$$\mathfrak{M} \models \mathsf{I}\,xy(U(x), V(y)) \text{ iff } |U^\mathfrak{M}| = |V^\mathfrak{M}|.$$

This is the form in which we shall usually introduce the semantics of quantifiers.

b) It is perfectly consistent with the definition of a quantifier that the quantifier may have empty vocabulary, even if structures for empty vocabulary consist only of the ground set without further structure. The defining class of such a quantifier is then determined by the class $S \subseteq \text{Card}$ of cardinalities of $\emptyset$-structures in the defining class. We reserve the symbol Ω_S for such a quantifier with defining class

$$K_{\Omega_S} = \{\, \mathfrak{M} \in \text{Str}(\emptyset) \mid \text{card}(\mathfrak{M}) \in S \,\}.$$

However, the width of Ω_S is zero, so Ω_S quantifies no variables in no formulas. In particular, quantifiers Ω_S cannot be nested. Thus, $\text{FO}(\Omega_S)$ is otherwise as first order logic, but there is a new sentence $\Omega_S()$ behaving like atomic sentences and expressing the fact that the cardinality of the structure is in S.

The *arity* of the quantifier Q is $\sup(\{\, n_R \mid R \in \tau_Q \,\} \cup \{1\})$ where n_R is the arity of the relation symbol R, for each $R \in \tau_Q$. We shall work exclusively with *unary* quantifiers, i.e., with quantifiers of arity 1. The *width* of Q is $\text{wd}(Q) = |\tau_Q|$. Q is *simple*, if it is of width one. Simple unary quantifiers are called *Mostowski quantifiers*. Q is called *universe-independent*, if we have $\mathfrak{A} \in K_Q$ iff $\mathfrak{B} \in K_Q$ whenever $\mathfrak{A}, \mathfrak{B} \in \text{Str}(\tau_Q)$ are such that for every $R \in \tau_Q$, it holds that $R^{\mathfrak{A}} = R^{\mathfrak{B}}$.

Example 2.2. We list some common unary quantifiers. Throughout this paper, we write $\upsilon = \{U\}$ where U is a unary relation symbol. Put $\tau = \{U, V\}$.

a) Universe-independent Mostowski quantifiers are here called *cardinality quantifiers*. The general form of a cardinality quantifier is C_S with $S \subseteq \text{Card}$ and vocabulary υ. For $\mathfrak{M} \in \text{Str}(\upsilon)$, we have
$$\mathfrak{M} \models \mathsf{C}_S\, xy\, U(x) \iff |U^{\mathfrak{M}}| \in S.$$
We note that $\exists$ can be identified with $\mathsf{C}_{\text{Card} \smallsetminus \{\emptyset\}}$. In 1960's and 1970's, the cardinality quantifiers $\mathsf{Q}_\alpha = \mathsf{C}_S$ with $S = \{\, \kappa \in \text{Card} \mid \kappa \geq \aleph_\alpha \,\}$ where intensively studied. See, e.g., [Fuh65, Kei70].

b) Another interesting class of Mostowski quantifiers are the *threshold quantifiers* T_f of vocabulary υ where the parameter f is a function with $\text{dom}(f) \cup \text{rg}(f) \subseteq \text{Card}$ (typically $f\colon \mathbb{N} \to \mathbb{N}$). Then for $\mathfrak{M} \in \text{Str}(\upsilon)$,

$$\mathfrak{M} \models \mathsf{T}_f\, xy\, U(x) \text{ iff } \text{card}(\mathfrak{M}) \in \text{dom}(f) \text{ and } |U^{\mathfrak{M}}| \geq f(\text{card}(\mathfrak{M})).$$

For example $\mathsf{Maj} = \mathsf{T}_f$ for $f\colon \mathbb{N} \to \mathbb{N}$, $f(n) = \lfloor n/2 \rfloor + 1$.

c) Linguistically motivated quantifiers include I_f, R_f, Most_f of vocabulary τ where f a function with $\text{dom}(f) \cup \text{rg}(f) \subseteq \text{Card}$, such that for $\mathfrak{M} \in \text{Str}(\tau)$, we have

$$\begin{aligned} \mathfrak{M} &\models \mathsf{I}_f\, xy\, (U(x), V(y)) \iff |V^{\mathfrak{M}}| = f\left(|U^{\mathfrak{M}}|\right), \\ \mathfrak{M} &\models \mathsf{R}_f\, xy\, (U(x), V(y)) \iff |V^{\mathfrak{M}}| \geq f\left(|U^{\mathfrak{M}}|\right), \\ \mathfrak{M} &\models \mathsf{Most}_f\, xy\, (U(x), V(y)) \iff |U^{\mathfrak{M}} \cap V^{\mathfrak{M}}| \geq f\left(|U^{\mathfrak{M}}|\right). \end{aligned}$$

The first two can be seen as generalizations of the Härtig quantifier I and the Rescher quantifier R:

$$\mathfrak{M} \models \mathsf{R}\,xy\,(U(x), V(y)) \iff |U^{\mathfrak{M}}| \leq |V^{\mathfrak{M}}|.$$

Ehrenfeucht–Fraïssé-games are a prolific tool for studying various logics. It is therefore a bit surprising that in this paper such games for quantifier logics are used only in the reduction theorem 2.8. of this section. Nevertheless, we define the appropriate systems of partial isomorphisms (i.e, the algebraic of Fraïssé's form of Ehrenfeucht–Fraïssé-games) and present the characterization result without proofs. The proofs are fairly standard, anyway, and for those who want to reconstruct the proofs by themselves, we give the following background information: Weese [Wee80] presented the Ehrenfeucht–Fraïssé-game for logics with monotone quantifiers in 1980. This is the key result, as every quantifier can be monotonized by Imhof's trick (see [HI98]). The only complication concerns the translations between sentences using the original quantifier and the monotonized one, as the quantifier rank should be preserved. This is satisfied by the following definition.

Definition 2.3. The *quantifier rank* $\mathrm{qr}(\varphi)$ of a sentence φ of a quantifier logic $\mathrm{FO}(\mathcal{Q})$ is defined inductively as follows:

$$\begin{aligned}
&\mathrm{qr}(\varphi) = 0, \text{ for } f \text{ atomic},\\
&\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi),\\
&\mathrm{qr}(\psi \wedge \vartheta) = \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\vartheta)\}\\
&\mathrm{qr}(Q\,(\boldsymbol{x}_R \psi_R(\boldsymbol{x}_R))_{R\in\tau_Q}) = \sup\{\, \mathrm{qr}(\psi_R) \mid R \in \tau_Q \,\} + \mathrm{ar}(Q).
\end{aligned}$$

The following is the algebraic equivalent to a single move in the Ehrenfeucht–Fraïssé-game.

Definition 2.4. Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are structures of the same vocabulary τ and let Q be quantifier. Let $p \in \mathrm{Part}(\mathfrak{M}, \mathfrak{N})$ and $I \subseteq \mathrm{Part}(\mathfrak{M}, \mathfrak{N})$ where $p \in \mathrm{Part}(\mathfrak{M}, \mathfrak{N})$ is the set of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$.

a) We say that p Q-*extends to* I, if for every $\mathfrak{A} \in K_Q$ with $\mathrm{Dom}(\mathfrak{A}) = \mathrm{Dom}(\mathfrak{M})$, one of the following holds:

1) There exists $\mathfrak{B} \in K_Q$ with $\mathrm{Dom}(\mathfrak{B}) = \mathrm{Dom}(\mathfrak{N})$ such that for every $R \in \tau_Q$ and $\boldsymbol{b} = (b_0, \ldots, b_{n_R-1}) \in \mathrm{Dom}(\mathfrak{N})^{n_R}$, there is an extension $q \in I$ of p such that $\{b_0, \ldots, b_{n_R-1}\} \subseteq \mathrm{rg}(q)$ and $\boldsymbol{b} \in R^{\mathfrak{N}} \iff q^{-1}\boldsymbol{b} \in R^{\mathfrak{M}}$.

2) There is $R \in \tau_Q$, $\boldsymbol{b} = (b_0, \ldots, b_{n_R-1}) \in \mathrm{Dom}(\mathfrak{N})^{n_R}$ and two extensions $q, r \in I$ of p such that $\{b_0, \ldots, b_{n_R-1}\} \subseteq \mathrm{rg}(q) \cap \mathrm{rg}(r)$ and $q^{-1}(\boldsymbol{b}) \in R^{\mathfrak{M}}$, but $r^{-1}(\boldsymbol{b}) \notin R^{\mathfrak{M}}$.

b) We say that p *Q-extends back-and-forth to* I if p Q-extends to I and $p^{-1} \in \mathrm{Part}(\mathfrak{N}, \mathfrak{M})$ Q-extends to $I^{-1} = \{\, q^{-1} \mid q \in I \,\}$.

There is a clear intuition behind this definition, which unfolds when we draft what an Q-extension means as a Q-move of an Ehrenfeucht–Fraïssé-game: Let p be the position in the play. Suppose $\forall$ makes a Q-move, so he first chooses $\mathfrak{A} \in K_Q$.

Then $\exists$ has two options: Either she complies to $\forall$'s move and picks $\mathfrak{B} \in K_Q$, or she protests against $\forall$'s choice (which corresponds to case 2 in the definition). In the first case, $\forall$ checks $\exists$'s choice $\mathfrak{B} \in K_Q$ by picking $R \in \tau_Q$ and $\boldsymbol{b} = (b_0, \dots, b_{n_R-1}) \in \mathrm{Dom}(\mathfrak{N})^{n_R}$. $\exists$ then answers by $\boldsymbol{a} = (a_0, \dots, a_{n_R-1}) \in \mathrm{Dom}(\mathfrak{M})^{n_R}$ with $\boldsymbol{a} \in R^{\mathfrak{M}}$ iff $\boldsymbol{b} \in R^{\mathfrak{N}}$, and the play continues at the position $p \cup \{(a_0, b_0), \dots, (a_{n_R-1}, b_{n_R-1})\}$ (with $\mathfrak{A}$ and $\mathfrak{B}$ now forgotten). In the second case, $\exists$ picks $R \in \tau_Q$, $\boldsymbol{b} \in \mathrm{Dom}(\mathfrak{N})^{n_R}$ and $\boldsymbol{a} \in R^{\mathfrak{M}}$, $\boldsymbol{a}' \notin R^{\mathfrak{M}}$ and $\forall$ has to choose if the play continues at $p \cup \{(a_0, b_0), \dots, (a_{n_R-1}, b_{n_R-1})\}$ or $p \cup \{(a'_0, b_0), \dots, (a'_{n_R-1}, b_{n_R-1})\}$. (Intuitively, $\exists$ tries to demonstrate that $\mathfrak{A}$ is not definable.)

Definition 2.5. Let $\mathfrak{M}, \mathfrak{N} \in \mathrm{Str}(\tau)$, $k \in \mathbb{N}$ and let $\mathcal{Q}$ be a set of quantifiers of finite width. Then $(I_0, \dots, I_k)$ is *a $\mathcal{Q}$-system of partial isomorphisms* between $\mathfrak{M}$ and $\mathfrak{N}$, if

1) $\mathrm{Part}(\mathfrak{M}, \mathfrak{N}) \supseteq I_0 \supseteq I_1 \supseteq \cdots \supseteq I_k \neq \emptyset$,

2) for every $i = 0, \dots, k-1$ and $Q \in \mathcal{Q} \cup \{\exists\}$ with $i + \mathrm{ar}(Q) \leq k$, every $p \in I_{i+\mathrm{ar}(Q)}$ Q-extends back-and-forth to I_i.

In symbols, $(I_0, \dots, I_k) \colon \mathfrak{M} \cong_{k;\mathcal{Q}} \mathfrak{N}$.

Note that $\exists$-back-and-forth extension can be shown to be equivalent to the normal back-and-forth criterion for first order logic.

We state the following characterization without a proof. We write $\mathfrak{M} \equiv_k \mathfrak{N}\,(\mathrm{FO}(\mathcal{Q}))$ if $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same sentences of $\mathrm{FO}(\mathcal{Q})$ up to quantifier rank k.

Theorem 2.6. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures of a finite vocabulary τ, $k \in \mathbb{N}$ and let $\mathcal{Q}$ be a set of quantifiers of finite width. Then*

$$\mathfrak{M} \equiv_k \mathfrak{N}\,(\mathrm{FO}(\mathcal{Q})) \quad \textit{iff} \quad \mathfrak{M} \cong_{k;\mathcal{Q}} \mathfrak{N}. \quad \square$$

The comparisons of expressive power of quantifiers are generally very hard. If we restrict the attention to unary quantifiers, the question becomes feasible, since unary structures are simple enough.

Let us say that $\mathcal{U}$ is a *possibly improper partition of* X, if $\mathcal{U} \setminus \{\emptyset\}$ is a partition of X, i.e., $\bigcup \mathcal{U} = X$ and $\mathcal{U}$ is a disjoint family of sets. Similarly, call an indexed family $(U_i)_{i \in I}$ *a possibly improper partition of* X if $\bigcup_{i \in I} U_i = X$ and $U_i \cap U_j = \emptyset$, for $i, j \in I$, $i \neq j$.

Let $\mathfrak{M}$ be a structure for a finite unary relational vocabulary τ. Then the relations of R generate a possibly improper partition of $\mathrm{Dom}(\mathfrak{M})$. More formally, put

$$U_{\mathfrak{M}}(\sigma) = \{\, a \in \mathrm{Dom}(\mathfrak{M}) \mid \sigma = \{\, R \in \tau \mid a \in R^{\mathfrak{M}} \,\}\,\},$$

for $\sigma \subseteq \tau$.

Then $(U_{\mathfrak{M}}(\sigma))_{\sigma\subseteq\tau}$ is a possibly improper partition of $\mathrm{Dom}(\mathfrak{M})$. Put

$$c_{\mathfrak{M}}\colon \mathcal{P}(\tau)\to \mathrm{Card},\ c_{\mathfrak{M}}(\sigma)=|U_{\mathfrak{M}}(\sigma)|.$$

Then $c_{\mathfrak{M}}$ is an invariant of $\mathfrak{M}$, i.e., for another $\mathfrak{N}\in\mathrm{Str}(\tau)$, we have and $\mathfrak{M}\cong\mathfrak{N}$ iff $c_{\mathfrak{N}}=c_{\mathfrak{M}}$.

Fix a bijection $f_\tau\colon \mathcal{P}(\tau)\to n$ where $n=2^{|\tau|}\in\mathrm{Card}^n$ and $\boldsymbol{\kappa}_{\mathfrak{M}}=c_{\mathfrak{M}}\circ {f_\tau}^{-1}\in\mathrm{Card}^n$. We make the convention that $f_\tau(\emptyset)=n-1$. For a unary quantifier Q of finite width, put

$$\mathcal{R}(Q)=\{\,\boldsymbol{\kappa}_{\mathfrak{M}}\mid \mathfrak{M}\in K_Q\,\}\subseteq\mathrm{Card}^n$$

where $n=2^{\mathrm{wd}(Q)}$. For C a class of cardinals, write

$$\mathcal{R}(Q,C)=\mathcal{R}(Q)\cap C^n.$$

Example 2.7. An easy calculation shows that $\mathcal{R}(\Omega_S)=S$, $\mathcal{R}(\mathsf{C}_S)=S\times\mathrm{Card}$,

$$\mathcal{R}(\mathsf{T}_f)=\{\,(\kappa,\lambda)\in\mathrm{Card}^2\mid \kappa+\lambda\in\mathrm{dom}(f),\kappa\geq f(\kappa+\lambda)\,\},$$

for f a function with $\mathrm{dom}(f)\cup\mathrm{rg}(f)\subseteq\mathrm{Card}$ and

$$\mathcal{R}(\mathsf{I})=\{\,(\kappa_0,\kappa_1,\kappa_2,\kappa_3)\in\mathrm{Card}^4\mid \kappa_0+\kappa_1=\kappa_0+\kappa_2\,\}$$

assuming the enumerations $\varphi_v(v)=0$, $\varphi_v(\emptyset)=1$, $\varphi_{\tau_{\mathsf{I}}}(\tau_{\mathsf{I}})=0$, $\varphi_{\tau_{\mathsf{I}}}(\{U\})=1$, $\varphi_{\tau_{\mathsf{I}}}(\{V\})=2$, $\varphi_{\tau_{\mathsf{I}}}(\emptyset)=3$.

We need some notation for manipulation of relations. For $n,l\in\mathbb{Z}_+$, $l\leq n$, $\mathcal{U}_{n,l}$ is the set of sequences $\boldsymbol{U}=(U_0,\dots,U_{l-1})$ of disjoint subsets of n and $\mathcal{V}_{n,l}$ is the set of possibly improper partitions $\boldsymbol{U}=(U_0,\dots,U_{l-1})$ of n. In addition,

$$J_{n,l}=\{\,(\boldsymbol{U},\boldsymbol{t})\in V_{n,l}\times\mathbb{Z}^l\mid \boldsymbol{t}=(t_0,\dots,t_{l-1})\text{ satisfies }\sum_{i\in l}t_i=0\,\}.$$

For a sequence $\boldsymbol{\kappa}\in\mathrm{Card}^n$ and $\boldsymbol{U}=(U_0,\dots,U_{l-1})\in\mathcal{U}_{n,l}$, we write

$$\boldsymbol{s}(\boldsymbol{\kappa},\boldsymbol{U})=\left(\sum_{i\in U_0}\kappa_i,\dots,\sum_{i\in U_{l-1}}\kappa_i\right).$$

For a finite tuple $\boldsymbol{\kappa}\in(\mathrm{Card}\cup\mathbb{Z})^n$ and $\lambda\in\mathrm{Card}\cup\mathbb{Z}$, we write

$$\boldsymbol{\kappa}\downarrow\lambda=(\min\{\kappa_0,\lambda\},\dots,\min\{\kappa_{l-1},\lambda\})$$

and

$$\boldsymbol{\kappa}\uparrow\lambda=(\max\{\kappa_0,\lambda\},\dots,\max\{\kappa_{l-1},\lambda\}).$$

Furthermore, we write

$$\boldsymbol{\kappa}\downarrow^+\lambda=(\boldsymbol{\kappa}\downarrow\lambda)^\frown\left(\sum_{i\in n}\kappa_i\right).$$

The following result is an obvious generalization of a similar result for cardinality quantifiers by Corredor [Cor86]. It is well-known among researchers of unary quantifiers, and variants of it have appeared in literature (see, e.g., [Vää97, Luo99]).

Proposition 2.8. *Let Q be a unary quantifier and $\mathcal{Q}$ a set of unary quantifiers, all of finite width. Let C be an infinite initial segment of the class of cardinals. Denote $n = 2^{\mathrm{wd}(Q)}$ and $l_q = 2^{\mathrm{wd}(q)}$, for $q \in \mathcal{Q}$. Then the following are equivalent:*

a) $\mathrm{FO}(Q) \leq \mathrm{FO}(\mathcal{Q}) / \mathcal{S}_C$*,*

b) $\mathcal{R}(Q, C)$ *is a (finite) Boolean combination of relations*

$$\{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q, C) \,\}$$

where $q \in \mathcal{Q} \cup \{\exists\}$ *and* $(\boldsymbol{U}, \boldsymbol{t}) \in J_{n,l_q}$*.*

Proof. We only sketch the direction from condition b to a, omitting some easy but tedious details. Let $\boldsymbol{U} = (U_0, \ldots, U_{l-1}) \in \mathcal{V}_{n,l}$ where $l = 2^{\mathrm{wd}(q)}$ for some $q \in \mathcal{Q}$. Then there is a quantifier-free interpretation $F \colon \mathrm{Str}(\tau_Q) \to \mathrm{Str}(\tau_q)$ such that for every $\mathfrak{M} \in \mathrm{Str}(\tau_Q)$, we have $\boldsymbol{\kappa}_{F(\mathfrak{M})} = \boldsymbol{s}(\boldsymbol{\kappa}_{\mathfrak{M}}, \boldsymbol{U})$. Here, quantifier-free interpretation means that there are quantifier-free formulas $\vartheta_R(x)$, $R \in \tau_q$, such that for every $\mathfrak{M} \in \mathrm{Str}(\tau_Q)$, it holds that $R^{F(\mathfrak{M})} = \vartheta_R^{\mathfrak{M}}$.

Consider the sentence $\varphi = q(x_R \vartheta_R(x_R))$. Then for $\mathfrak{M} \in \mathrm{Str}(\tau_Q)$, we have

$$\mathfrak{M} \models \varphi \text{ iff } F(\mathfrak{M}) \in K_Q \text{ iff } \boldsymbol{s}(\boldsymbol{\kappa}_{\mathfrak{M}}, \boldsymbol{U}) = \boldsymbol{\kappa}_{F(\mathfrak{M})} \in \mathcal{R}(q).$$

Put in another way,

$$\{\, \boldsymbol{\kappa}_{\mathfrak{M}} \mid \mathfrak{M} \in \mathrm{Str}(\tau_Q),\ \mathfrak{M} \models \varphi \,\} = \{\, \boldsymbol{\kappa} \in \mathrm{Card}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) \in \mathcal{R}(q) \,\}.$$

If $\boldsymbol{t} = (t_0, \ldots, t_{l-1}) \in \mathbb{Z}^l$ with $\sum_{i \in l} t_i = 0$, then modifying the sentence φ slightly we get $\varphi' \in \mathrm{FO}(q)[\tau_Q]$ such that

$$\{\, \boldsymbol{\kappa}_{\mathfrak{M}} \mid \mathfrak{M} \in \mathrm{Str}(\tau_Q),\ \mathfrak{M} \models \varphi' \,\} = \{\, \boldsymbol{\kappa} \in \mathrm{Card}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q) \,\}.$$

Suppose now condition b holds, i.e., $\mathcal{R}(Q, C)$ is a Boolean combination of relations $R_i = \{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i) + \boldsymbol{t}_i \in \mathcal{R}(q_i) \,\}$, $i \in I$, where $q_i \in \mathcal{Q} \cup \{\exists\}$ and $(\boldsymbol{U}_i, \boldsymbol{t}) \in J_{n,l_{q_i}}$. By previous paragraph, for each $i \in I$, there is $\varphi_i \in \mathrm{FO}(\mathcal{Q})[\tau_Q]$ such that for $\mathfrak{M} \in \mathrm{Str}(\tau_Q)$, we have $\mathfrak{M} \models \varphi_i$ iff $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i) + \boldsymbol{t}_i \in \mathcal{R}(q_i)$. If ψ is an appropriate Boolean combination of sentences φ_i, $i \in I$, then for every $\mathfrak{M} \in \mathrm{Str}(\tau_Q) \cap \mathcal{S}_C$, we have $\mathfrak{M} \models \psi$ iff $\mathfrak{M} \in K_Q$, so that Q is definable in $\mathrm{FO}(\mathcal{Q})$ on $\mathcal{S}_C$. By semi-regularity of $\mathrm{FO}(\mathcal{Q})$, this implies condition a.

Suppose now condition a holds, i.e., $\mathrm{FO}(Q) \leq \mathrm{FO}(\mathcal{Q}) / \mathcal{S}_C$ and so Q is definable in $\mathrm{FO}(\mathcal{Q})$ on the class $\mathcal{S}_C$. Fix a finite $\mathcal{Q}_0 \subseteq \mathcal{Q}$ and $\varphi \in \mathrm{FO}(\mathcal{Q}_0)[\tau_Q]$ defining K_Q on $\mathcal{S}_C$. Write $r = \mathrm{qr}(\varphi)$. Our claim is now that $\mathcal{R}(Q, C)$ is a Boolean combination of relations of form $\{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q, C) \,\}$ where $q \in \mathcal{Q}_0 \cup \{\exists\}$, $(\boldsymbol{U}, \boldsymbol{t}) \in J_{n,l_q}$ and $\boldsymbol{t} \in \{-r, -r+1, \ldots, r-1, r\}^{l_q}$. Note that the set $\mathfrak{r}$ of such relations is finite.

Toward contradiction, suppose $\mathcal{R}(Q, C)$ is not a Boolean combination of relations in $\mathfrak{r}$. Pick tuples $\boldsymbol{\lambda} \in \mathcal{R}(Q, C)$ and $\boldsymbol{\mu} \in C^n \setminus \mathcal{R}(Q, C)$ such that for every $R \in \mathfrak{r}$ we have $\boldsymbol{\lambda} \in R$ iff $\boldsymbol{\mu} \in R$. In particular, $\boldsymbol{\lambda} \in R_{si}$ iff $\boldsymbol{\mu} \in R_{si}$ holds for the relations $R_{si} = \{\, \boldsymbol{\kappa} \in$

$C^n \mid \boldsymbol{s}(\boldsymbol{\kappa},\boldsymbol{U})+\boldsymbol{t} \in \mathcal{R}(\exists, C)\,\}$ where $s \in \{0,\dots,r\}$, $i \in n$ with $\boldsymbol{U} = (\{i\}, n \smallsetminus \{i\})$ and $\boldsymbol{t} = (s,-s)$. These particular cases imply that $\boldsymbol{\lambda}{\downarrow}(r+1) = \boldsymbol{\mu}{\downarrow}(r+1)$. Pick structures $\mathfrak{M},\mathfrak{N} \in \mathrm{Str}(\tau_Q)$ with $\boldsymbol{\kappa}_{\mathfrak{M}} = \boldsymbol{\lambda}$ and $\boldsymbol{\kappa}_{\mathfrak{N}} = \boldsymbol{\mu}$; then $\mathfrak{M} \in K_Q$, but $\mathfrak{N} \notin K_Q$.

Building a system of partial isomorphisms is now simple: put $I_i = \{\, p \in \mathrm{Part}(\mathfrak{M},\mathfrak{N}) \mid |p| \le r-i \,\}$, $i = 0,\dots,r$. As $\boldsymbol{\kappa}_{\mathfrak{M}}{\downarrow}(r+1) = \boldsymbol{\kappa}_{\mathfrak{N}}{\downarrow}(r+1)$, the sequence $(I_0,\dots,I_r)$ clearly satisfies the normal back-and-forth conditions, i.e., $(I_0,\dots,I_r)$ is a decreasing sequence of nonempty subsets of $\mathrm{Part}(\mathfrak{M},\mathfrak{N})$ and every $p \in I_{i+1}$ $\exists$-extends back-and-forth to I_i, for $i = 0,\dots,r-1$. Let us check that for every $p \in I_{i+1}$ and $q \in \mathcal{Q}_0$ the partial isomorphism p q-extends back-and-forth to I_i, too. By symmetry, it is enough to consider forth extensions.

So suppose $q \in \mathcal{Q}_0$, $p \in I_{i+1}$ for some $i \in r$ and $\mathfrak{A} \in K_q$ with $\mathrm{Dom}(\mathfrak{A}) = \mathrm{Dom}(\mathfrak{M})$. For every $j \in l_q = 2^{\mathrm{wd}(q)}$, put

$$U'_j = \{\, i \in n \mid U_{\mathfrak{M}}(f_{\tau_Q}^{-1}(i)) \cap U_{\mathfrak{A}}(f_{\tau_q}^{-1}(j)) \not\subseteq \mathrm{dom}(p) \,\}.$$

Note that for $i \in n$, we have $i \notin \bigcup_{j \in l_q} U'_j$ iff $U_{\mathfrak{M}}(f_{\tau_Q}^{-1}(i)) \subseteq \mathrm{dom}(p)$. We have two cases.

1) The sets U'_j, $j \in l_q$, are disjoint. We note in passing that then $\mathfrak{A}$ is actually quantifier-free definable in $\mathfrak{M}$ with parameters in $\mathrm{dom}(p)$. Choose $U_j \supseteq U'_j$, $j \in l_q$, such that $\boldsymbol{U} = (U_j)_{j \in l_q} \in V_{n,l_q}$. For $j \in l_q$, we now have $U_{\mathfrak{A}}(f_{\tau_q}^{-1}(j)) \smallsetminus \mathrm{dom}(p) = \bigcup_{i \in U_j} U_{\mathfrak{M}}(f_{\tau_Q}^{-1}(i)) \smallsetminus \mathrm{dom}(p)$. For every $j \in l_q$, put $A_j^+ = U_{\mathfrak{A}}(f_{\tau_q}^{-1}(j)) \cap \mathrm{dom}(p)$, $A_j^- = \left(\bigcup_{i \in U_j} U_{\mathfrak{M}}(f_{\tau_Q}^{-1}(i))\right) \cap \mathrm{dom}(p)$, $B_j^+ = p[A_j^+]$, $B_j^- = p[A_j^-]$. $\exists$'s natural reply is now $\mathfrak{B} \in \mathrm{Str}(\tau_q)$ such that $\mathrm{Dom}(\mathfrak{B}) = \mathrm{Dom}(\mathfrak{N})$ and for every $j \in l_q$

$$U_{\mathfrak{B}}(f_{\tau_q}^{-1}(j)) = \left(\bigcup_{i \in U_j} U_{\mathfrak{N}}(f_{\tau_Q}^{-1}(i)) \smallsetminus \mathrm{rg}(p)\right) \cup B_j^+.$$

Then for $j \in l_q$, we have

$$U_{\mathfrak{A}}(f_{\tau_q}^{-1}(j)) = \left(\bigcup_{i \in U_j} U_{\mathfrak{M}}(f_{\tau_Q}^{-1}(i) \smallsetminus A_j^-)\right) \cup A_j^+$$

and

$$U_{\mathfrak{B}}(f_{\tau_q}^{-1}(j)) = \left(\bigcup_{i \in U_j} U_{\mathfrak{N}}(f_{\tau_Q}^{-1}(i) \smallsetminus B_j^-)\right) \cup B_j^+.$$

The main point is now showing that $\mathfrak{B} \in K_q$. Indeed, if we put $t_j = |A_j^+| - |A_j^-|$, for $j \in l_q$, we see that $\boldsymbol{\kappa}_{\mathfrak{A}} = \boldsymbol{s}(\boldsymbol{\kappa}_{\mathfrak{M}},\boldsymbol{U}) + \boldsymbol{t}$ and $\boldsymbol{\kappa}_{\mathfrak{B}} = \boldsymbol{s}(\boldsymbol{\kappa}_{\mathfrak{N}},\boldsymbol{U}) + \boldsymbol{t}$ with $\boldsymbol{t} = (t_j)_{j \in l_q}$. As $|t_j| \le r$, for each $j \in l_q$, we see that $R = \{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa},\boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q,C) \,\} \in \mathfrak{r}$. Now $\mathfrak{A} \in K_q$ implies $\boldsymbol{s}(\boldsymbol{\kappa}_{\mathfrak{M}},\boldsymbol{U}) + \boldsymbol{t} = \boldsymbol{\kappa}_{\mathfrak{A}} \in \mathcal{R}(q,C)$, so $\boldsymbol{\kappa}_{\mathfrak{M}} \in R$, and by our choice of tuples $\boldsymbol{\kappa}_{\mathfrak{M}}$ and $\boldsymbol{\kappa}_{\mathfrak{N}}$, also $\boldsymbol{\kappa}_{\mathfrak{N}} \in R$, whence $\boldsymbol{\kappa}_{\mathfrak{B}} \in \mathcal{R}(q,C)$, so indeed we have $\mathfrak{B} \in K_q$. The remaining conditions are now easy to check.

2) There are distinct $j,k \in l_q$ such that $U'_j \cap U'k \neq \emptyset$. There is $P \in f^{-1}_{\tau_q}(j) \triangle f^{-1}_{\tau_q}(k) \subseteq \tau_q$, say, $P \in f^{-1}_{\tau_q}(j)$ and $P \notin f^{-1}_{\tau_q}(k)$. Pick $i \in U'_j \cap U'_k \subseteq n$, $a_0 \in U_{\mathfrak{M}}(f^{-1}_{\tau_Q}(i)) \cap U_{\mathfrak{A}}(f^{-1}_{\tau_q}(j)) \setminus \mathrm{dom}(p)$ and $a_1 \in U_{\mathfrak{M}}(f^{-1}_{\tau_Q}(i)) \cap U_{\mathfrak{A}}(f^{-1}_{\tau_q}(k)) \setminus \mathrm{dom}(p)$. Then $a_0 \notin P^{\mathfrak{A}}$, but $a_1 \in P^{\mathfrak{A}}$ and it is readily seen that there is $b \in \mathrm{Dom}(\mathfrak{N})$ such that both $p \cup \{(a_0,b)\} \in I_i$ and $p \cup \{(a_1,b)\} \in I_i$ hold. So $\exists$ can reject $\mathfrak{A}$ as in Definition 2.4 a, case 2.

By cases 1 and 2, we have that p q-extends to I_i. In conclusion, $(I_0,\dots,I_r)\colon \mathfrak{M} \cong_{r;\mathcal{Q}_0} \mathfrak{N}$. By Theorem 2.6, this means that $\mathfrak{M} \equiv_k \mathfrak{N}\,(\mathrm{FO}(\mathcal{Q}_0))$, and, in particular, $\mathfrak{M} \models \varphi$ iff $\mathfrak{N} \models \varphi$. This is in contradiction with the assumption that q defines Q on $\mathcal{S}_C$, as we have $\mathfrak{M} \in K_Q$ and $\mathfrak{N} \notin K_Q$. □

In particular, for $S \subseteq T$ we have that $\mathrm{FO}(\mathsf{C}_S) \leq \mathrm{FO}(\mathsf{C}_T)$ iff S is a Boolean combination of classes $T+m$ and $\{\,\kappa+m \mid \kappa > 0\,\}$ where $m \in \mathbb{N}$. The latter classes result from the effect of the existential quantifier, the effect which can be eliminated by the following reformulation: $\mathrm{FO}(\mathsf{C}_S) \leq \mathrm{FO}(\mathsf{C}_T)$ iff there exists $S' \subseteq \mathrm{Card}$ such that $S \triangle S' \subseteq \omega$ is finite and S' is a Boolean combination of classes $T+m$, $m \in \mathbb{Z}$.

Corredor [Cor86] found this simple fact and used it to show that the partial order of cardinality quantifiers according to the expressive power is already very rich, containing, among others, infinite descending chains and antichains.

3 Bounded quantifiers

We start exploring the hierarchy of unary quantifiers from the weakest of quantifiers, proceeding from bottom to top. The trivial starting point is then the first order definable unary quantifiers. Although there is nothing really new in the characterization of FO-definable unary quantifiers, we go through this easy case for reasons of comparison, and to acquaint the reader with notation and methods.

There is a class of unary quantifiers almost as trivial as the FO-definable one, which is called the class of bounded quantifiers. The notion, as it appears here, is a variant of many similar notions appearing in the literature, e.g, in [Vää97, DHS98]. Different boundedness notions are usually not generalizations of each other, but they try to convey the same idea of inexpressibility. We shall find out that bounded quantifiers are equivalent to unary quantifiers of empty vocabulary, which appears to be a case which has not been considered at all, probably due to its triviality.

Proposition 3.1. *Let Q be a unary quantifier with $n = 2^{\mathrm{wd}(Q)} \in \mathbb{N}$. Then Q is first-order definable iff there are $m \in \mathbb{N}$ and $R \subseteq \{0,\dots,m\}^n$ such that*

$$\mathcal{R}(Q) = \{\, \boldsymbol{\kappa} \mid \boldsymbol{\kappa} \downarrow m \in R \,\}.$$

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ_Q-structures. Then a standard application of Ehrenfeucht–Fraïssé-games shows that $\mathfrak{A} \equiv_r \mathfrak{B}$ (FO) iff $\boldsymbol{\kappa}_{\mathfrak{A}} \downarrow r = \boldsymbol{\kappa}_{\mathfrak{B}} \downarrow r$. Suppose now the condition is not met. Then for each $m \in \mathbb{Z}_+$, there are $\boldsymbol{\kappa} \in \mathcal{R}(Q)$, $\boldsymbol{\lambda} \in \mathrm{Card}^n \setminus \mathcal{R}(Q)$ such that $\boldsymbol{\kappa} \downarrow m = \boldsymbol{\lambda} \downarrow m$. Picking τ_Q-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\boldsymbol{\kappa}_{\mathfrak{A}} = \boldsymbol{\kappa}$ and $\boldsymbol{\kappa}_{\mathfrak{B}} = \boldsymbol{\lambda}$ we observe that $\mathfrak{A} \in K_Q$, $\mathfrak{B} \notin K_Q$ and $\mathfrak{A} \equiv_m \mathfrak{B}$ (FO). This implies that Q is not FO-definable.

Suppose then that the condition holds: $\mathcal{R}(Q) = \{\, \boldsymbol{\kappa} \mid \boldsymbol{\kappa} \downarrow m \in R \,\}$ with some $m \in \mathbb{N}$ and $R \subseteq \{0, \ldots, m\}^n$. Then K_Q is closed under $\equiv_m$-equivalence of FO: If $\mathfrak{A} \in K_Q$ and $\mathfrak{B}$ is a τ_Q-structure with $\mathfrak{A} \equiv_m \mathfrak{B}$ (FO), then $\boldsymbol{\kappa}_{\mathfrak{B}} \downarrow m = \boldsymbol{\kappa}_{\mathfrak{A}} \downarrow m \in R$, whence $\boldsymbol{\kappa}_{\mathfrak{B}} \in \mathcal{R}(Q)$ and $\mathfrak{B} \in K_Q$. As τ_Q is finite, this implies that Q is FO-definable. □

Hence, in this context, FO-definability corresponds to the way birds count: 0, 1, 2 up to m as "many". When we enhance this ability just a bit, changing $\boldsymbol{\kappa} \downarrow m$ to $\boldsymbol{\kappa} \downarrow^+ m$, we get a concept which is called boundedness of quantifiers here. Note that for appropriate unary relational structures $\mathfrak{A}$ we have $\boldsymbol{\kappa}_{\mathfrak{A}} \downarrow^+ m = (\boldsymbol{\kappa}_{\mathfrak{A}} \downarrow m)\hat{\ }(\mathrm{card}(\mathfrak{A}))$, so that this change entails adding the information about the size of the structure.

Definition 3.2. A unary quantifier Q of finite width is *bounded* on class $C \subseteq \mathrm{Card}$, if there are $m \in \mathbb{N}$ and $R \subseteq \{0, \ldots, m\}^n \times \mathrm{Card}$ with $n = 2^{\mathrm{wd}}(Q)$ such that

$$\mathcal{R}(Q, C) = \{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{\kappa} \downarrow^+ m \in R \,\}.$$

The default is $C = \mathrm{Card}$: Then we simply say that Q is *bounded* without referring to the class C.

Note that the FO-definability of a unary quantifier of finite width implies that it is bounded.

Example 3.3. a) Let $S \subseteq \mathrm{Card}$ and consider the cardinality quantifier C_S with

$$\mathcal{R}(\mathsf{C}_S) = \{\, (\kappa, \lambda) \in \mathrm{Card}^2 \smallsetminus \{(0,0)\} \mid \kappa \in S \,\}.$$

Suppose C_S is bounded, and pick $m \in \mathbb{Z}_+$ and $R \subseteq \{0, \ldots, m\}^2 \times \mathrm{Card}$ such that

$$\mathcal{R}(\mathsf{C}_S) = \{\, (\kappa, \lambda) \in \mathrm{Card}^2 \mid (\kappa, \lambda) \downarrow m \in R \,\}.$$

For every $\mu, \nu \geq m$, we have $(\mu, \nu) \downarrow^+ m = (m, m, \mu + \nu) = (\nu, \mu) \downarrow^+ m$, which implies $\mu \in S$ iff $(\mu, \nu) \in \mathcal{R}(\mathsf{C}_S)$ iff $(\mu, \nu) \downarrow^+ m = (\nu, \mu) \downarrow^+ m \in R$ iff $(\nu, \mu) \in \mathcal{R}(\mathsf{C}_S)$ iff $\nu \in S$. Hence, either $S \subseteq m$ or $\mathrm{Card} \smallsetminus S \subseteq m$.

Since $m \in \mathbb{Z}_+$ was arbitrary, boundedness of C_S implies that either S is finite or co-finite ($\mathrm{Card} \smallsetminus S$ finite) and this finite part is included in ω. Conversely, if this condition holds, C_S is easily seen to be FO-definable, which implies boundedness. Hence, C_S is bounded iff it is FO-definable.

b) Let $f \colon \mathbb{N} \to \mathbb{N}$ be such that $f(0) = 1$ and $f(n) \leq n + 1$, for $n \in \mathbb{N}$. Consider the threshold quantifier T_f with

$$\mathcal{R}(\mathsf{T}_f) = \{\, (k, l) \in \mathbb{N} \times \mathbb{N} \mid k \geq f(k + l) \,\}.$$

Let $g \colon \mathbb{N} \to \mathbb{N}$, $g(n) = \min\{f(n), n + 1 - f(n)\}$. If g is unbounded, then for each $m \in \mathbb{Z}_+$ there is $n \in \mathbb{N}$ such that $g(n) > m$, so $m < f(m) \leq n - m$. Then $(m, n - m) \notin \mathcal{R}(\mathsf{T}_f)$ and $(n - m, m) \in \mathcal{R}(\mathsf{T}_f)$, but

$$(m, n - m) \downarrow^+ m = (m, m, n) = (n - m, m) \downarrow^+ m.$$

This implies that T_f is not bounded.

Suppose then g is bounded. Denote by $m \in \mathbb{N}$ the maximum value of g, and put

$$R = \{ \boldsymbol{\kappa}\downarrow^+ m \mid \boldsymbol{\kappa} \in \mathcal{R}(\mathsf{T}_f) \}.$$

Let $\boldsymbol{\kappa} \in \mathrm{Card}^2$. Trivially, $\boldsymbol{\kappa} \in \mathcal{R}(\mathsf{T}_f)$ implies $\boldsymbol{\kappa}\downarrow^+ m \in R$. In the other direction, suppose that $\boldsymbol{\kappa}\downarrow^+ m \in R$. Choose $\boldsymbol{\lambda} \in \mathcal{R}(\mathsf{T}_f)$ such that $\boldsymbol{\lambda}\downarrow^+ m = \boldsymbol{\kappa}\downarrow^+ m \in R$. We observe that $\boldsymbol{\lambda} = (i,j) \in \mathbb{N}\times\mathbb{N}$ where $i \geq f(i+j)$. We may assume that $\boldsymbol{\kappa} \neq \boldsymbol{\lambda}$; then $\boldsymbol{\kappa}\downarrow^+ m = \boldsymbol{\lambda}\downarrow^+ m$ is possible only if $\boldsymbol{\kappa}\downarrow^+ m = (m,m,i+j) = \boldsymbol{\lambda}\downarrow^+ m$. Then $i,j \geq m$. Consequently, we have $i+j+1-f(i+j) > j+i-f(i+j) \geq j \geq m$. However, since g is bounded by m, we must have $f(i+j) \leq m$. Now if $\boldsymbol{\kappa} = (r,s)$, we get $r \geq \min\{r,m\} = \min\{i,m\} = m \geq f(i+j) = f(r+s)$. Therefore, we have $\boldsymbol{\kappa} \in \mathcal{R}(\mathsf{T}_f)$.

All in all, T_f is bounded iff g is bounded.

c) For the Härtig quantifier I we have

$$\mathcal{R}(\mathsf{I},\omega) = \{ (\kappa_0,\kappa_1,\kappa_2,\kappa_3) \in \omega^4 \mid k_1 = k_2 \}.$$

Now for every $m \in \mathbb{Z}_+$, we may choose $\boldsymbol{\kappa} = (0,m,m+2,0)$ and $\boldsymbol{\lambda} = (0,m+1,m+1,0)$, and then $\boldsymbol{\kappa} \notin \mathcal{R}(\mathsf{I})$, $\boldsymbol{\lambda} \in \mathcal{R}(\mathsf{I})$ and $\boldsymbol{\kappa}\downarrow^+ m = (0,m,m,0,2m+2) = \boldsymbol{\lambda}\downarrow^+ m$. Therefore, I is not bounded.

The general appearance of bounded quantifiers is deceptive: The following result shows that they can be reduced to the simplest kind of unary quantifiers.

Proposition 3.4. *For a unary quantifier Q of finite width and an infinite initial segment C of* Card*, the following are equivalent:*

a) Q is bounded on C.

b) There are finitely many classes $C_0,\ldots,C_{s-1} \subseteq C$ such that

$$\mathrm{FO}(Q) \equiv \mathrm{FO}(\Omega_{C_0},\ldots,\Omega_{C_{s-1}}) \;/\mathcal{S}_C.$$

c) There are finitely many classes $C_0,\ldots,C_{s-1} \subseteq C$ such that

$$\mathrm{FO}(Q) \leq \mathrm{FO}(\Omega_{C_0},\ldots,\Omega_{C_{s-1}}) \;/\mathcal{S}_C.$$

Proof. Let $n = 2^{\mathrm{wd}(Q)}$. Obviously condition b implies condition c, so suppose now c holds, i.e., there are classes $C_0,\ldots,C_{s-1} \subseteq C$ with $s \in \mathbb{N}$ such that $\mathrm{FO}(Q) \leq \mathcal{L} \;/\mathcal{S}_C$ where $\mathcal{L} = \mathrm{FO}(\Omega_{C_0},\ldots,\Omega_{C_{s-1}})$. In particular, Q is $\mathcal{L}$-definable on $\mathcal{S}_C$; let $\varphi \in \mathcal{L}[\tau_Q]$ be the defining sentence. Partition C according to the classes C_i: For each $I \subseteq s$, let D_I be the set of cardinals $\kappa \in C$ such that $I = \{ i \in s \mid \kappa \in C_i \}$. Since quantifiers of vocabulary $\emptyset$ cannot be nested but can appear only in front of subformulas of atomic form $\Omega_{C_i}()$ which trivialize in classes $\mathcal{S}_{D_i}$, we have $\mathcal{L} \equiv \mathrm{FO} \;/\mathcal{S}_{D_I}$. Let $\varphi_I \in \mathrm{FO}[\tau_q]$ be a translation of $\varphi \in \mathcal{L}[\tau_Q]$ on the class $\mathcal{S}_{D_I}$. Now the previous proposition implies that there is $m_I \in \mathbb{N}$ and $R_I \subseteq \{0, \dots, m\}$ such that

$$\begin{aligned} \mathcal{R}_{\tau_Q}(\varphi_I) &= \{\, \boldsymbol{\kappa}_{\mathfrak{M}} \mid \mathfrak{M} \in \mathrm{Str}(\tau_Q), \mathfrak{M} \models \varphi_I \,\} \\ &= \{\, \boldsymbol{\kappa} \in \mathrm{Card}^n \mid \boldsymbol{\kappa} \downarrow m_I \in R_I \,\}. \end{aligned}$$

A moment's reflection shows that all m_I, $I \subseteq s$, can be assumed to be equal, i.e., $m_I = m$ is constant. Put $R = \bigcup_{I \subseteq s}(R_I \times D_I)$. It is then straightforward to show that for every $\boldsymbol{\kappa} \in C^n$, we have $\boldsymbol{\kappa} \in \mathcal{R}(Q, C)$ iff $\boldsymbol{\kappa} \downarrow^+ m \in R$. Hence, Q is bounded on C.

Suppose then condition a holds, i.e., Q is bounded. Choose $m \in \mathbb{Z}_+$ and $R \subseteq \{0, \dots, m\}^n \times \mathrm{Card}$ such that $\mathcal{R}(Q, C) = \{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{\kappa} \downarrow^+ m \in R \,\}$. We judiciously choose classes of cardinals such that condition b will hold. Put $B = \{\, (\kappa_0, \dots, \kappa_{n-1}) \in \mathbb{N}^n \mid \max\{\kappa_0, \dots, \kappa_{n-1}\}) = m \,\} \subseteq \{0, \dots, m\}^n$. By the preceding proposition, we know that for every $A \subseteq B$ there is $\psi_A \in \mathrm{FO}[\tau_Q]$ such that

$$\mathcal{R}_{\tau_Q}(\psi_A) = \{\, \boldsymbol{\kappa} \in \mathrm{Card}^n \mid \boldsymbol{\kappa} \downarrow m \in A \,\}.$$

For every $\lambda \in C$, define $R_\lambda = \{\, \boldsymbol{\kappa} \in \{0, \dots, m\}^n \mid \boldsymbol{\kappa}\hat{\ }(\lambda) \in R \,\}$. Now for every $A \subseteq B$, put

$$C_A = \{\, \lambda \in C \mid A = R_\lambda \cap B \,\}.$$

We aim to show that $\mathrm{FO}(Q) \equiv \mathcal{L} \,/\mathcal{S}_C$ where $\mathcal{L} = \mathrm{FO}(\{\, \Omega_{C_A} \mid A \subseteq B \,\})$.

For $\boldsymbol{\kappa} \in \mathrm{Card}^n$, write $\boldsymbol{s}(\boldsymbol{\kappa}) = \boldsymbol{s}(\boldsymbol{\kappa}, (n))$ for the sum of components of $\boldsymbol{\kappa}$, for short. Note that $\mathcal{R}(\Omega_{C_A}) = C_A$ and put

$$S_A = \{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}) \in \mathcal{R}(\Omega_{C_A}) \,\} = \{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}) \in C_A \,\},$$

for $A \subseteq B$. Let us compare the relation $\mathcal{R}(Q, C)$ to the relation

$$\mathcal{R}' = \bigcup_{A \subseteq B} \left(\mathcal{R}_{\tau_Q}(\psi_A) \cap S_A \right),$$

so let $\boldsymbol{\kappa} = (\kappa_0, \dots, \kappa_n) \in C^n$. Denote $\lambda = \boldsymbol{s}(\boldsymbol{\kappa})$. Let A be the unique subset of B for which $\lambda \in C_A$, or equivalently, $\boldsymbol{\kappa} \in S_A$. Then $\boldsymbol{\kappa} \in \mathcal{R}'$ iff $\boldsymbol{\kappa} \in \mathcal{R}_{\tau_Q}(\psi_A)$ iff $\boldsymbol{\kappa} \downarrow m \in A$. If $\lambda \geq mn$, then by the pigeonhole principle, we have $\boldsymbol{\kappa} \downarrow m \in B$ so that $\boldsymbol{\kappa} \downarrow m \in A$ iff $\boldsymbol{\kappa} \downarrow m \in R_\lambda$ iff $\boldsymbol{\kappa} \downarrow^+ m = (\boldsymbol{\kappa} \downarrow m)\hat{\ }(\lambda) \in R$ iff $\boldsymbol{\kappa} \in \mathcal{R}(Q, C)$. Hence, $\boldsymbol{\kappa} \in \mathcal{R}'$ iff $\boldsymbol{\kappa} \in \mathcal{R}(Q, C)$ provided that $\lambda \geq mn$. Since there are only finitely many tuples with $\lambda < mn$, this means that the symmetric difference $\mathcal{R}(Q, C) \triangle \mathcal{R}'$ is finite and included in $\mathbb{N}^n$. So $\mathcal{R}(Q, C) \triangle \mathcal{R}' = \mathcal{R}_{\tau_Q}(\varphi)$ for some $\varphi \in \mathrm{FO}[\tau_Q]$. Consequently, $\mathcal{R}(Q, C)$ is a Boolean combination of relations $\mathcal{R}_{\tau_Q}(\varphi)$, $\mathcal{R}_{\tau_Q}(\psi_A)$ and S_A, for $A \subseteq B$. Applying Theorem 2.8 first to FO case, we see that $\mathcal{R}(Q, C)$ is a Boolean combination of relations of form $\{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q, C) \,\}$ where $q \in \{\, \Omega_{C_A} \mid A \subseteq B \,\} \cup \{\exists\}$ and $(\boldsymbol{U}, \boldsymbol{t}) \in V_{n, 2^{\mathrm{wd}(q)}}$. Another application of that theorem now gives that Q is $\mathcal{L}$-definable on $\mathcal{S}_C$ which implies $\mathrm{FO}(Q) \leq \mathcal{L} \,/\mathcal{S}_C$.

In the other direction, we still need to show that $\mathcal{L} \leq \mathrm{FO}(Q) / \mathcal{S}_C$, so let us show that each Ω_{C_A}, $A \subseteq B$, is $\mathrm{FO}(Q)$-definable on the class C. For every $\boldsymbol{\kappa} = (\kappa_0, \dots, \kappa_{n-1}) \in B$, pick $j(\boldsymbol{\kappa})$ such that $\kappa_{j(\boldsymbol{\kappa})} = m$, and let $\boldsymbol{t}(\boldsymbol{\kappa}) = (t_0, \dots, t_{n-1}) \in \mathbb{Z}^n$ be the tuple with $\sum_{i \in n} t_i = 0$ and such that $\boldsymbol{\kappa}$ and $\boldsymbol{t}(\boldsymbol{\kappa})$ differ only on one coordinate: $t_{j(\boldsymbol{\kappa})} \neq \boldsymbol{\kappa}_{j(\boldsymbol{\kappa})}$. Put $\boldsymbol{U} = (U_0, \dots, U_{n-1})$ where $U_{j(\boldsymbol{\kappa})} = \{0\}$ and $U_i = \emptyset$, otherwise. Then for every $\lambda \in C$, $\lambda \geq mn$, we have

$$(\boldsymbol{s}((\lambda), \boldsymbol{U}) + \boldsymbol{t}(\boldsymbol{\kappa})) \downarrow m = \boldsymbol{\kappa}.$$

Put

$$\mathcal{R}_{\boldsymbol{\kappa}} = \{\, \lambda \in C \mid \boldsymbol{s}((\lambda), \boldsymbol{U}) + \boldsymbol{t}(\boldsymbol{\kappa}) \in \mathcal{R}(Q, C) \,\}.$$

For $A \subseteq B$, consider the relation

$$\mathcal{R}_A = \bigcap_{\boldsymbol{\kappa} \in A} \mathcal{R}_{\boldsymbol{\kappa}} \smallsetminus \bigcup_{\boldsymbol{\kappa} \in B \smallsetminus A} \mathcal{R}_{\boldsymbol{\kappa}}$$

as a unary relation on C. For every $\lambda \in C$, $\lambda \geq mn$, we have that $\lambda \in \mathcal{R}_A$ iff

$$\begin{aligned} A &= \{\, \boldsymbol{\kappa} \in B \mid \lambda \in \mathcal{R}_{\boldsymbol{\kappa}} \,\} \\ &= \{\, \boldsymbol{\kappa} \in B \mid \boldsymbol{s}((\lambda), \boldsymbol{U}) + \boldsymbol{t}(\boldsymbol{\kappa}) \in \mathcal{R}(Q, C) \,\} \\ &= B \cap R_\lambda, \end{aligned}$$

as $(\boldsymbol{s}((\lambda), \boldsymbol{U}) + \boldsymbol{t}(\boldsymbol{\kappa})) \downarrow^+ m = \boldsymbol{\kappa}\hat{\ }(\lambda)$ and $\mathcal{R}(Q, C) = \{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{\kappa} \downarrow^+ m \in R \,\}$. So $\lambda \in \mathcal{R}_A$ iff $\lambda \in C_A$, provided that $\lambda \geq mn$. We have shown that $\mathcal{R}_A \triangle C_A \subseteq \{0, \dots, mn\}$ which implies that $C_A = \mathcal{R}(\Omega_{C_A}, C)$ is an appropriate Boolean combination. Hence, Ω_{C_A} is $\mathrm{FO}(Q)$-definable on the class C.

Since Q is $\mathcal{L}$-definable and each Ω_{C_A}, $A \subseteq B$, $\mathrm{FO}(Q)$-definable on the class $\mathcal{S}_C$, we get $\mathrm{FO}(Q) \equiv \mathcal{L} / \mathcal{S}_C$. □

4 Regularity and small width

In this section, we study the borderline between Mostowski quantifiers and other unary quantifiers. We thus concentrate on unary quantifiers of width one (Mostowski quantifiers) and width two. We approach this issue in a most concrete manner: We repeat three problems by Jouko Väänänen [Vää97] and provide answers in this and the next section. In the process, we shall learn that regularity is an important factor considering such problems.

Let $f \colon \mathbb{N} \to \mathbb{N}$ and put $g \colon \mathbb{N} \to \mathbb{Z}$, $g(n) = \min\{f(n), n + 1 - f(n)\}$. Väänänen asked the following questions:

a) If f is unbounded, does it imply that I_f is not definable by Mostowski quantifiers?

b) Assume that $f(n) \leq n + 1$, for every $n \in \mathbb{N}$. If g is unbounded, does it imply that Most_f is not definable by Mostowski quantifiers?

c) If f is unbounded, does it imply that R_f is not definable by Mostowski quantifiers?

Väänänen showed that the answer for c is affirmative and conjectured that this is the case for a and b two. Here, these conjectures are shown to be false and a slightly strengthened version of c is presented.

Obviously, $\mathrm{FO}(\mathsf{I}_f) \leq \mathrm{FO}(\mathsf{R}_f)$ and $\mathrm{FO}(\mathsf{Most}_f) \leq \mathrm{FO}(\mathsf{R}_f)$. Note that one should not confuse the concept of boundedness of quantifiers to boundedness of f: The fact that f is bounded does not imply that any of I_f, Most_f of R_f would be bounded, it rather implies that

$$\mathrm{FO}(\mathsf{I}_f) \equiv \mathrm{FO}(\mathsf{R}_f) \equiv \mathrm{FO}(\mathsf{Most}_f) \equiv \mathrm{FO}(\mathsf{C}_{S_0}, \ldots, \mathsf{C}_{S_m})$$

for $S_i = f^{-1}\{i\}$, $i = 0, \ldots, m = \max\{\, f(n) \mid n \in \mathbb{N} \,\}$.

An impatient reader may now proceed right away to the two examples of this section where negative solutions to the questions a and b are provided. These solutions look like as if they could be constructed from the scratch, without any special knowledge of unary quantifiers. However, in reality, I found the constructions since I knew where to look for the solution. Therefore, we develop some general theory first.

We devise an appropriate topological space for studying unary quantifiers of fixed width, essentially as points of the space. For technical reasons, relations of the quantifiers rather than quantifiers themselves are used as points. Note first that for any set X, the power set $\mathcal{P}(X)$ is in a natural way a topological space: Consider the canonical bijection $\mathcal{P}(X) \to {}^X 2 = {}^X\{0,1\}$, $A \mapsto \chi_A$ where χ_A is the characteristic function of the set A, i.e.,

$$\chi_A \colon X \to 2,\ \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

Now, if we regard 2 as a discrete space of two points, then ${}^X 2$ is naturally endowed with the product topology, and the canonical bijection induces a topology in $\mathcal{P}(X)$. Thus, $\mathcal{P}(X)$ and ${}^X 2$ are homeomorphic. A natural basis for $\mathcal{P}(X)$ is $\mathcal{B} = \{\, U(A,B) \mid A, B \subseteq X, A \cap B = \emptyset \,\}$ where

$$U(A,B) = \{\, Y \subseteq X \mid A \subseteq Y,\ Y \cap B = \emptyset \,\}.$$

Note also that $\mathcal{P}(X)$ is a Hausdorff space, and, by Tychonoff's theorem, a compact space.

Now fix a finite unary vocabulary τ and a class C of cardinals with $\omega \subseteq C$ for studying unary quantifiers of vocabulary τ. Put $n = 2^{|\tau|}$. Then the appropriate topological space is $\mathfrak{X}_n = \mathcal{P}((C \cup \mathbb{Z})^n)$. Note that the notation makes sense, even if n is not a power of 2.

For every relation $R \subseteq (C \cup \mathbb{Z})^n$ and finite $A \subseteq C \cup \mathbb{Z}$, define

$$N(R,A) = \{\, S \in \mathfrak{X}_n \mid S \cap A^n = R \cap A^n \,\}.$$

It is readily seen that, not only $N(R,A) = U(R \cap A^n, A^n \smallsetminus R)$ and thus $N(R,A)$ is a member of the canonical basis, but also that

$$\mathcal{B}_n = \{\, N(R,A) \mid R \subseteq (C \cup \mathbb{Z})^n,\ A \subseteq C \cup \mathbb{Z} \text{ finite} \,\}$$

is another basis for $\mathfrak{X}_n$.

An important, but easy to see, technical point is that sets

$$K_1 = \{\, \mathcal{R}(q, C) \mid q \text{ is a quantifier of vocabulary } \tau \,\}$$

and

$$K_2 = \{\, \mathcal{R}(q, C) \mid q \text{ is a universe-independent quantifier of vocabulary } \tau \,\}$$

are closed in $\mathfrak{X}_n$. Since we deal with relations, it is helpful to observe that Boolean operations are continuous. In specific, $f\colon \mathfrak{X}_n \to \mathfrak{X}_n$, $f(R) = A^n \smallsetminus R$ is even a homeomorphism and $g\colon \mathfrak{X}_n \times \mathfrak{X}_n \to \mathfrak{X}_n$, $g(R, S) = R \cap S$, is continuous. A further typical way of manipulating relations is the following:

Lemma 4.1. *Let $t\colon (C \cup \mathbb{Z})^n \to (C \cup \mathbb{Z})^l$ where $n, l \in \mathbb{Z}_+$. Then $h\colon \mathfrak{X}_l \to \mathfrak{X}_n$, $h(R) = t^{-1}[R]$ is continuous.*

Proof. It is enough to check that h is continuous at any $R \in \mathfrak{X}_l$. Let $B \subseteq C \cup \mathbb{Z}$ be finite. Then there is finite $B' \subseteq C \cup \mathbb{Z}$ such that $t[B^n] \subseteq (B')^l$. Now for every $S \in N(R, B')$ we have $h(S) \cap B^n = h(R) \cap B^n$, so $h(S) \in N(h(R), B)$ and h is continuous at R. □

Lemma 4.2. *Let q be a unary quantifier of finite vocabulary τ, $l = 2^{|\tau|}$, $n \in \mathbb{Z}_+$ and $(\boldsymbol{U}, \boldsymbol{t}) \in J_{n,l}$. Then there is a quantifier q' of the same vocabulary τ and $(\boldsymbol{U}', \boldsymbol{t}') \in J_{n,l}$ where $\boldsymbol{U}' = (U'_0, \ldots, U'_{l-1})$ such that*

1) $n - 1 \in U'_{l-1}$ and

2) for every $\boldsymbol{\kappa} \in \mathrm{Card}^n$, we have $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q)$ iff $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}') + \boldsymbol{t}' \in \mathcal{R}(q')$.

Proof. Let $\boldsymbol{U} = (U_0, \ldots, U_{l-1})$. Pick any permutation σ of $l = \{0, \ldots, l-1\}$ such that $n - 1 \in U_{\sigma(l-1)}$. This is possible, since $\{U_0, \ldots, U_{l-1}\} \smallsetminus \{\emptyset\}$ is a partition of n. Consider $R' = \{\, \boldsymbol{\kappa} \circ \sigma \mid \boldsymbol{\kappa} \in \mathcal{R}(q) \,\}$. Obviously $R' \subseteq \mathrm{Card}^n \smallsetminus \{\boldsymbol{0}\}$, so there is a quantifier q' of vocabulary τ such that $R' = \mathcal{R}(q')$. Put $\boldsymbol{U}' = \boldsymbol{U} \circ \sigma$ and $\boldsymbol{t}' = \boldsymbol{t} \circ \sigma$. Then it is easy to see that the conditions of the lemma are satisfied. □

In plain terms, the following results convey the idea that irregularity of a logic does not help in defining universe-independent quantifiers.

Theorem 4.3. *Let Q be a universe-independent unary quantifier of finite width and let C be a set of cardinals which is an infinite initial segment of the class C. Suppose*

$$\mathrm{FO}(Q) \leq \mathrm{FO}(\mathcal{Q}) \,/\mathcal{S}_C$$

for some finite set $\mathcal{Q}$ of unary quantifiers of width at most $k \in \omega$. Then there is a finite set $\mathcal{Q}^*$ of unary universe-independent quantifiers of width k such that

$$\mathrm{FO}(Q) \leq \mathrm{FO}(\mathcal{Q}^*) \,/\mathcal{S}_C.$$

Proof. Let σ be a vocabulary of Q and $n = 2^{|\sigma|} \in \mathbb{N}$. By renaming and adding dummy symbols, we may assume that the quantifiers $q \in \mathcal{Q}$ have a common vocabulary τ. Put $l = 2^k$. The plan of the proof is as follows: It is first showed that we can write

$$\mathcal{R}(Q, C) = f(\mathcal{R}(q_0, C), \ldots, \mathcal{R}(q_{r-1}, C))$$

with $r \in \mathbb{N}$, $q_0, \ldots, q_{r-1} \in \mathcal{Q}$ and $f\colon \mathfrak{X}_l^r \to \mathfrak{X}_n$ continuous. Then it is showed that it is possible to approximate quantifiers $q_0, \ldots, q_{r-1}$ by universe-independent $\tilde{q}_0, \ldots, \tilde{q}_{r-1}$ so that $f(\mathcal{R}(\tilde{q}_0, C), \ldots, \mathcal{R}(\tilde{q}_{r-1}, C))$ is arbitrarily close to $\mathcal{R}(Q, C)$. Finally, topological reasoning demonstrates that this implies the claim of the theorem.

For the first point, we first apply Theorem 2.8 which implies that $\mathcal{R}(Q, C)$ is a Boolean combination of relations of form $\{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q, C) \,\}$ where $q \in \mathcal{Q} \cup \{\exists\}$ and $(\boldsymbol{U}, \boldsymbol{t}) \in J_{n,l}$. More formally, let us write the Boolean combination in disjunctive normal form. This means finding $r \in \mathbb{Z}_+$, $\mathcal{I} \subseteq \mathcal{P}(r)$, quantifiers $q_0, \ldots, q_{r-1} \in \mathcal{Q}$ and $(\boldsymbol{U}_i, \boldsymbol{t}_i) \in J_{n,l}$ $(i \in r)$ such that

$$\begin{aligned}\mathcal{R}(Q, C) = \bigcup_{I \in \mathcal{I}} \Big(\bigcap_{i \in I} \{\, \boldsymbol{\kappa} \in C^n \mid h_i(\boldsymbol{\kappa}) \in \mathcal{R}(q_i, C) \,\} \\ \smallsetminus \bigcup_{i \in r \smallsetminus I} \{\, \boldsymbol{\kappa} \in C^n \mid h_i(\boldsymbol{\kappa}) \in \mathcal{R}(q_i, C) \,\} \Big)\end{aligned}$$

where $h_i\colon (C \cup \mathbb{Z})^n \to C \cup \mathbb{Z}$, $h_i(\boldsymbol{\kappa}) = \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i) + \boldsymbol{t}_i$. Hence

$$\mathcal{R}(Q, C) = f(\mathcal{R}(q_0, C), \ldots, \mathcal{R}(q_{r-1}, C))$$

where $f\colon \mathfrak{X}_l^r \to \mathfrak{X}_n$,

$$f(R_0, \ldots, R_{r-1}) = \bigcup_{I \in \mathcal{I}} \Big((C^n \smallsetminus \{\boldsymbol{0}\}) \cap \bigcap_{i \in I} h_i^{-1}[R_i] \smallsetminus \bigcup_{i \in r \smallsetminus I} h_i^{-1}[R_i] \Big).$$

By Lemma 4.1, mappings $R \mapsto h_i^{-1}[R]$ are continuous. Mapping f can be composed using these, projections, constant functions and Boolean functions, so f is continuous.

For the second point, let $B \subseteq C \cup \mathbb{Z}$ be finite and non-empty. Let us find universe-independent quantifiers $\tilde{q}_0, \ldots, \tilde{q}_{r-1}$ with vocabulary τ such that

$$f(\mathcal{R}(\tilde{q}_0, C), \ldots, \mathcal{R}(\tilde{q}_{r-1}, C)) \in N(\mathcal{R}(Q, C), B).$$

By Lemma 4.2, we may assume that $n - 1$ is a member of the last component of the sequence $\boldsymbol{U}_i$, for each $i \in r$. Choose $\mu \in C$ such that $\mu \geq n \cdot \kappa + 1$, for each $\kappa \in B$. Fix $i \in r$ for a moment, and let us choose an appropriate quantifier $\tilde{q}_i$. We essentially fix the semantics according to the behaviour of q_i on cardinality μ. The defining class $K_{\tilde{q}_i}$ of $\tilde{q}_i$ consists thus of all τ-structures $\mathfrak{M}$ for which there is $\mathfrak{N} \in K_{q_i}$ such that $\operatorname{card}(\mathfrak{N}) = \mu$, for every $R \in \tau$ we have $R^{\mathfrak{M}} = R^{\mathfrak{N}}$, and $|\operatorname{Dom}(\mathfrak{N}) \smallsetminus \bigcup_{R \in \tau} R^{\mathfrak{N}}| = \mu$ provided that $\mu \geq \omega$. Note that $\mathfrak{N}$ is unique up to isomorphism, if it exists. Clearly, $\tilde{q}_i$ is universe-independent.

Let us compare relation $\mathcal{R}(Q, C)$ to $f(\mathcal{R}(\tilde{q}_0, C), \ldots, \mathcal{R}(\tilde{q}_{r-1}, C))$. Write $\widetilde{R}_i = \mathcal{R}(\tilde{q}_i, C)$, for short. Let $\boldsymbol{\kappa} \in B^n$. If $\boldsymbol{\kappa} \notin C^n \smallsetminus \{\boldsymbol{0}\}$, then clearly $\boldsymbol{\kappa} \notin \mathcal{R}(Q, C)$ and $\boldsymbol{\kappa} \notin f(\widetilde{R}_0, \ldots, \widetilde{R}_{r-1})$, so suppose $\boldsymbol{\kappa} = (\kappa_0, \dots, \kappa_{r-1}) \in C^n \setminus \{\mathbf{0}\}$. Since $\sum_{i\in n} \kappa_i \le \mu$, we can pick $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_{r-1}) \in C^n$ such that $\boldsymbol{\lambda}\restriction(n-1) = \boldsymbol{\kappa}\restriction(n-1)$ and $\sum_{i\in n} \lambda_i = \mu$, and $\lambda_{n-1} = \mu$ provided that $\mu \ge \omega$. Our hypothesis on the form of $\boldsymbol{U}_i$ implies now that $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i)\restriction(l-1) = \boldsymbol{s}(\boldsymbol{\lambda}, \boldsymbol{U}_i)\restriction(l-1)$ and if $\mu \ge \omega$, then $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i)(l-1) = \mu$. Switching now to the corresponding τ-structures $\mathfrak{M}$, $\mathfrak{N}$ such that $\boldsymbol{\kappa}_{\mathfrak{M}} = \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i) + \boldsymbol{t}_i = h_i(\boldsymbol{\kappa})$ and $\boldsymbol{\kappa}_{\mathfrak{N}} = h_i(\boldsymbol{\lambda})$ it is easy to see that $h_i(\boldsymbol{\kappa}) \in \widetilde{R}_i$ iff $\mathfrak{M} \in K_{\tilde{q}_i}$ iff $\mathfrak{N} \in K_{q_i}$ iff $h_i(\boldsymbol{\lambda}) \in \mathcal{R}(q_i, C)$, as $\mathfrak{M}$ and $\mathfrak{M}$ have, up to isomorphism, the same mutual relation as in the definition of $\tilde{q}_i$. This, in turn, implies that $\boldsymbol{\kappa} \in f(\widetilde{R}_0, \dots, \widetilde{R}_{r-1})$ iff $\boldsymbol{\lambda} \in f(\mathcal{R}(q_0, C), \dots, \mathcal{R}(q_{r-1}, C)) = \mathcal{R}(Q, C)$. As Q is universe-independent, we get $\boldsymbol{\kappa} \in \mathcal{R}(Q, C)$ iff $\boldsymbol{\lambda} \in \mathcal{R}(Q, C)$ iff $\boldsymbol{\kappa} \in f(\widetilde{R}_0, \dots, \widetilde{R}_{r-1})$. We have proved that $f(\widetilde{R}_0, \dots, \widetilde{R}_{r-1}) \in N(\mathcal{R}(Q, C), B)$.

Finally, we draw topological conclusions. Denote the set of all $\mathcal{R}(\tilde{q}, C)$ where $\tilde{q}$ is universe-independent quantifier of vocabulary τ by K_2. In the previous paragraph we showed that $\mathcal{R}(Q, C)$ is in the closure of $f[K_2^r]$. We know that K_2 is closed in $\mathfrak{X}_l$. As $\mathfrak{X}_l$ is compact, K_2 is also compact as a closed subset. Consequently, the image $f[K_2^r]$ is also compact, as f is continuous. Hence, $f[K_2^r]$ is closed and $\mathcal{R}(Q, C) \in f[K_2^r]$. This means that there are universe-independent $q_0^*, \dots, q_{r-1}^*$ having vocabulary τ such that

$$\mathcal{R}(Q, C) = f(\mathcal{R}(q_0^*, C), \dots, \mathcal{R}(q_{r-1}^*, C)).$$

By Theorem 2.8, we have $\mathrm{FO}(Q) \le \mathrm{FO}(\mathcal{Q}^*) / \mathcal{S}_C$ with $\mathcal{Q}^* = \{ q_i^* \mid i \in r \}$. □

In the case $k = 1$, we get:

Corollary 4.4. *Let Q be a universe-independent unary quantifier of finite width, $\mathcal{Q}$ a finite set of Mostowski quantifiers and C infinite initial segment of cardinals. Suppose $\mathrm{FO}(Q) \le \mathrm{FO}(\mathcal{Q}) / \mathcal{S}_C$. Then there is a finite set of cardinality quantifiers $\widetilde{\mathcal{Q}}$ such that $\mathrm{FO}(Q) \le \mathrm{FO}(\widetilde{\mathcal{Q}}) / \mathcal{S}_C$.* □

Next I provide two counterexamples:

Example 4.5. Let us construct an unbounded $f\colon \mathbb{N} \to \mathbb{N}$ such that I_f is definable by Mostowski quantifiers. The last corollary shows that if this is really possible, then I_f has to be definable by cardinality quantifiers. Without further ado, we put $f\colon \mathbb{N} \to \mathbb{N}$,

$$f(n) = \begin{cases} 3n, & \text{for } n \in S \\ 0, & \text{otherwise} \end{cases}$$

where $S = \{ 9^n \mid n \in \mathbb{N} \}$. This example can be varied in many ways, but the critical idea is that S is sparse enough. We intend to show that $\mathrm{FO}(I_f) \le \mathrm{FO}(\mathsf{C}_{\mathbb{N}}, \mathsf{C}_S, \mathsf{C}_T, \mathsf{C}_{\widetilde{T}}, \mathsf{C}_{T^*})$ where

$$\begin{aligned} T &= \{ 3s \mid s \in S \}, \\ \widetilde{T} &= \{ m \in \mathbb{N} \mid m/n \in [1, 4/3] \text{ for some } n \in T \} \text{ and} \\ T^* &= \{ m \in \mathbb{N} \mid m/n \in [2/3, 5/6] \cup [7/6, 4/3] \text{ for some } n \in T \}. \end{aligned}$$

Consider the sentences

$$\varphi_0 = \mathsf{C}_S\, xU(x) \land \mathsf{C}_T\, xV(x) \land \mathsf{C}_{\widetilde{T}}\, x(U(x) \lor V(x)) \\ \land \big(\mathsf{C}_{T^*}\, x(U(x) \lor V(x)) \lor \mathsf{C}_{T^*}\, x(\neg U(x) \land V(x))\big)$$

and

$$\varphi = \mathsf{C}_{\mathbb{N}}\, U(x) \land ((\neg\, \mathsf{C}_S\, xU(x) \land \neg\exists x V(x)) \lor \varphi_0)\,.$$

We claim that φ defines I_f.

Let $\mathfrak{M}$ be a $\{U, V\}$-structure. If $|U^{\mathfrak{M}}| \notin S$ or $|V^{\mathfrak{M}}| \notin T$, then clearly $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \in K_{\mathsf{I}_f}$. Suppose then $|U^{\mathfrak{M}}| \in S$ and $|V^{\mathfrak{M}}| \in T$, i.e., $|U^{\mathfrak{M}}| = 3^{2m}$ and $|V^{\mathfrak{M}}| = 3^{2n+1}$ for some $m, n \in \mathbb{N}$. Then $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi_0$ and we have several cases:

1) If $m > n$, then $3^{2m} \le |U^{\mathfrak{M}} \cup V^{\mathfrak{M}}| \le \frac{4}{3} \cdot 3^{2m}$. Then for every $t \in T$, we have either $|U^{\mathfrak{M}} \cup V^{\mathfrak{M}}| \ge 3t > \frac{4}{3}t$ or $|U^{\mathfrak{M}} \cup V^{\mathfrak{M}}| \le \frac{4}{9}t < t$. Hence, $\mathfrak{M} \notin K_{\mathsf{I}_f}$ and $\mathfrak{M} \not\models \mathsf{C}_{\widetilde{T}}\, x(U(x) \lor V(x))$ implying $\mathfrak{M} \not\models \varphi$.

2) If $m \le n$, then $3^{2n+1} \le |U^{\mathfrak{M}} \cup V^{\mathfrak{M}}| \le \frac{4}{3} \cdot 3^{2n+1}$, so $|U^{\mathfrak{M}} \cup V^{\mathfrak{M}}| \in \widetilde{T}$. Write $v_- = |V^{\mathfrak{M}} \setminus U^{\mathfrak{M}}|$, $v = |V^{\mathfrak{M}}|$ and $v_+ = |U^{\mathfrak{M}} \cup V^{\mathfrak{M}}|$. Then $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi_0$ iff $v_+ \in T^*$ or $v_- \in T^*$. We also see that $v_- \le v \le v_+$ and $\left|\frac{v_+ - v_-}{v}\right| = \frac{|U^{\mathfrak{M}}|}{|V^{\mathfrak{M}}|} = \frac{1}{3} \cdot 3^{2(m-n)}$.

2a) If $m < n$, then $\left|\frac{v_+ - v_-}{v}\right| \le \frac{1}{27} < \frac{1}{6}$, so $1 \le \frac{v_+}{v} \le 1 + \frac{v_+ - v_-}{v} < \frac{7}{6}$ and $1 \ge \frac{v_-}{v} \ge 1 - \left|\frac{v_+ - v_-}{v}\right| > 1 - \frac{1}{6} = \frac{5}{6}$. Consequently, $v_+, v_- \notin T^*$. Hence, $\mathfrak{M} \notin K_{\mathsf{I}_f}$ and $\mathfrak{M} \not\models \varphi$.

2b) If $m = n$, then $\frac{v_+ - v}{v} + \frac{v - v_-}{v} = \frac{v_+ - v_-}{v} = \frac{1}{3}$ implying $\frac{1}{6} \le \frac{v_+ - v}{v} \le \frac{1}{3}$ or $\frac{1}{6} \le \frac{v - v_-}{v} \le \frac{1}{3}$. In the first case, it holds that $\frac{7}{6} \le \frac{v_+}{v} \le \frac{4}{3}$ and $v_+ \in T^*$, in the second case, it holds that $\frac{2}{3} \le \frac{v_-}{v} \le \frac{5}{6}$ and $v_- \in T^*$, respectively. So $\mathfrak{M} \in K_{\mathsf{I}_f}$ and $\mathfrak{M} \models \varphi$.

We have showed that φ defines I_f, so $\mathrm{FO}(\mathsf{I}_f) \le \mathrm{FO}(\mathsf{C}_{\mathbb{N}}, \mathsf{C}_S, \mathsf{C}_T, \mathsf{C}_{\widetilde{T}}, \mathsf{C}_{T^*})$ even if f is unbounded.

Example 4.6. Consider $f\colon \mathbb{N} \to \mathbb{N}$,

$$f(n) = \begin{cases} \lceil n/2 \rceil, & \text{if } n \in S \\ n+1, & \text{otherwise} \end{cases}$$

where $S = \{\, 2^k - 1 \mid k \in \mathbb{Z}_+ \,\}$. Note that $g\colon \mathbb{N} \to \mathbb{N}$, $g(n) = \min\{f(n), n+1-f(n)\}$ is unbounded. Let us construct set $T_+, T_- \subseteq \mathbb{N}$ such that Most_f is definable in $\mathrm{FO}(\mathsf{C}_S, \mathsf{C}_{T^+}, \mathsf{C}_{T^-})$. By induction on $k \in \mathbb{N}$, put $T_0^+ = T_0^- = \emptyset$ and

$$T_{k+1}^+ = T_k^+ \cup \{\, 2^{k+1} - 1 - n \mid n \in T_k^- \,\},$$
$$T_{k+1}^- = T_k^- \cup \{\, 2^{k+1} - 1 - n \mid n \in \{0, \dots, 2^k - 1\} \setminus T_k^+ \,\}.$$

Put $T^+ = \bigcup_{k \in \mathbb{N}} T_k^+$ and $T^- = \bigcup_{k \in \mathbb{N}} T_k^-$. The relevant properties of sets T^+ and T^- are: For $n \in \mathbb{Z}_+$ and $k = \lceil \mathrm{lb}(n+1) \rceil$,

(1) $$n \in T^+ \text{ iff } 2^k - 1 - n \in T^-$$

and

(2) $$n \in T^- \text{ iff } 2^k - 1 - n \notin T^+.$$

The idea behind the sets T^+ and T^- is akin to the well-known Morse-Thue sequence. Consider the sentence

$$\varphi = \mathsf{C}_S\, xU(x) \wedge \big(\mathsf{C}_{T^+}\, x(U(x) \wedge V(x)) \iff \mathsf{C}_{T^+}\, x(U(x) \wedge \neg V(x))\big).$$

Let $\mathfrak{M}$ be a τ-structure. If $|U^{\mathfrak{M}}| \notin S$, then $\mathfrak{M} \notin K_{\mathsf{Most}_f}$ and $\mathfrak{M} \not\models \varphi$. Suppose now $|U^{\mathfrak{M}}| \in S$ and pick $k \in \mathbb{Z}_+$ such that $|U^{\mathfrak{M}}| = 2^k - 1$. Write $n = |U^{\mathfrak{M}} \cap V^{\mathfrak{M}}|$; then $U^{\mathfrak{M}} \smallsetminus V^{\mathfrak{M}} = 2^k - 1 - n$. If $n \geq f(2^k - 1) = 2^{k-1}$, then $k = \lceil \mathrm{lb}(n+1) \rceil$ and by (1), we have $\mathfrak{M} \models \varphi$. If $n < f(2^k - 1) = 2^{k-1}$, then $2^k - 1 - n \geq 2^{k-1}$ and applying (2) to $2^k - 1 - n$ instead of n we get $\mathfrak{M} \not\models \varphi$. Hence, $\mathfrak{M} \in K_{\mathsf{Most}_f}$ iff $|U^{\mathfrak{M}} \cap V^{\mathfrak{M}}| \geq f(|U^{\mathfrak{M}}|)$ iff $\mathfrak{M} \models \varphi$. We have seen that Most_f is definable in $\mathrm{FO}(\mathsf{C}_S, \mathsf{C}_{T^+}, \mathsf{C}_{T^-})$ even if g is unbounded.

5 Hierarchy

Width of a quantifier is a syntactical notion. On the semantic side, it is clear that large width by itself does not guarantee strong expressive power. From the onset, it is not obvious at all that there is a unary quantifier of width $n+1$ which is not definable by means of unary quantifiers of width n, for every $n \in \mathbb{N}$. This hierarchy result can be established in various ways: Per Lindström demonstrated this by a simple but clever counting argument without going to concrete examples. Nešetřil and Väänänen [NV96] provided examples of quantifiers and showed the undefinability by some other combinatorial means.

Here, we follow [Luo00] were combinatorial theory was developed for solving the hierarchy problem. This entails introducing the concept of unary dimension for unary quantifiers and a quite systematic procedure for calculating it. Unary dimension is the semantic counterpart for the width of a quantifier, except for the fact that maximal dimension for unary quantifiers of width l is 2^l. Not accidentally, 2^l is the maximal number of parts to which l predicates may partition unary structure. The exposition is rather sketchy in this section, relying a lot on intuition. The reader is advised to consult [Luo00] for further details and explanations.

As a by-product of the theory, we get an affirmative answer to question c in the preceding section.

Since definability problems of unary quantifiers can be reduced to combinatorial problems of relations, we start with some notions of relations. We need a dimensional notion that reflects the structure of the relation better than the arity. Intuitively, the idea is clear: The relation may or may not depend simultaneously on all of its variables. It is apparent that there is no such dependence if some of the variables are dummy (futile). For a more elaborate example, let $f, g\colon \mathbb{N} \to \mathbb{N}$ be (non-trivial) mappings and consider the relation $R = \{\, (x, f(x), g(f(x))) \mid x \in \mathbb{N} \,\}$. Then we can test if $(x, y, z) \in R$ by just checking if $y = f(x)$ and $z = g(y)$, i.e., checking some binary relations. So, even if R is ternary, it has essentially binary structure.

The following is an implementation of this idea:

Definition 5.1. A relation $R \subseteq A^n$ is *congruent with* a function $f\colon A^n \to B$, if for all $\boldsymbol{a}, \boldsymbol{b} \in A^n$, we have that $\boldsymbol{a} \in R$ and $f(\boldsymbol{a}) = f(\boldsymbol{b})$ imply $\boldsymbol{b} \in R$. If $(f_J)_{J \in \mathcal{J}}$ is a family of functions $f_J = {}^J A \to X_J$, then $\nabla_{J\in\mathcal{J}} f_J$ stands for the function $f\colon {}^I A \to \prod_{J\in\mathcal{J}} X_J$, $f(\boldsymbol{a}) = (f_J(\boldsymbol{a}\restriction J))_{J\in\mathcal{J}}$ where $I = \cup\mathcal{J}$ and $X_J = \mathrm{rg}(f_J)$, for $J \in \mathcal{J}$. The *rank* of the relation R is the least $k \in \mathbb{Z}_+$ such that there are finite colourings $\chi_I\colon {}^I A \to F_I$, $I \in [n]^k$ such that R is congruent with $\chi = \nabla_{I\in[n]^k}\chi_I\colon A^n \to \prod_{I\in[n]^k} F_I$.

However, this notion is sufficient for our purposes only as far as relations on infinite cardinals are concerned. For finite cardinals, we need to elaborate the idea further. This need results from the fact that we can define unions of predicates, which corresponds to taking sums of cardinals.

Definition 5.2. Let C be an initial segment of cardinals and let $R \subseteq (C \cup \mathbb{Z})^n$. Then the *rank of R relative to addition*, in symbols $r_+(R)$, is the least $l \in \mathbb{Z}_+$ for which the following holds: There are finite colourings $\chi_{\boldsymbol{U}}\colon (C\cup\mathbb{Z})^l \to F_{\boldsymbol{U}}$, for $\boldsymbol{U} \in \mathcal{U}_{n,l}$, such that R is congruent with the colouring $\chi\colon (C \cup \mathbb{Z})^n \to \prod_{\boldsymbol{U}\in\mathcal{U}_{n,l}} F_{\boldsymbol{U}}$, $\chi(\boldsymbol{a}) = \big(\chi_{\boldsymbol{U}}(\boldsymbol{s}(\boldsymbol{a},\boldsymbol{U}))\big)_{\boldsymbol{U}\in\mathcal{U}_{n,l}}$. The function χ is denoted by $\nabla^+_{\boldsymbol{U}\in\mathcal{U}_{n,l}}\chi_{\boldsymbol{U}}$.

We list some of the properties of ranks r and r_+. The proofs are easy but technical and can be found in [Luo00, Sections 2 and 3]. For $R, S \subseteq (C \cup \mathbb{Z})^n$,

(1) $1 \le r_+(R) \le r(R) \le n$,

(2) $r((C \cup \mathbb{Z})^n \smallsetminus R) = r(R)$ and similarly $r_+((C \cup \mathbb{Z})^n \smallsetminus R) = r_+(R)$,

(3) $r(R \cap S) \le \max\{r(R), r(S)\}$ and $r_+(R \cap S) \le \max\{r_+(R), r_+(S)\}$,

(4) $r(R) = r_+(R) = 1$ if R is finite,

(5) $r_+(R) = r_+(R + \boldsymbol{t})$ for any $\boldsymbol{t} \in \mathbb{Z}^n$,

(6) if $R = \{\, \boldsymbol{a} \in (C \cup \mathbb{Z})^n \mid \boldsymbol{s}(\boldsymbol{a},\boldsymbol{U}) \in T \,\}$ for some $\boldsymbol{U} \in \mathcal{U}_{n,l}$ and $T \subseteq A^l$, then $r_+(R) \le r_+(T)$.

Returning to quantifiers, we define:

Definition 5.3. Let Q be a unary quantifier of finite width and C be a set of cardinals with $C \supseteq \omega$. The *unary dimension* of Q *relative to* C is $\mathrm{udim}_C(Q) = r_+(\mathcal{R}(Q, C))$. The *unary dimension* $\mathrm{udim}(Q)$ of Q is $\mathrm{udim}(Q) = \max\{\, \mathrm{udim}_{\kappa\cap\mathrm{Card}}(Q) \mid \kappa \in \mathrm{Card}, \kappa \ge \omega \,\}$.

Let C be an infinite initial segment of Card and let $R \subseteq (C \smallsetminus \mathbb{N})^n \subseteq (C \cup \mathbb{Z})^n$. Then it is fairly easy to show (cf. [Luo00, Theorem 3.5]) that $r(R) \le \max\{r_+(R), 2\}$. Consequently, if Q is a unary quantifier of finite width, $\mathcal{R}(Q, \omega) = \emptyset$ and $r(\mathcal{R}(Q, \kappa)) \ge 2$ for some infinite cardinal κ, then $\mathrm{udim}(Q) = \max\{\, r(\mathcal{R}(Q, \kappa\cap\mathrm{Card})) \mid \kappa \in \mathrm{Card}, \kappa \ge \omega \,\}$. This means that if the expressive power of a quantifier trivializes on finite structures, but $\mathrm{udim}(Q) \ge 2$, then we may use the simpler rank r instead of r_+ in determining the unary dimension of Q.

Reformulating the results of [Luo00, Theorems 4.5 and 4.7] we have:

Theorem 5.4. *Let Q be a unary quantifier of finite width, and let C be an infinite initial set of cardinals.*

a) Suppose $\mathrm{FO}(Q) \leq \mathrm{FO}(\mathcal{Q}) / \mathcal{S}_C$ *where $\mathcal{Q}$ is a finite set of cardinals. Then* $\mathrm{udim}_C(Q) \leq \max_{q \in \mathcal{Q} \cup \{\exists\}} \mathrm{udim}_C(q)$.

b) Suppose $\mathrm{udim}_C(Q) < 2^k$ *with $k \in \mathbb{Z}_+$. Then there is a finite set $\mathcal{Q}$ of unary universe-independent quantifiers of width k such that Q is definable by* $\mathrm{FO}(\mathcal{Q})$ *on $\mathcal{S}_C$.*

Proof. (Sketch) a) By Theorem 2.8, we can write $\mathcal{R}(Q, C)$ as a Boolean combination of relations of form $\{\, \boldsymbol{\kappa} \in C^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathcal{R}(q, C) \,\}$ where $q \in \mathcal{Q} \cup \{\exists\}$, $n = 2^{\mathrm{wd}}(Q)$ and $(\boldsymbol{U}, \boldsymbol{t}) \in J_{n, l_q}$, $l_q = 2^{\mathrm{wd}(Q)}$. The basic properties of the rank relative to addition now imply that $r_+(\mathcal{R}(Q, C)) \leq \max_{q \in \mathcal{Q} \cup \{\exists\}} r_+(\mathcal{R}(q, C))$, or equivalently, $\mathrm{udim}_C(Q) \leq \max_{q \in \mathcal{Q} \cup \{\exists\}} \mathrm{udim}_C(q)$.

b) As $l = r_+(\mathcal{R}(Q, C)) = \mathrm{udim}_C(Q) \leq 2^k - 1$, there are finite colourings $\chi_{\boldsymbol{U}}$, $\boldsymbol{U} \in \mathcal{U}_{n,l}$ witnessing this. For each $\boldsymbol{U} \in \mathcal{U}_{n,l}$ and colour $c \in \mathrm{rg}(\chi_{\boldsymbol{U}})$ we pick a universe-independent $Q_{\boldsymbol{U},c}$ with $\mathcal{R}(Q_{\boldsymbol{U},c}) = \chi_{\boldsymbol{U}}^{-1}\{c\}$. It is then routine to show that $\mathcal{R}(Q, C)$ is a Boolean combination of appropriate kind, so that then Theorem 2.8 implies $\mathrm{FO}(Q) \leq \mathrm{FO}(\mathcal{Q}) / \mathcal{S}_C$ for the collection $\mathcal{Q}$ of quantifiers $Q_{\boldsymbol{U},c}$. □

Example 5.5. It is known that $\mathrm{udim}(\mathsf{C}_S) = 1$, for every $S \subsetneq \mathrm{Card}$, and $\mathrm{udim}(\mathsf{I}) = \mathrm{udim}(\mathsf{R}) = \mathrm{udim}(\mathsf{Maj}) = 2$. These facts are intuitively obvious, e.g., $\mathcal{R}(\mathsf{Maj})$ clearly depends on both of its variables, but also combinatorially easy to prove (see [Luo00]). The previous theorem now implies that I, R and Maj are not definable in the logic $\mathrm{FO}(\mathcal{C})$ where $\mathcal{C}$ is the collection of all cardinal quantifiers. We can no even better: I (and R) is universe-independent, so Corollary 4.4 shows that I is not definable in $\mathrm{FO}(\mathcal{M})$ where $\mathcal{M}$ is the collection of Mostowski quantifiers. This was first proved by Kolaitis and Väänänen [KV95]. They actually showed that I is not definable even in $\mathrm{FVL}(\mathcal{Q})$ for any finite $\mathcal{Q} \subseteq \mathcal{M}$, but this is only seemingly stronger, since $\mathrm{FVL}(\mathcal{Q})$ and $\mathrm{FO}(\mathcal{Q})$ can be shown to have the same expressive power on unary structures for finite vocabularies.

As a more elaborate application, we find an answer to question c in the preceding section by Väänänen. The combinatorial content needed for the result is extracted in the following lemma.

Lemma 5.6. *Let C be an infinite set of cardinals with $\omega \subseteq C$. Suppose $f \colon C \rightharpoonup C$, i.e., f is a partial function on C having values in C. Then the following are equivalent for $R = \{\, (\kappa, \lambda) \in C \times C \mid \kappa \in \mathrm{dom}(f),\, \lambda \geq f(\kappa) \,\}$:*

a) $\mathrm{rg}(f)$ *is finite.*

b) $r(R) = 1$.

c) $r_+(R) = 1$.

Proof. Since $1 \le r_+(R) \le r(R)$, it is immediate that condition b implies c.

Suppose $\mathrm{rg}(f)$ is finite. We choose as colours $F = \mathrm{rg}(f) \cup \{0, \mu\}$ with $\mu = \sup\{\, \kappa^+ \mid \kappa \in C \,\}$. Extend f to colouring $\chi_0\colon C \to F$ so that $\chi_0(\kappa) = \mu$, for $\kappa \in C \setminus \mathrm{dom}(f)$. Define $\chi_1\colon C \to F$, $\chi_1(\lambda) = \max\{\, \kappa \in F \mid \kappa \le \lambda \,\}$, and $\chi\colon C \times C \to F \times F$, $\chi(\kappa, \lambda) = (\chi_0(\kappa), \chi_1(\lambda))$. Obviously χ is a finite colouring. For $(\kappa, \lambda) \in C \times C$, it is easy to see that $(\kappa, \lambda) \in R$ iff $\chi(\kappa, \lambda) = (c_0, c_1)$ with $c_0 \le c_1$. Hence, R is congruent with χ, which implies $r(R) = 1$. So condition a implies condition b.

The true contents of the lemma is the implication from condition c to condition a. Suppose $\chi_i\colon C \to F_i$, $i = 0, 1, 2$ are finite colourings such that R is congruent with the colouring $\chi\colon C \times C \to F_0 \times F_1 \times F_2$, $\chi(\kappa, \lambda) = (\chi_0(\kappa), \chi_1(\lambda), \chi_2(\kappa + \lambda))$. We proceed by a sequence of claims.

Consider an arbitrary non-empty $I \subseteq \mathrm{dom}(f)$ and pick $\mu \in I$ such that $f(\mu) = \min f[I]$ and put $\nu = \min I$.

Claim 1: If $\chi_0 \restriction I$ is a constant function and $f(\mu) \ge \sup(I \cup \{\omega\})$, then $f \restriction I$ is a constant function.

This is because for arbitrary $\kappa \in I$, we have by our assumptions $\chi(\mu, f(\mu)) = (\chi_0(\mu), \chi_1(f(\mu)), \chi_2(\mu + f(\mu))) = (\chi_0(\kappa), \chi_1(f(\mu)), \chi_2(\kappa + f(\mu))) = \chi(\kappa, f(\mu))$. As $(\mu, f(\mu)) \in R$ and R is congruent with χ, we get $(\kappa, f(\kappa)) \in R$, i.e., $f(\mu) \ge f(\kappa)$. As $f(\mu) = \min f[I]$, this implies $f(\mu) = f(\kappa)$.

Claim 2: If $\chi_0 \restriction I$ and $\chi_2 \restriction I$ are constant functions, $\nu \ge \omega$ and $\nu \ge f(\mu)$, then $f \restriction I$ is a constant function.

Similarly than in Claim 1 we get, for $\kappa \in I$, that $\chi(\mu, f(\mu)) = (\chi_0(\mu), \chi_1(f(\mu)), \chi_2(\mu + f(\mu))) = (\chi_0(\mu), \chi_1(f(\mu)), \chi_2(\mu)) = (\chi_0(\kappa), \chi_1(f(\mu)), \chi_2(\kappa)) = (\chi_0(\kappa), \chi_1(f(\mu)), \chi_2(\kappa + f(\mu))) = \chi(\kappa, f(\mu))$. Again, it holds that $f(\kappa) = f(\mu)$.

Claim 3: If $\omega \le \nu < f(\mu) < \sup I$, then $\chi_1 \restriction I$ and $\chi_2 \restriction I$ cannot both be constant functions.

Choose $\nu' \in I$ with $\nu' > f(m) > \nu$. Then $(\mu, \nu) \notin R$ and $(\mu, \nu') \in R$ so that congruence of R with χ implies $\chi(\mu, \nu) = (\chi_0(\mu), \chi_1(\nu), \chi_2(\mu + \nu)) = (\chi_0(\mu), \chi_1(\nu), \chi_2(\nu)) \ne (\chi_0(\mu), \chi_1(\nu'), \chi_2(\mu + \nu')) = \chi(\mu, \nu')$. Hence, $\chi_1(\nu) \ne \chi_1(\nu')$ or $\chi_2(\mu) \ne \chi_2(\mu + \nu') = \chi_2(\nu')$.

Claim 4: If $\chi_0 \restriction I$ and $\chi_2 \restriction I$ are constant functions and $f(\mu) < \omega$, then $f[I]$ is finite.

For $l \in I \subseteq \omega$ and $l \ge f(\mu)$, we have $\chi(\mu, l) = (\chi_0(\mu), \chi_1(l), \chi_2(\mu + l)) = (\chi_0(l), \chi_1(\mu), \chi_2(l + \mu)) = \chi(l, \mu)$. Combined with $(\mu, l) \in R$ we get $(l, \mu) \in R$, i.e., $m \ge f(l)$. Consequently, $f[I] \subseteq f[f(\mu) \cap I] \cup \{0, \dots, \mu\}$ is finite.

Now partition $\mathrm{dom}(f)$ in finitely many parts so that for each part I, either $I \subseteq \omega$ or $I \cap \omega = \emptyset$, and $\chi_0 \restriction I$, $\chi_1 \restriction I$ and $\chi_2 \restriction I$ are constant functions. Let $\mathcal{A}$ be the arising partition. Let $I \in \mathcal{A}$. If $I \subseteq \omega$, then either $\min f[I] \ge \omega$ and Claim 1 implies that $f \restriction I$ is constant, or $\min f[I] < \omega$ and Claim 4 implies that $f[I]$ is finite. On the other hand, if $I \cap \omega = \emptyset$, then by Claim 3, either $\min f[I] \le \min I$ or $\min f[I] \ge \sup I = \sup(I \cup \{\omega\})$, whence Claim 1 and 2 imply that $f \restriction I$ is constant. In any case, $f[I]$ is finite. Summing up, we get that $\mathrm{rg}(f) = \bigcup_{I \in \mathcal{A}} f[I]$ is finite. □

Now we can answer the question c in the preceding section in even a more general framework than it was originally posed.

Theorem 5.7. *Let f be a function with* $\mathrm{dom}(f) \cup \mathrm{rg}(f) \subseteq \mathrm{Card}$. *Then the following are equivalent:*

a) R_f *is definable by Mostowski quantifiers, i.e., there is a finite set of Mostowski quantifiers* $\mathcal{Q}$ *such that* R_f *is definable in the logic* $\mathrm{FO}(\mathcal{Q})$.

b) R_f *is definable by cardinality quantifiers.*

c) $\mathrm{udim}(\mathsf{R}_f) = 1$.

d) $\mathrm{rg}(f)$ *is finite.*

Proof. Condition b trivially implies a, and condition a implies b, by Corollary Moscard. Furthermore, conditions b and c are equivalent, by Theorem 5.4, so we need to prove only the equivalence of c and d.

Suppose first that $\mathrm{rg}(f)$ is infinite. We know that

$$\mathcal{R}(\mathsf{R}_f) = \{\, (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \in C^4 \smallsetminus \{\mathbf{0}\} \mid \kappa_0 + \kappa_1 \in \mathrm{dom}(f),\ \kappa_0 + \kappa_2 \geq f(\kappa_0 + \kappa_1) \,\}$$

where C is any initial segment of Card that is a set and contains $\omega \cup \mathrm{dom}(f)$. Put $S = \{\, (0, \kappa_1, \kappa_2, \kappa_3) \mid \kappa_1, \kappa_2, \kappa_3 \in C \,\}$. Clearly $r_+(S) = 1$ (even $r(S) = 1$). Now

$$\begin{aligned}\mathcal{R}(\mathsf{R}_f) \cap S &= \{\, (0, \kappa_1, \kappa_2, \kappa_3) \in C^4 \smallsetminus \{\mathbf{0}\} \mid \kappa_1 \in \mathrm{dom}(f),\ \kappa_2 \geq f(\kappa_1) \,\} \\ &= \{\, (0)\hat{\ }(\kappa, \lambda)\hat{\ }(\mu) \mid (\kappa, \lambda) \in R, \mu \in C \,\} \smallsetminus \{\mathbf{0}\}\end{aligned}$$

where R is as in the previous lemma. By the basic properties of the rank relative to addition, it holds that

$$r_+(R) = r_+(\mathcal{R}(\mathsf{R}_f) \cap S) \leq \max\{r_+(\mathcal{R}(\mathsf{R}_f)), r_+(S)\} = r_+(\mathcal{R}(\mathsf{R}_f)).$$

However, since $\mathrm{rg}(f)$ is infinite, the previous lemma maintains that

$$1 < r_+(R) \leq r_+(\mathcal{R}(\mathsf{R}_f)) = \mathrm{udim}(\mathsf{R}_f),$$

so that condition c does not hold.

Assume then that $\mathrm{rg}(f)$ is finite. Then

$$\mathcal{R}(\mathsf{R}_f) = \mathcal{R}(\mathsf{R}_f, C) = \bigcup_{\mu \in \mathrm{rg}(f)} (S' \cap S_\mu \cap T_\mu)$$

where $S' = \{\, (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \in C^4 \smallsetminus \{\mathbf{0}\} \mid \kappa_0 + \kappa_1 \in \mathrm{dom}(f) \,\}$, $S_\mu = \{\, (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \in C^4 \mid \kappa_0 + \kappa_1 \in f^{-1}\{\mu\} \,\}$ and $T_\mu = \{\, (\kappa_0, \kappa_1, \kappa_2, \kappa_3) \in C^4 \mid \kappa_0 + \kappa_2 \geq \mu \,\}$. It is easily seen that $r_+(S') = r_+(S_\mu) = r_+(T_\mu) = 1$, for all $m \in \mathrm{rg}(f)$, so that $r_+(\mathcal{R}(\mathsf{R}_f)) = 1$ as Boolean combination of such relations. Hence, $\mathrm{udim}(\mathsf{R}_f) = r_+(\mathcal{R}(\mathsf{R}_f)) = 1$. □

This far we have been only dealing with concrete quantifiers of low unary dimension. The main point of the combinatorial theory is, however, that unary dimension can be arbitrarily large, i.e, there is a whole hierarchy of unary quantifiers.

Theorem 5.8. *For every $n \in \mathbb{Z}_+$ there is a unary quantifier Q of width $\lceil \mathrm{lb}\, n \rceil$ such that* $\mathrm{udim}(Q) = n$.

Proof. (Sketch) This is essentially [Luo00, Theorem 5.3]. For a fixed $\boldsymbol{c} \in \mathbb{Q}^n$, consider

$$\mathcal{R}_{\boldsymbol{c}} = \{\, \boldsymbol{x} \in \mathbb{N}^n \mid \boldsymbol{c} \cdot \boldsymbol{x} = 0 \,\}.$$

Thus, $\mathcal{R}_{\boldsymbol{c}}$ is the set of nonnegative lattice points of a hyperplane that is perpendicular to $\boldsymbol{c}$. The rank $r_+(\mathcal{R}_{\boldsymbol{c}})$ varies with $\boldsymbol{c}$, but under some natural conditions it holds that $r_+(\mathcal{R}_{\boldsymbol{c}}) = n$. The proof of this fact uses some advanced combinatorics, a result of Ramsey theory called multidimensional van der Waerden's theorem or Gallai's and Witt's theorem. Now the rank is robust to small changes, so $r_+(\mathcal{R}_{\boldsymbol{c}} \smallsetminus \{\boldsymbol{0}\}) = n$ and also $r_+(R) = n$ for the relation where we have added dummy variables to increase the arity of R to 2^l with $l = \lceil \mathrm{lb}\, n \rceil$. Then there is a unary quantifier Q of width l such that $\mathcal{R}(Q) = R$ and $\mathrm{udim}(Q) = r_+(\mathcal{R}(Q)) = n$. □

As a final point, note that the unary dimension hierarchy is actually finer than the hierarchy that would have been based only on the definability of universe-independent unary quantifiers of fixed width. Furthermore, we can still refine this a bit. Let $n \in \mathbb{Z}_+$. By the previous theorem, there is a unary quantifier of dimension 2^n and width n. This quantifier cannot be universe-independent, since the relation of a universe-independent unary quantifier has at least one dummy variable. However, modifying the proof of the previous theorem we get a unary quantifier Q' of dimension 2^n and width $n+1$ that is universe-independent. By Theorem 4.3, this Q' cannot be definable by means of quantifiers of width n. Hence, one can find an irregular layer of unary quantifiers of dimension 2^n and width n which is lower than the corresponding regular layer.

In the next section, we put the irregular logics generated by unary quantifiers under magnifying glass.

6 Regularity gap

Regularity is a very practical property of logics, analogous to field axioms in algebra. In a field, calculating expressions is easy; similarly, in a regular logic, forming sentences is easy. Unfortunately, not all of the interesting logics are regular, e.g., the logic Σ_1^1 is not even closed under negation. Therefore, it may be interesting to study how irregular a particular logic is. Note that if we restrict ourselves to finite structures, this is a notoriously open question for Σ_1^1, related to Fagin's characterization theorem ($\Sigma_1^1 \equiv \mathrm{NP} \,/\mathcal{F}$) and NP vs. PTIME problem.

Before going to business, let us sketch a measure of irregularity. Let $\mathcal{L}$ be any logic, and let $\mathcal{Q}$ be the collection of $\mathcal{L}$-definable quantifiers Q. Then $\mathrm{FO}(\mathcal{Q})$ is the *semiregular* *closure of* $\mathcal{L}$, i.e., the least semiregular logic $\mathcal{L}'$ with $\mathcal{L}' \geq \mathcal{L}$. Furthermore, we can regularize each Q: Q^{reg} is the quantifier of vocabulary $\sigma = \tau_Q \cup \{P\}$ with P a new unary relation symbol, having defining class

$$K_{Q^{\mathrm{reg}}} = \{\, \mathfrak{M} \in \mathrm{Str}(\sigma) \mid (\mathfrak{M}|P^{\mathfrak{M}})\restriction\tau_Q \in K_Q \,\}.$$

Clearly Q^{reg} is universe-independent. Put $\mathcal{Q}^{\mathrm{reg}} = \{\, Q^{\mathrm{reg}} \mid Q \in \mathcal{Q} \}$; then $\mathcal{L}^{\mathrm{reg}} = \mathrm{FO}(\mathcal{Q}^{\mathrm{reg}})$ is the *regular closure of* $\mathcal{L}$.

For the lower bound, we need to assume $\mathcal{L}$ is semiregular already. Let $\mathcal{Q}^{\circ}$ be the collection of all $\mathcal{L}$-definable universe-independent quantifiers Q. Then $\mathcal{L}^{\circ} = \mathrm{FO}(\mathcal{Q}^{\circ})$ is, by semi-regularity of $\mathcal{L}$, included in $\mathcal{L}$, and is, in fact, the largest regular logic $\mathcal{L}'$ such that $\mathcal{L}' \leq \mathcal{L}$. We call $\mathcal{L}^{\circ}$ the *regular interior* of $\mathcal{L}$. Now the pair $(\mathcal{L}^{\circ}, \mathcal{L}^{\mathrm{reg}})$ is a reasonable measure of irregularity of $\mathcal{L}$; the larger the gap between $\mathcal{L}^{\circ}$ and $\mathcal{L}^{\mathrm{reg}}$, the more irregular is $\mathcal{L}$.

Returning to our primary interest, we note that quantifier logics are always semiregular, if the base logic (usually FO) is regular. This far we have found a unary quantifier Q such that for every universe-independent quantifier q definable in $\mathrm{FO}(Q)$, we have $\mathrm{udim}(q) < \mathrm{udim}(Q)$. Investigating this example further, we shall learn that the situation is even worse: it may happen that such q is always FO-definable (and hence $\mathrm{udim}(q) = 1$). Consequently, $\mathrm{FO}(Q)^{\circ} = \mathrm{FO}$. We shall extract a property of the example, *weakness against padding*, which is sufficient for the phenomenon to happen. We shall call this phenomenon *a regularity gap*.

Definition 6.1. A relation $R \subseteq \mathbb{N}^n$ is *weak against padding*, if there is a set $I \subseteq n$ such that the following holds. For every $\boldsymbol{\kappa} = (\kappa_0, \ldots, \kappa_{n-1}) \in \mathbb{N}^n$ there is a threshold $r \in \mathbb{N}$ such that if $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{N}^n$ differs from $\boldsymbol{\kappa}$ in only one coordinate, say $\lambda_i \neq \kappa_i$, and $\lambda_i \geq r$, then $\boldsymbol{\lambda} \in R$ iff $i \in I$. We may stress the set I by saying that R is *weak against padding of type* I. Unary quantifier of finite width Q is *weak against padding* if $\mathcal{R}(Q, \omega)$ is.

Example 6.2. Let $f\colon \mathbb{N} \to \mathbb{N}$ and suppose $f(n) \leq n + 1$, for every $n \in \mathbb{N}$. Suppose further that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty}(n + 1 - f(n)) = \infty$. Recall that the relation of T_f is

$$\mathcal{R}(\mathsf{T}_f) = \mathcal{R}(\mathsf{T}_f, \omega) = \{\, (k, l) \in \mathbb{N}^2 \mid k \geq f(k + l) \,\}.$$

We show that T_f is weak against padding of type $\{0\}$: For $(k, l) \in \mathbb{N}^2$, choose $r \in \mathbb{N}$ such that for $n \in \mathbb{N}$, $n \geq r$, we have $f(n) > k$ and $n + 1 - f(n) > l$. Then for $(k', l) \in \mathbb{N}^2$ with $k' \geq r$, it holds that $k'+l+1-f(k'+l) > l$ implying $k' \geq f(k'+l)$, i.e., $(k'+l) \in \mathcal{R}(\mathsf{T}_f)$. Respectively, for $(k, l') \in \mathbb{N}^2$, $l' \geq r$, we get $f(k + l') > k$ and $(k, l') \notin \mathcal{R}(\mathsf{T}_f)$.

In particular, Maj is weak against padding. Also $\mathsf{T}_{\lceil \mathrm{lb} \rceil}$ is weak against padding.

The result on T_f can be reversed: If T_f is weak against padding of type $\{0\}$, then $\lim_{n\to\infty} f(n) = \lim_{n\to\infty}(n + 1 - f(n)) = \infty$. We leave it as an easy exercise to show that other types than $\{0\}$ are possible, but result in FO-definable cases.

We start the technical work by showing that weakness against padding is almost preserved under appropriate operations.

Lemma 6.3. *Let $n, l \in \mathbb{Z}_+$, $(\boldsymbol{U}, \boldsymbol{t}) \in J_{n,l}$ and let $R \subseteq \mathbb{N}^l$ be weak against padding. Then there is $S \subseteq \mathbb{N}^n$ which is weak against padding and satisfies the equality*

$$\{\, \boldsymbol{\kappa} \in \mathbb{N}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in R \,\} = S \cap \{\, \boldsymbol{\kappa} \in \mathbb{N}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathbb{N}^l \,\}.$$

Proof. We proceed in two steps. Suppose the type of weakness of R is $I \subseteq n$. Write $R' = \{\, \boldsymbol{\kappa} \in \mathbb{N}^l \mid (\boldsymbol{\kappa} + \boldsymbol{t}) \uparrow 0 \in R \,\}$. Then a moment's reflection shows that R' is also weak against padding of type I. Obviously we have the equality

$$\{\, \boldsymbol{\kappa} \in \mathbb{N}^l \mid \boldsymbol{\kappa} + \boldsymbol{t} \in R \,\} = R' \cap \{\, \boldsymbol{\kappa} \in \mathbb{N}^l \mid \boldsymbol{\kappa} + \boldsymbol{t} \in \mathbb{N}^l \,\}.$$

Put $S = \{\, \boldsymbol{\kappa} \in \mathbb{N}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) \in R' \,\}$ and $J = \bigcup_{i \in I} U_i$ where $(U_0, \ldots, U_{l-1}) = \boldsymbol{U}$. Then S is weak against padding of type J and for $\boldsymbol{\kappa} \in \mathbb{N}^n$, we have $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in R$ iff $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) \in R'$ and $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathbb{N}^l$ iff $\boldsymbol{\kappa} \in S$ and $\boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}) + \boldsymbol{t} \in \mathbb{N}^l$, whence the claimed equality between the relations holds. □

Next we see how the regularity gap opens up.

Theorem 6.4. *Let $\mathcal{Q}$ be a set of quantifiers which are weak against padding and let Q be a universe-independent unary quantifier of finite width. Assume Q is definable in* FO($\mathcal{Q}$) *on $\mathcal{F}$. Then Q is definable in* FO *on $\mathcal{F}$.*

Proof. As Q is definable in FO($\mathcal{Q}$), we have (by Theorem 2.8) that $\mathcal{R}(Q, \omega)$ is a Boolean combination of relations R_i, $i \in I$, where $R_i = \{\, \boldsymbol{\kappa} \in \mathbb{N}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i) + \boldsymbol{t}_i \in \mathcal{R}(q_i, \omega) \,\}$ for some $q_i \in \mathcal{Q} \cup \{\exists\}$, $(\boldsymbol{U}_i, \boldsymbol{t}_i) \in J_{n, l_{q_i}}$, $n = 2^{\mathrm{wd}(Q)}$ and $l_{q_i} = 2^{\mathrm{wd}(q_i)}$, I finite. For clarity, let us fix a disjunctive normal form: For some $\mathcal{K} \subseteq \mathcal{P}(I)$, we have

$$\mathcal{R}(Q, \omega) = \bigcup_{K \in \mathcal{K}} \left(\bigcap_{i \in K} R_i \cap \bigcap_{i \in I \smallsetminus K} (\omega^n \smallsetminus R_i) \right). \qquad (*)$$

Put $J = \{\, i \in I \mid q_i \in \mathcal{Q} \,\}$. Previous lemma enables us to write, for $i \in J$,

$$R_i = S_i \cap \{\, \boldsymbol{\kappa} \in \mathbb{N}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i) + \boldsymbol{t}_i \in \mathbb{N}^{l_{q_i}} \,\}$$

with S_i weak against padding of type I_i. For $i \in I$, put

$$R'_i = \begin{cases} R_i, & \text{for } i \in I \smallsetminus J \\ \emptyset, & \text{for } i \in J \text{ and } n-1 \notin I_i \\ \{\, \boldsymbol{\kappa} \in \mathbb{N}^n \mid \boldsymbol{s}(\boldsymbol{\kappa}, \boldsymbol{U}_i) + \boldsymbol{t}_i \in \mathbb{N}^{l_{q_i}} \,\}, & \text{for } i \in J \text{ and } n-1 \in I_i \end{cases}$$

and

$$R' = \bigcup_{K \in \mathcal{K}} \left(\bigcap_{i \in K} R'_i \cap \bigcap_{i \in I \smallsetminus K} (\omega^n \smallsetminus R'_i) \right).$$

For each $i \in I$, there is obviously some $m_i \in \mathbb{N}$ such that for every $\boldsymbol{\kappa} \in \mathbb{N}^n$, we have $\boldsymbol{\kappa} \in R'_i$ iff $\boldsymbol{\kappa} \downarrow m \in R'_i$. Hence, for $m = \max\{\, m_i \mid i \in I \,\}$ and for some $R_0 \subseteq \{0, \ldots, m\}^n$ we have that $R' = \{ \boldsymbol{\kappa} \in \mathbb{N}^n \mid \boldsymbol{\kappa}\!\downarrow\! m \in R_0 \}$. By Theorem 3.1, there is some FO-definable Q' such that $R' = \mathcal{R}(Q', \omega)$.

Put

$$g\colon \mathrm{Card}^n \to \mathrm{Card}^n,\ g(\boldsymbol{\kappa}) = \boldsymbol{\kappa}\!\restriction\!(n-1)\hat{\ }(m),$$
$$R'' = \{ \boldsymbol{\kappa} \in \mathrm{Card}^n \mid g(\boldsymbol{\kappa}) \in \mathcal{R}(Q') \}$$

and

$$R_1 = \{ \boldsymbol{\kappa} \in \{0, \ldots, m\}^n \mid g(\boldsymbol{\kappa}) \in R_0 \}.$$

Thus, g replaces the last component of a tuple by m. For every $\boldsymbol{\kappa} \in \mathrm{Card}^n$, we have $g(\boldsymbol{\kappa})\!\downarrow\! m = g(\boldsymbol{\kappa}\!\downarrow\! m)$, so $\boldsymbol{\kappa} \in R''$ iff $g(\boldsymbol{\kappa}) \in \mathcal{R}(Q')$ iff $g(\boldsymbol{\kappa}\!\downarrow\! m) = g(\boldsymbol{\kappa})\!\downarrow\! m \in R_0$ iff $\boldsymbol{\kappa}\!\downarrow\! m \in R_1$. Consequently, also the quantifier Q'' of vocabulary τ_Q such that $\mathcal{R}(Q'') = R''$ is FO-definable.

Clearly, Q'' is universe-independent. The main point to prove here is that $\mathcal{R}(Q, \omega) = \mathcal{R}(Q'', \omega)$, so that Q is FO-definable on $\mathcal{F}$. Indeed, let $\boldsymbol{\kappa} = (\kappa_0, \ldots, \kappa_{n-1}) \in \mathbb{N}^n$. For each $i \in J$, pick a threshold $r_i \in \mathbb{N}$ such that if $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{n-1}) \in \mathbb{N}^n$ differs from $\boldsymbol{\kappa}$ on only one component, say $\lambda_t \neq \kappa_t$, and $\lambda_t > r_i$, then $\lambda \in S_i$ iff $t \in I_i$. Pick r such that $r \geq r_i$, for every $i \in J$, that $r > \kappa_t$, for each $t \in n$, and $r > m$.

Consider $\boldsymbol{\kappa}' = \boldsymbol{\kappa}\!\restriction\!(n-1)\hat{\ }(r) \in \mathbb{N}^n$, i.e., the last component of $\boldsymbol{\kappa}$ is replaced by r. Then $\boldsymbol{\kappa}' \in S_i$ iff $n - 1 \in I_i$, for $i \in J$. By $(*)$, we have thus $\boldsymbol{\kappa}' \in R_i$ iff $\boldsymbol{\kappa}' \in R'_i$, for each $i \in I$, implying $\boldsymbol{\kappa}' \in \mathcal{R}(Q, \omega)$ iff $\boldsymbol{\kappa}' \in R'$. As Q and Q'' are universe-independent, we get the following sequence of equivalent statements: $\boldsymbol{\kappa} \in \mathcal{R}(Q, \omega)$ iff $\boldsymbol{\kappa}' \in \mathcal{R}(Q, \omega)$ iff $\boldsymbol{\kappa}' \in R'$ iff $g(\boldsymbol{\kappa}'\!\downarrow\! m) = g(\boldsymbol{\kappa}')\!\downarrow\! m \in R_0$ iff $\boldsymbol{\kappa}'\!\downarrow\! m \in R_1$ iff $\boldsymbol{\kappa}' \in \mathcal{R}(Q'')$ iff $\boldsymbol{\kappa}' \in \mathcal{R}(Q'', \omega)$. Hence, $\mathcal{R}(Q, \omega) = \mathcal{R}(Q'', \omega)$. □

Corollary 6.5. *Let $\mathcal{Q}$ be a set of quantifiers which are weak against padding and let $\mathcal{L}$ be a regular logic such that $\mathcal{L} \leq \mathrm{FO}(\mathcal{Q})/\mathcal{U}$ where $\mathcal{U}$ is the class of all finite unary relational structures. Then $\mathcal{L} \equiv \mathrm{FO}/\mathcal{U}$.*

Proof. Let Q be a $\mathcal{L}$-definable unary quantifier of finite width. By the previous theorem, Q^{reg} is FO-definable on $\mathcal{U}$. However, $\mathrm{FO}(Q) \leq \mathrm{FO}(Q^{\mathrm{reg}})$, so also Q is FO-definable on $\mathcal{U}$. Hence, $\mathcal{L} \leq \mathrm{FO}/\mathcal{U}$ and by regularity of $\mathcal{L}$, we get $\mathcal{L} \equiv \mathrm{FO}/\mathcal{U}$. □

The main theorem of the section shows that the regularity gap can be arbitrarily large in the realm of unary quantifiers.

Theorem 6.6. *For every $n \in \mathbb{Z}_+$, there is a quantifier Q which is weak against padding and for which $\mathrm{udim}(Q) \geq n$.*

Proof. Choose $l \in \mathbb{Z}_+$ with $2^l \geq n$. From the proof of 5.8 we know that there is $\boldsymbol{c} = (c_0, \ldots, c_{2^l-1}) \in \mathcal{Q}^{2^l}$ such that $r_+(\mathcal{R}_{\boldsymbol{c}}) = 2^l$ where $\mathcal{R}_{\boldsymbol{c}} = \{ \boldsymbol{x} \in \mathbb{N}^n \mid \boldsymbol{c} \cdot \boldsymbol{x} = 0 \}$. All of the components of $\boldsymbol{c}$ are non-zero, since otherwise some variables of $\mathcal{R}_{\boldsymbol{c}}$ would be dummy implying $r_+(\mathcal{R}_{\boldsymbol{c}}) < 2^l$.

Let us show that $\mathcal{R}_{\boldsymbol{c}}$ is weak against padding. The argument is geometric: If we make a shift big enough, we are away from the hyperplane. Let $\boldsymbol{\kappa} = (\kappa_0, \ldots, \kappa_{2^l-1}) \in \mathbb{N}^{2^l}$. Put

$u = \boldsymbol{c} \cdot \boldsymbol{\kappa}$, $\gamma = \min\{\, |c_i| \mid i \in 2^l \,\} > 0$ and $r = \max\{\kappa_0, \ldots, \kappa_{2^l-1}\} + \lceil |u|/\gamma \rceil + 1 \in \mathbb{N}$. Let $\lambda = (\lambda_0, \ldots, \lambda_{2^l-1}) \in \mathbb{N}^{2^l}$ be a sequence that differs from $\boldsymbol{\kappa}$ on only one component, say $\lambda_i \neq \kappa_i$. Suppose also $\lambda_i \geq r$. Then $\lambda_i - \kappa_i \geq \lceil |u|/\gamma \rceil + 1$, so

$$\begin{aligned} |\boldsymbol{c} \cdot \boldsymbol{\lambda}| &= |u + c_i(\lambda_i - \kappa_i)| \geq |c_i||\lambda_i - \kappa_i| - |u| \\ &\geq \gamma \cdot \left(\frac{|u|}{\gamma} + 1 \right) - |u| = \gamma > 0. \end{aligned}$$

Hence, $\boldsymbol{\lambda} \notin \mathcal{R}_{\boldsymbol{c}}$, so $\mathcal{R}_{\boldsymbol{c}}$ is weak against padding of type $\emptyset$.

Let $Q_{\boldsymbol{c}}$ be the quantifier with $\mathcal{R}(Q_{\boldsymbol{c}}) = \mathcal{R}_{\boldsymbol{c}} \smallsetminus \{\mathbf{0}\}$. Then $Q_{\boldsymbol{c}}$ is weak against padding (removing $\mathbf{0}$ does not change that) and $\operatorname{udim}(Q_{\boldsymbol{c}}) = 2^l \geq n$. □

7 Conclusion

Hopefully I have managed to convince the reader that unary quantifier definability theory on all structures or on all finite structures can be developed in a systematic way. There is of course plenty of room for new results, for the simple reason that many natural combinatorial questions can be formulated as unary quantifier definability problems (for more on this point, see [Luo99]). In fact, this is the very nature of the field and the landscape of the field seems to be well understood.

Let us take a look at the future, then. From a traditional model-theoretic perspective, unary quantifiers may seem to be clumsy and lacking many desirable properties, but from the finite model theory perspective this is not a problem. On the contrary, unary quantifiers have an interesting relation to built-in relations of the structures and some of the quantifiers have also been studied for their own sake in descriptive complexity theory (for built-in relations, see [Imm99] or [Sch01] and about the relation to quantifiers, see [Luo04]). However, descriptive complexity seems to suggest that the focus should be on restricted classes of structures, such as ordered structures, and on vectorizations of quantifiers.

A lot of attention has been paid to ordered structures, but even questions related to first order logic are often very involved. As to quantifiers, interesting things start to happen [Nur96, Luo04]. For example, recall that the Härtig quantifier I has unary dimension two. Consequently, I is not definable by cardinality quantifiers, and this holds even on finite structures. On the other hand, the class of cardinality quantifiers that are capable of expressing I on $\mathcal{O}$ is large and combinatorially characterizable [Luo04]. There are some indications that unary quantifier theory may be very complex.

There seems to be a qualitative change in the landscape when we move from finite structures to ordered structures. The landscape of unary quantifiers on all or finite structures is quite homogeneous: If we fix a unary quantifier Q and restrict the considerations to stronger unary quantifiers Q', there is no qualitative change. On the other hand, there seem to be thresholds of logics on ordered structures, e.g., it is more difficult to separate quantifiers that are stronger than I than those that cannot define I.

The picture is even more unclear for vectorizations. Without going to details, vectorization is a process where elements are replaced by k-tuples, for some $k \in \mathbb{Z}_+$. In particular, the k-th vectorization of Q is k-ary $Q^{(k)}$. The importance of vectorizations is demonstrated, among other results, by Dawar's result on PTIME-characterizability [Daw95]. The generic question is as follows: Given unary quantifiers q and Q, is it true that q is definable in $\mathrm{FO}(\{ Q^{(n)} \mid n \in \mathbb{Z}_+ \})$. It is conceivable that unary vectorized quantifiers theory has some similarities with the theory without vectorizations.

The results of this paper are unlikely to be directly perused if one tries to develop theory in the case of ordered structures or vectorizations. However, I think they would help indirectly: Quantifier definability theory, as presented in this paper, should serve as a point of departure and of comparison for future research.

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