A RAMSEY THEOREM WITH AN APPLICATION TO SEQUENCES IN BANACH SPACES

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ABSTRACT. The notion of a maximally conditional sequence is introduced for sequences in a Banach space. It is then proved using Ramsey theory, that every basic sequence in a Banach space has a subsequence which is either an unconditional basic sequence, or a maximally conditional sequence. An apparently novel, purely combinatorial lemma in the spirit of Galvin's theorem is used in the proof. An alternative, quicker proof of the dichotomy result for sequences in Banach spaces is also sketched.

1. INTRODUCTION

A well-known and significant problem in Banach space theory that went back at least to the 1958 paper of Bessaga and Pełczyński [3] was whether every infinite-dimensional Banach space contains an unconditional basic sequence. This was finally settled in the negative in 1991 by Gowers and Maurey with their construction in [6] of a hereditarily indecomposable space. Earlier an important partial negative result was given when Maurey and Rosenthal constructed in [9] a weakly null basic sequence without any unconditional subsequence. By contrast, it was known already to Banach that any weakly null normalized sequence in a Banach space always has a basic subsequence. The example of Maurey and Rosenthal prompted a line of research begun by Elton in [5] where the property of unconditionality is replaced by some weaker, but closely related property in order to obtain positive results. Thus Elton defined the concept of a near-unconditional sequence and showed that every weakly null normalized sequence has a near-unconditional subsequence. Later, other types of partial unconditionality conditions and corresponding results were given by Odell [11] and Argyros et al [2]. The surveys [12] and [1] serve as good introductions to the subject. A common thread is the use of a Ramsey theoretic result at some point of the proof.

This paper has two objectives. The first is to address a natural question regarding basic sequences which fail to have any unconditional subsequences. More specifically, it is proved that every such sequence has a subsequence which is maximally conditional, a concept defined in this paper as follows:

Definition 1.1. A basic sequence (x_k) in a Banach space is maximally conditional, if given any two infinite disjoint sets $E, F \subset \mathbf{N}$ and any positive real number $C < \infty$ there exists a finitely supported sequence $(a_k)_{k\in E\cup F}$ of scalars such that

$$\|\sum_{k\in E} a_k x_k\| > C\|\sum_{k\in E\cup F} a_k x_k\|.$$

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More intuitively speaking, any two subspaces spanned by subsequences of (x_k) have angle 0. The result that any basic sequence has either an unconditional subsequence or a maximally conditional subsequence appears below as Theorem 2.2. Although much easier to prove, this theorem should be viewed as an analogue of Gowers' dichotomy theorem which says that every infinite-dimensional Banach space contains either an unconditional basic sequence or a hereditarily indecomposable subspace. In fact, this result of Gowers' can be rephrased as follows: any basic sequence either has an unconditional block sequence or a block sequence (y_k) such that any two block subspaces of (y_k) have angle 0. The reader is referred to the relatively easygoing exposition in [8] for a proof of this fact and to [7] for a more complete treatment of Gowers' original proof.

The second purpose of this paper is to draw attention to an apparently novel Ramsey theoretic lemma used in the proof of Theorem 2.2 on maximally conditional subsequences. The lemma in question is related to Galvin's theorem and expresses the fact that for any set $\mathcal{A}$ of pairs (A, B) of disjoint finite subsets of $\mathbf{N}$ there exists an infinite set $M \subset \mathbf{N}$ such that either there is no $(A, B) \in \mathcal{A}$ such that $A \cup B \subset M$, or else M is densely saturated by pairs $(A, B) \in \mathcal{A}$ in a specific sense. This purely combinatorial result, which appears as Lemma 3.1 below, seems natural enough to merit independent interest and should have potential applications beyond those presented in this paper.

2. Maximally conditional sequences

Recall that a sequence (x_k) in a Banach space X is called a basis for X, if for every $x \in X$ there exists a unique sequence (a_k) of scalars such that $x = \sum_{k=1}^{\infty} a_k x_k$ with the series converging in the norm of X. A basic sequence is a sequence (x_k) which is a basis for the closed linear span $[(x_k)]$. A basic sequence (x_k) is unconditional, if each convergent series of the form $\sum_{k=1}^{\infty} a_k x_k$ is unconditionally convergent i.e. $\sum_{k=1}^{\infty} a_{\pi(k)} x_{\pi(k)}$ converges for every permutation $\pi : \mathbf{N} \to \mathbf{N}$.

The following characterisation of unconditionality is convenient for the purposes of this paper. A good reference for any of the facts about Banach spaces mentioned without proof below is Megginson's textbook [10] .

Lemma 2.1. *A basic sequence (x_k) in a Banach space X is unconditional, if and only if there exists a constant $C < \infty$ such that for all finite sets $E \subset \mathbf{N}$ and all sequences of scalars (a_k) one has*

$$\|\sum_{k \in E} a_k x_k\| \leq C \|\sum_{k=1}^{\infty} a_k x_k\|. \tag{2.1}$$

Recall that if (x_k) is any basic sequence, then for any finite set $E \subset \mathbf{N}$ one can define a bounded linear projection P_E (depending on the sequence (x_k)) on the closed linear span $[(x_k)]$ by setting

$$P_E(\sum_{k=1}^{\infty} a_k x_k) = \sum_{k \in E} a_k x_k. \tag{2.2}$$

For an infinite set E the series $\sum_{k \in E} a_k x_k$, where the elements of E are enumerated in ascending order, need not converge if (x_k) is not an unconditional basic sequence.

Next we introduce some notation which will be used for the rest of this paper. For any basic sequence (x_k), let

$$\mathcal{B}_{(x_k)} = \{E \subset \mathbf{N} : P_E \text{ is a bounded projection on } [(x_k)]\}.$$

For disjoint $E, F \subset \mathbf{N}$ it holds that $P_{E\cup F} = P_E + P_F$, while for any $E, F \subset \mathbf{N}$ we also have the identities $P_{E\cap F} = P_E P_F = P_F P_E$ and $P_{\mathbf{N}\setminus E} = I - P_E$, where I is the identity operator on $[(x_k)]$. It follows from the above that $\mathcal{B}_{(x_k)}$ is in fact a Boolean algebra of sets.

If $\mathcal{B}_{(x_k)} = \mathcal{P}(\mathbf{N})$, then it can be deduced with little effort from the principle of uniform boundedness that the condition of Lemma 2.1 holds. Thus $\mathcal{B}_{(x_k)} = \mathcal{P}(\mathbf{N})$ if and only if (x_k) is an unconditional basic sequence.

If (x_k) is not unconditional, how small can $\mathcal{B}_{(x_k)}$ in fact be? It follows from the discussion above that each finite and co-finite subset of $\mathbf{N}$ must be an element of $\mathcal{B}_{(x_k)}$. A well-known space demonstrates that

$$\mathcal{B}_{(x_k)} = \{F \subset \mathbf{N} : |F| < \infty \text{ or } |\mathbf{N} \setminus F| < \infty\} \tag{2.3}$$

is possible for a basic sequence (x_k).

Namely, let (y_k) be the summing basis, formally defined as the standard vector basis of c_{00} in the completion of c_{00} given by the norm

$$\|(a_k)\|_\Sigma = \sup_{n\in\mathbf{N}} |\sum_{k=1}^{n} a_k y_k|.$$

The fact that equation (2.3) holds with (y_k) in place of (x_k) is easily verified directly, a task left to the reader.

The definition of a maximally conditional sequence given in the introduction can be rephrased as follows: (x_k) is maximally conditional, if P_E fails to define a bounded operator on $[(x_k)_{E\cup F}]$ for any pair of disjoint, infinite $E, F \subset \mathbf{N}$. Now we can state the main theorem of this paper.

Theorem 2.2. *Every basic sequence in a Banach space has a subsequence, which is either an unconditional basic sequence, or a maximally conditional sequence.*

Note that if (x_k) is maximally conditional, then (2.3) holds. The converse is not true in general. Before proving Theorem 2.2, we give an example showing that the property of being maximally conditional is in fact a strictly stronger property for a sequence (x_k) than the minimality property defined by (2.3).

Consider the basis (z_k) which is again formally the sequence of natural basis vectors in c_{00}. The ambient space is the completion of c_{00} with respect to the norm

$$\|(a_k)\|_v = |a_1| + \sum_{k=1}^{\infty} |a_{k+1} - a_k|.$$

Note that $\|z_k\| = 2$ for all k. We claim that $\mathcal{B}_{(z_k)} = \{F \subset \mathbf{N} : |F| < \infty \text{ or } |\mathbf{N}\setminus F| < \infty\}$. Take $E \subset \mathcal{P}(\mathbf{N})$ such that $|E| = \infty$ and $|\mathbf{N} \setminus E| = \infty$. Now $\| \sum_{k=1}^{n} z_k\| = 2$ for all $n \geq 1$, while

$$\|P_E(\sum_{k=1}^{n} z_k)\|_v \geq |\{k < n : k \in E, k+1 \notin E\}| \to \infty$$

as $n \to \infty$, so P_E cannot be a bounded projection.

On the other hand, if $E, F \subset \mathbf{N}$ are disjoint sets such that $E \cup F$ contains no two consecutive numbers then in fact

$$\|\sum_{k\in E} a_k z_k\|_v \leq 2\sum_{k\in E}|a_k| \leq \|\sum_{k\in E\cup F} a_k z_k\|_v,$$

so the sequence (z_k) is not maximally conditional.

3. Proof of the main theorem

We present two proofs of Theorem 2.2. This is justified by the fact that the longer of the two proofs is based on a combinatorial lemma which seems natural enough to merit independent interest. A more direct proof, relying only on well known combinatorial results is presented at the end of the paper, with the permission of Edward Odell who suggested it. At this point we follow [4] and introduce some convenient notation related to sets. For a set $M \subset \mathbf{N}$ we let

$$M^{(\infty)} = \{E \subset M : |E| = \infty\}$$

and

$$M^{(<\infty)} = \{E \subset M : |E| < \infty\}.$$

In practice this notation is used only for countably infinite sets M, so that $M^{(\infty)}$ will always be an uncountable set and $M^{(<\infty)}$ a countably infinite set.

We start preparing for the first proof of Theorem 2.2 by noting that (x_k) is maximally conditional if and only if, for any two disjoint sets $E, F \in \mathbf{N}^{(\infty)}$ and any $C > 0$, there exist finite sets $A \subset E$ and $B \subset F$ and a sequence of scalars (a_k) such that

$$\|\sum_{k\in A} a_k x_k\| > C\|\sum_{k\in A\cup B} a_k x_k\|.$$

This serves to motivate the Ramsey-theoretic Lemma 3.1 below, whose proof relies on Galvin's theorem (see [4] for a proof of this fundamental result). Galvin's theorem states that if $\mathcal{A} \subset \mathbf{N}^{(<\infty)}$ and $R \in \mathbf{N}^{(\infty)}$, then there exists a subset $S \in R^{(\infty)}$ such that either

(1) There exists no $C \in \mathcal{A}$ such that $C \subset S$, or
(2) For every $S' \in S^{(\infty)}$ there exists an initial segment C of S' such that $C \in \mathcal{A}$.

We now come to the key lemma on which our first proof of Theorem 2.2 is based.

Lemma 3.1. *Let $\mathcal{A}$ be a set of pairs of the form (A, B), where $A, B \in \mathbf{N}^{(<\infty)}$ and $A \cap B = \emptyset$. Let $R \in \mathbf{N}^{(\infty)}$. Then there exists a subset $S \in R^{(\infty)}$ such that one of the following alternatives holds:*

(1) There exists no $(A, B) \in \mathcal{A}$ such that $A \cup B \subset S$, or
(2) For any two sets $E, F \in S^{(\infty)}$ there exists a pair $(A, B) \in \mathcal{A}$ such that $A \subset E$ and $B \subset F$.

Proof. To start off, we apply Galvin's theorem with the family

$$\mathcal{A}' = \{A \cup B : (A, B) \in \mathcal{A}\}.$$

This gives us a a subset $T \in R^{(\infty)}$ whith one of the following properties:

(i) There exists no $C \in \mathcal{A}'$ such that $C \subset T$, or
(ii) For all $T' \in T^{(\infty)}$ there exists an initial segment C of T' such that $C \in \mathcal{A}'$.

In the first case we can set $S = T$ and the conclusion of the lemma follows. We suppose then that (ii) holds. In this case simply setting $S = T$ is not enough, but rather we will find a set $S \in T^{(\infty)}$ such that alternative (2) in the statement of the theorem holds.

Let $\mathcal{C}$ be the family of all finite sets of the form $\{a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n\} \subset \mathbf{N}$ for some $n \geq 1$, such that there exists $(A, B) \in \mathcal{A}$ with $A \subset \{a_1, \ldots .a_n\}$ and $B \subset \{b_1, \ldots .b_n\}$. We apply Galvin's theorem again, this time with $\mathcal{C}$ and T.

Let us show that for every $T' \in T^{(\infty)}$ we can find a set $C \in \mathcal{C}$ with $C \subset T'$, thus ruling out the first alternative given by Galvin's theorem. Let $T' = \{t_1 < t_2 < \cdots\}$ and denote the set of even terms $\{t_{2k}\}_{k=1}^{\infty}$ by T''. Now $T'' \in T^{(\infty)}$, so by our assumption about T, there exists an initial segment $\{t_{2k}\}_{k=1}^{n} = A \cup B$ with $(A, B) \in \mathcal{A}$. Since $A \cap B = \emptyset$ there exists a (unique) set C satisfying

$$\{t_{2k}\}_{k=1}^{n} \subset C \subset \{t_k\}_{k=1}^{2n+1} \subset T',$$

and such that, enumerating C in increasing order, elements of A appear only as odd terms, elements of B appear only as even terms and, in addition $|C|$ is even. Thus we have found $E \subset T'$ with $C \in \mathcal{C}$ as claimed.

Now having ruled out the first alternative given by Galvin's theorem, we find a set $S = \{s_1 < s_2 < \cdots\} \in T^{(\infty)}$ such that every infinite subset of S has an initial segment in $\mathcal{C}$. If $E, F \in S^{(\infty)}$, pick an increasing sequence of terms $a_1 < b_1 < a_2 < b_2 < \cdots$ with $a_k \in E$ and $b_k \in F$ for all k. The resulting set must have an initial segment in $\mathcal{C}$. Hence there exists a pair $(A, B) \in \mathcal{A}$ with $A \subset E$ and $B \subset F$. $\square$

Let $C > 0$. We call a basic sequence (x_k) projection-unconditional with constant C, if (2.1) holds with C. The content of Lemma 2.1 is precisely that a sequence is unconditional if and only if it is projection-unconditional with some constant $C < \infty$. We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Fix the basic sequence (x_k). Let us assume that (x_k) does not have any unconditional subsequence. In particular, for each $C < \infty$ it fails to have a subsequence which is projection-unconditional with constant C. For a particular value of C, let $\mathcal{A}_C$ consist of all pairs (A, B) where $A, B \in \mathbf{N}^{(<\infty)}$ such that $A \cap B = \emptyset$ and there exists a sequence of scalars (a_k) such that

$$\|\sum_{k\in A} a_k x_k\| > C\|\sum_{k\in A\cup B} a_k x_k\|.$$

Note that a subsequence $(x_k)_{k\in S}$ related to an infinite set $S \subset \mathbf{N}$ of indices is projection-unconditional with constant C if and only if there is no pair $(A, B) \in \mathcal{A}_C$ such that $A \cup B \subset S$. Therefore we conclude from Lemma 3.1 and our assumption that (x_k) has no unconditional subsequence, that for every $R \in \mathbf{N}^{(\infty)}$ there exists $S \in R^{(\infty)}$ such that for any $E, F \in S^{(\infty)}$ there exists a pair $(A, B) \in \mathcal{A}_C$ with $A \subset E$ and $B \subset F$.

Using this fact we construct a sequence (S_k) of sets $S_k \in \mathbf{N}^{(\infty)}$ such that $S_{k+1} \subset S_k$ for all k and also that for any $E, F \in S_k^{(\infty)}$ there exists a pair $(A, B) \in \mathcal{A}_k$ with $A \subset E$ and $B \subset F$. Finally, construct a set $S = \{s_k\}_{k=1}^{\infty}$ by picking an increasing sequence (s_k) of integers with $s_k \in S_k$ for each k. Now it is easy to verify that the related subsequence $(x_k)_{k\in S}$ is maximally conditional, using the fact that for any $E, F \in S^{(\infty)}$ we have $|E \cap S_k| = |F \cap S_k| = \infty$ for all k. $\square$

Having given a proof of Theorem 2.2 via Lemma 3.1 let us sketch an alternative proof which uses a slightly stronger Ramsey result than Galvin's theorem, namely the Galvin-Prikry theorem. This alternative, and somewhat more direct proof is based on an outline suggested by Edward Odell, and is included with his kind permission. We use standard terminology and results from infinite-dimensional Ramsey theory to the extent found in the final chapter of the book [4].

Fix a basic sequence (x_k). Call $M \in \mathbf{N}^{(\infty)}$ an even-projection set if, writing M as an increasing sequence (m_k), the formula

$$\sum_{k=1}^{\infty} a_k x_{m_k} \mapsto \sum_{k=1}^{\infty} a_{2k} x_{m_{2k}}$$

defines a bounded projection on $[(x_{m_k})]$. For each $C < \infty$ the family

$$\{\{m_k\} \in \mathbf{N}^{(\infty)} : \| \sum_{k=1}^{\infty} a_{2k} x_{m_{2k}} \| \leq C \| \sum_{k=1}^{\infty} a_k x_{m_k} \| \text{ for all } (a_k) \subset \mathbf{R}\}$$

where the sets $\{m_k\}$ are indexed as increasing sequences, is a closed subset of $\mathbf{N}^{(\infty)}$ in the classical topology (recalled in Section 4 below), so the set of all even-projection sets is in fact an F_σ set. By the Galvin-Prikry theorem one can then conclude that there exists a set $S \in \mathbf{N}^{(\infty)}$ such that either every $S' \in S^{(\infty)}$ is an even-projection set or none is.

Suppose first that every $S' \in S^{(\infty)}$ is an even-projection set. Writing S as an increasing sequence (s_k), let $T = \{s_{2k}\}_{k=1}^{\infty}$. Let us show that the associated sequence $(x_k)_{k \in T}$ is unconditional by showing that for any $E \subset T$ the projection

$$\sum_{k \in T} a_k x_k \mapsto \sum_{k \in E} a_k x_k$$

is always well-defined. Fix $E \subset T$. Now we can find a set $R = \{r_k\}_{k=1}^{\infty}$ indexed as an increasing sequence such that $E \subset T \subset R \subset S$ and elements of E appear only as even terms in the sequence (r_k). Now any sequence $(a_k)_{k \in T}$ can be extended to a sequence $(a_k)_{k \in R}$ by setting $a_k = 0$ for $k \in R \setminus T$. Since R is an even-projection set, the projection

$$\sum_{k \in T} a_k x_k = \sum_{k \in R} a_k x_k \mapsto \sum_{k \in E} a_k x_k$$

is well-defined.

Suppose now that no $S' \in S^{(\infty)}$ is an even-projection set. Now we can show directly that $(x_k)_{k \in S}$ is maximally conditional. Let $E, F \in S^{(\infty)}$. Pick an increasing sequence (r_k) with odd terms in E and even terms is F. Then $R = \{r_k\}_{k=1}^{\infty}$ is not an even-projection set, so for any $C > 0$ we can find scalars $(a_k)_{k \in E \cup F}$ with $a_k = 0$ for $k \notin R$ such that $\| \sum_{k \in E} a_k x_k \| > C \| \sum_{k \in E \cup F} a_k x_k \|$.

4. An open problem

We conclude by stating an interesting question regarding the family $\mathcal{B}_{(x_k)}$ introduced in Section 2. The Cantor topology on $\mathcal{P}(\mathbf{N})$ is defined by taking as basic open sets all sets of the form

$$U_{A,B} = \{M \in \mathcal{P}(\mathbf{N}) : A \subset M, B \cap M = \emptyset\},$$

where $A, B \subset \mathbf{N}$ are finite sets. The restriction of this topology to the subspace $\mathbf{N}^{(\infty)}$ is variously called the classical topology or the Baire topology on $\mathbf{N}^{(\infty)}$. Now it is an elementary observation, that for any $C > 0$ the set

$$\mathcal{B}_C = \{M \in \mathcal{B}_{(x_k)} : \|P_E\| \leq C\}$$

is a closed subset of $\mathcal{P}(\mathbf{N})$ with respect to the Cantor topology. Since we can write

$$\mathcal{B}_{(x_k)} = \bigcup_{n=1}^{\infty} \mathcal{B}_n,$$

it follows, that $\mathcal{B}_{(x_k)}$ is in fact an F_σ subset of $\mathcal{P}(\mathbf{N})$.

The unanswered question regards the structure of $\mathcal{B}_{(x_k)}$. It is conjectured by the author that for any Boolean subalgebra $\mathcal{B} \subset \mathcal{P}(\mathbf{N})$, which is an F_σ set with respect to the Cantor topology and which contains all finite sets, one can find a basic sequence (x_k) in some Banach space such that $\mathcal{B}_{(x_k)} = \mathcal{B}$.

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