OPERATOR-WEIGHTED COMPOSITION OPERATORS ON VECTOR-VALUED ANALYTIC FUNCTION SPACES

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Abstract. We study qualitative properties of the operator-weighted composition maps $W_{\psi,\varphi}\colon f \mapsto \psi(f\circ\varphi)$ on the vector-valued spaces $H^\infty_v(X)$ of X-valued analytic functions $f:\mathbb{D}\to X$, where $\mathbb{D}$ is the unit disk, X is a complex Banach space, φ is an analytic self-map of $\mathbb{D}$, ψ is an analytic operator-valued function on $\mathbb{D}$, and v is a bounded continuous weight on $\mathbb{D}$. Boundedness and compactness properties of $W_{\psi,\varphi}$ are characterized on $H^\infty_v(X)$ for infinite-dimensional X. It turns out that the (weak) compactness of $W_{\psi,\varphi}$ also involves properties of the auxiliary operator $T_\psi : x \mapsto \psi(\cdot)x$ from X to $H^\infty_v(X)$, in contrast to the familiar scalar-valued setting $X=\mathbb{C}$.

1. Introduction

Let X and Y be complex Banach spaces, $L(X,Y)$ the Banach space of all bounded linear operators from X to Y, and $H(X)$ the linear space of analytic functions $f\colon\mathbb{D}\to X$, where $\mathbb{D}$ is the open unit disk in the complex plane. If φ is an analytic map $\mathbb{D}\to\mathbb{D}$, ψ is an analytic operator-valued function $\mathbb{D}\to L(X,Y)$ and $f\in H(X)$, then $z\mapsto\psi(z)(f(\varphi(z))$ defines an analytic Y-valued map. Hence the "operator-weighted" composition

$$W_{\psi,\varphi}\colon f\mapsto\psi(f\circ\varphi) \tag{1.1}$$

is well defined as a linear map from $H(X)$ to $H(Y)$.

These operators contain various classes of concrete linear operators which have been intensively studied in the literature. For example, if $X=Y=\mathbb{C}$ and ψ is an analytic map $\mathbb{D}\to\mathbb{C}$, then the resulting weighted composition operators $f\mapsto\psi\cdot(f\circ\varphi)$ combine the analytic composition operators $C_\varphi : f\mapsto f\circ\varphi$ and the pointwise multiplication operators $M_\psi : f\mapsto\psi\cdot f$. There is an extensive and well-developed theory of the composition operators C_φ (see e.g. [12] or [23]), as well as of the multiplication operators M_ψ (see e.g. [8], [20] and their references) on many Banach spaces of (scalar-valued) analytic functions on the unit disk. In the vector-valued case, where X and Y are arbitrary Banach spaces, the class (1.1) is considerably larger as it also contains the general operator-valued multipliers M_ψ, where $M_\psi f(z) = \psi(z)(f(z))$ for $z\in\mathbb{D}$. In particular, any operator $U\in L(X,Y)$ factors through $W_{\psi,\varphi}$ for suitable choices of ψ and φ. Recently interest has arisen in composition operators and operator-valued multipliers on many vector-valued analytic function spaces, see e.g. [19], [4], [6], [20], [14], [16], [15], and [17]. Moreover, for a large class of infinite-dimensional Banach spaces X the linear onto isometries $T : H^\infty(X)\to H^\infty(X)$ have precisely the form

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$T = W_{\psi,\varphi}$, where $\psi(z) \equiv U$ is a fixed onto isometry of X and φ is a conformal automorphism of $\mathbb{D}$, see [18] and [7].

In this paper we discuss basic qualitative properties, such as boundedness, compactness and weak compactness, of the operator-weighted composition operators $W_{\psi,\varphi}$ on vector-valued H^∞ spaces. Our study is in particular motivated by the following questions:

(i) Does there exist a reasonable general theory of the operators $W_{\psi,\varphi}$ for Banach spaces X and Y?

(ii) Are there new phenomena in the operator-weighted situation?

Our setting will be that of the weighted vector-valued $H_v^\infty(X)$ spaces, that is,

$$H_v^\infty(X) = \{f \in H(X) \colon \|f\|_{H_v^\infty(X)} = \sup_{z\in\mathbb{D}} v(z)\|f(z)\|_X < \infty\},$$

where $v : \mathbb{D} \to (0,\infty)$ is a bounded continuous weight function and X is any complex Banach space. We will abbreviate $H_v^\infty = H_v^\infty(\mathbb{C})$ for $X = \mathbb{C}$. Here $H_v^\infty(X)$ are Banach spaces, which especially in the case $X = \mathbb{C}$ appear in the study and applications of growth conditions of analytic functions, see e.g. [1] and its references. The constant weight $v \equiv 1$ yields the space $H^\infty(X)$ of bounded analytic functions $\mathbb{D} \to X$. Weighted composition operators, as well as pointwise multipliers and composition operators, have earlier been investigated on the scalar-valued weighted spaces H_v^∞ in (among others) [5], [2], [3], [21], [9], [10], and [24].

Section 2 characterizes the bounded operator-weighted composition operators $W_{\psi,\varphi} : H_v^\infty(X) \to H_w^\infty(Y)$. In Section 3 precise conditions are obtained for the compactness and weak compactness of $W_{\psi,\varphi} : H_v^\infty(X) \to H_w^\infty(Y)$ in the case of radial weights v and w. Here the auxiliary operators

$$T_\psi \colon x \mapsto \psi(\cdot)x; \quad X \to H_w^\infty(Y)$$

turn out to be crucial, in contrast to the case $X = Y = \mathbb{C}$. Section 4 contains some further results and examples concerning these operators.

2. Operator-weighted compositions on $H_v^\infty(X)$

Results about the spaces $H_v^\infty(X)$ and their linear operators are often formulated in terms of so-called associated weights, see e.g. [1] and [5]. The associated weight $\tilde{v}$ of a weight function $v : \mathbb{D} \to (0,\infty)$ is defined as

$$\tilde{v}(z) = (\sup\{|f(z)| \colon \|f\|_{H_v^\infty} \le 1\})^{-1}, \quad z \in \mathbb{D}.$$

It follows that $v \le \tilde{v}$ on $\mathbb{D}$ and $\|f\|_{H_v^\infty} = \|f\|_{H_{\tilde{v}}^\infty}$ for any $f \in H(\mathbb{C})$. Hence

$$\|f\|_{H_v^\infty(X)} = \|f\|_{H_{\tilde{v}}^\infty(X)}, \quad f \in H(X), \tag{2.1}$$

since $\|f\|_{H_v^\infty(X)} = \sup_{\|x^*\|_{X^*}\le 1} \|x^* \circ f\|_{H_v^\infty}$ for any f. Moreover, for many concrete weights v there is a constant $C > 0$ so that $v \le \tilde{v} \le Cv$. For example, $v_p = \tilde{v}_p$ for $0 \le p < \infty$, where $v_p(z) = (1 - |z|^2)^p$ for $z \in \mathbb{D}$.

We first characterize the boundedness of the weighted compositions $W_{\psi,\varphi}$ between weighted vector-valued H^∞ spaces, where the (bounded continuous) weight functions are arbitrary. The case $X = Y = \mathbb{C}$ is contained in [21, Thm. 2.1] or [10, Prop. 3.1]. Below we will use the notation B_E for the closed unit ball of a Banach space E.

Theorem 2.1. *Let X, Y be complex Banach spaces and let v, w be weight functions $\mathbb{D} \to (0, \infty)$. Assume that $\psi \colon \mathbb{D} \to L(X, Y)$ and $\varphi \colon \mathbb{D} \to \mathbb{D}$ are analytic maps. Then*

$$\|W_{\psi,\varphi} \colon H_v^\infty(X) \to H_w^\infty(Y)\| = \sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)}. \tag{2.2}$$

In particular, $\psi \in H_w^\infty(L(X, Y))$ if $W_{\psi,\varphi}$ is bounded $H_v^\infty(X) \to H_w^\infty(Y)$.

Proof. If $f \in H_v^\infty(X)$ and $z \in \mathbb{D}$, then we get from (2.1) that

$$\begin{aligned} w(z)\|\psi(z)(f(\varphi(z)))\|_Y &\le w(z)\|\psi(z)\|_{L(X,Y)}\|f(\varphi(z))\|_X \\ &\le \frac{w(z)}{\tilde{v}(\varphi(z))}\|\psi(z)\|_{L(X,Y)}\|f\|_{H_v^\infty(X)}. \end{aligned}$$

Hence $\|W_{\psi,\varphi}\| \le \sup_{z\in\mathbb{D}} \|\psi(z)\|_{L(X,Y)} w(z)/\tilde{v}(\varphi(z))$.

We next verify the converse inequality

$$\|W_{\psi,\varphi}\| \ge \sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))}\|\psi(z)\|_{L(X,Y)}. \tag{2.3}$$

We may assume that $\psi \ne 0$, since (2.3) is obvious otherwise. Suppose to the contrary that (2.3) does not hold, that is,

$$\|W_{\psi,\varphi}\| < \sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))}\|\psi(z)\|_{L(X,Y)}.$$

Hence there are $\gamma > 1$, $z_0 \in \mathbb{D}$ and $x_0 \in X$ satisfying $\|x_0\| = 1$ so that

$$\begin{aligned} &\frac{w(z_0)}{\tilde{v}(\varphi(z_0))}\|\psi(z_0)\|_{L(X,Y)} \ge \gamma^3 \cdot \|W_{\psi,\varphi}\|, \\ &\|\psi(z_0)x_0\|_Y \ge \gamma^{-1} \cdot \|\psi(z_0)\|_{L(X,Y)}. \end{aligned}$$

According to the definition of $\tilde{v}(\varphi(z_0))$ there is a function $f_0 \in B_{H_v^\infty}$ for which $|f_0(\varphi(z_0))|\tilde{v}(\varphi(z_0)) \ge 1/\gamma$. Let $g_0 := f_0(\cdot)x_0 \in B_{H_v^\infty(X)}$. We get that

$$\begin{aligned} \|W_{\psi,\varphi}\| &\ge \|\psi(z_0)(g_0(\varphi(z_0)))\|_Y w(z_0) = \|\psi(z_0)x_0\|_Y |f_0(\varphi(z_0))| w(z_0) \\ &\ge \frac{\|\psi(z_0)\|_{L(X,Y)} w(z_0)}{\gamma^2 \cdot \tilde{v}(\varphi(z_0))} \ge \gamma \cdot \|W_{\psi,\varphi}\|. \end{aligned}$$

This estimate contradicts the fact that $\gamma > 1$ (note here that $W_{\psi,\varphi} \ne 0$ since $(W_{\psi,\varphi}g_x)(z) = \psi(z)x$ for the constant maps g_x, where $g_x(z) = x$ for $z \in \mathbb{D}$ and fixed $x \in X$). Hence (2.3) has been established. □

Remark 2.2. Alternatively, the boundedness of $W_{\psi,\varphi} \colon H_v^\infty(X) \to H_w^\infty(Y)$ can be expressed as a pointwise condition for a related family of weighted composition operators between scalar-valued spaces. For this note that $\psi(y^*, x) \colon z \mapsto y^*\psi(z)x$ are analytic maps $\mathbb{D} \to \mathbb{C}$ for $x \in X$ and $y^* \in Y^*$ under the assumptions of Theorem 2.1. We get the following fact, where (2.4) states that all the weighted composition operator $W_{\psi(y^*,x),\varphi}$ are bounded $H_v^\infty \to H_w^\infty$ by the scalar version of Theorem 2.1.

Fact. *$W_{\psi,\varphi}$ is bounded $H_v^\infty(X) \to H_w^\infty(Y)$ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z) \cdot |y^*\psi(z)x|}{\tilde{v}(\varphi(z))} < \infty \tag{2.4}$$

for all $y^* \in Y^*$ and $x \in X$.

Suppose above that (2.4) holds for all $y^* \in Y^*$ and $x \in X$. Fix $x \in X$ and consider the operators $T_z\colon Y^* \to \mathbb{C}$ defined by

$$T_z y^* = y^*\psi(z)x \cdot w(z)/\tilde{v}(\varphi(z)), \quad y^* \in Y^*, z \in \mathbb{D}.$$

Since $\sup_{z\in\mathbb{D}} |T_z y^*| < \infty$ for any $y^* \in Y^*$ by (2.4) it follows from the uniform boundedness principle that

$$\sup_{z\in\mathbb{D}} \|T_z\| = \sup_{z\in\mathbb{D}} \frac{w(z)\cdot\|\psi(z)x\|_Y}{\tilde{v}(\varphi(z))} < \infty.$$

By considering the operators $S_z\colon X \to Y$, where $S_z x = \psi(z)x\cdot w(z)/\tilde{v}(\varphi(z))$ for $x \in X$ and $z \in \mathbb{D}$, one gets similarly that

$$\sup_{z\in\mathbb{D}} \|S_z\| = \sup_{z\in\mathbb{D}} \|\psi(z)\|_{L(X,Y)} w(z)/\tilde{v}(\varphi(z)) < \infty.$$

Hence $W_{\psi,\varphi}$ is bounded $H_v^\infty(X) \to H_w^\infty(Y)$ according to Theorem 2.1.

3. Compactness properties of $W_{\psi,\varphi}$ on $H_v^\infty(X)$

Recall that if X is an infinite-dimensional Banach space, then the composition operator $C_\varphi\colon f \mapsto f\circ\varphi$ is never compact on $H_v^\infty(X)$, since C_φ fixes the constant maps $f_x(z) = x$ for any $x \in X$. By contrast, there are plenty of compact weighted composition operators $W_{\psi,\varphi}$ on $H_v^\infty(X)$ for infinite-dimensional X. For example, if S_0 is a fixed compact operator $X \to X$ then the constant maps $\psi(z) = S_0$ and $\varphi(z) = z_0$ yield compact operators $W_{\psi,\varphi}$.

In this section we characterize the compactness and weak compactness of the operators $W_{\psi,\varphi} : H_v^\infty(X) \to H_w^\infty(Y)$ for arbitrary complex Banach spaces. We will assume in this section for technical simplicity that v and w are radial weights, that is, $v(z) = v(|z|)$ and $w(z) = w(|z|)$ for $z \in \mathbb{D}$. We denote the class of compact operators $X \to Y$ by $K(X,Y)$, and by $W(X,Y)$ the respective class of weakly compact operators. If $X = Y$ we abbreviate $K(X) = K(X,X)$ and $W(X) = W(X,X)$.

A fundamental ingredient of our characterization will be the condition

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)} = 0, \tag{3.1}$$

which connects the behaviour of φ and ψ close to the boundary in analogy with the scalar case. In (3.1) the convention is that supremum over the empty set is zero. Our characterization will in addition involve the operator

$$T_\psi\colon x \mapsto \psi(\cdot)x;\ X \to H_w^\infty(Y),$$

which is a new ingredient in the vector-valued context. Note that

$$\|T_\psi\| = \sup_{\|x\|_X\le 1} \sup_{z\in\mathbb{D}} \|\psi(z)x\|_Y w(z) = \|\psi\|_{H_w^\infty(L(X,Y))}. \tag{3.2}$$

In particular, if $W_{\psi,\varphi}$ is bounded $H_v^\infty(X) \to H_w^\infty(Y)$, then T_ψ is bounded $X \to H_w^\infty(Y)$ by (2.2). We postpone a more careful discussion of the properties of T_ψ to Section 4.

The following main theorem extends results for the case $X = Y = \mathbb{C}$ from [9, Prop. 2.3], [21, Thm. 2.1], and [10, Cor. 4.3 and Thm. 5.2] to the vector-valued setting.

Theorem 3.1. *Let v and w be radial weight functions $\mathbb{D} \to (0,\infty)$, and X, Y be arbitrary complex Banach spaces.*

(a) The operator $W_{\psi,\varphi}\colon H_v^\infty(X) \to H_w^\infty(Y)$ is compact if and only if
(a1) $T_\psi\colon X \to H_w^\infty(Y)$ is compact, and
(a2) condition (3.1) holds.

(b) $W_{\psi,\varphi}\colon H_v^\infty(X) \to H_w^\infty(Y)$ is weakly compact if and only if
(b1) $T_\psi\colon X \to H_w^\infty(Y)$ is weakly compact, and
(b2) condition (3.1) holds.

The operators T_ψ, which only depend on the map ψ, are irrelevant for these questions in the case $X = Y = \mathbb{C}$, so their appearance in Theorem 3.1 is perhaps unexpected. Note also that the mere boundedness of $W_{\psi,\varphi}$ does not imply the (weak) compactness of T_ψ in the case of arbitrary spaces X and Y (see Section 4).

Proof of necessity in Theorem 3.1. Observe first that the compactness properties of $W_{\psi,\varphi}$ are inherited by T_ψ, since one may factor $T_\psi = W_{\psi,\varphi}A$, where $A\colon X \to H_v^\infty(X)$ is defined by $A(x) = f_x$, with $f_x(z) = x$ for $z \in \mathbb{D}$ and $x \in X$. The remaining part of the argument for the necessity part follows from the next lemma, which extends a scalar argument from [10, Thm. 5.2] or [2, Thm. 1]. Below we denote the unit sphere by $S_Z = \{x \in Z : \|x\| = 1\}$ for any Banach space Z.

Lemma 3.2. *If* (3.1) *does not hold, then there is a closed subspace $M \subset H_v^\infty(X)$, where M is linearly isomorphic to ℓ^∞, so that the restriction $W_{\psi,\varphi}|_M$ is an isomorphism $M \to W_{\psi,\varphi}(M)$. In particular, if $W_{\psi,\varphi}$ is a weakly compact operator $H_v^\infty(X) \to H_w^\infty(Y)$, then* (3.1) *is satisfied.*

Proof. If condition (3.1) fails to hold then there are $c > 0$ and a sequence $(z_n) \subset \mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$ and

$$\frac{w(z_n)}{\tilde{v}(\varphi(z_n))}\|\psi(z_n)\|_{L(X,Y)} > c, \quad n \in \mathbb{N}.$$

By passing to a subsequence, if necessary, we may assume that $(\varphi(z_n))$ is an interpolating sequence for H^∞. Hence, by the proof of [25, Thm. III.E.4], there are $C < \infty$ and a sequence $(h_k) \subset H^\infty$ so that $h_k(\varphi(z_n)) = 0$ for all $n \neq k$, $h_k(\varphi(z_k)) = 1$ for $k \in \mathbb{N}$ and $\sum_{n=1}^\infty |h_n(z)| \leq C$ for $z \in \mathbb{D}$.

Pick normalized elements $x_n \in S_X$, $y_n^* \in S_{Y^*}$ and $g_n \in S_{H_v^\infty}$ for $n \in \mathbb{N}$ so that $|y_n^*\psi(z_n)x_n| \geq 2^{-1}\|\psi(z_n)\|_{L(X,Y)} > 0$ and $|g_n(\varphi(z_n))|\tilde{v}(\varphi(z_n)) \geq \frac{1}{2}$. Put $f_n(z) = h_n(z)g_n(z)x_n$ for $z \in \mathbb{D}$. Define linear maps $T\colon \ell^\infty \to H_v^\infty(X)$ and $U\colon H_w^\infty(Y) \to \ell^\infty$ by

$$T\xi = \sum_{n=1}^\infty \xi_n f_n, \quad \xi = (\xi_n) \in \ell^\infty,$$
$$Uf = \left(\frac{y_k^* f(z_k)}{g_k(\varphi(z_k)) \cdot y_k^*\psi(z_k)x_k}\right)_{k\in\mathbf{N}}, \quad f \in H_w^\infty(Y).$$

Then T and U are bounded operators, since

$$\|T\xi\|_{H_v^\infty(X)} = \sup_{z\in\mathbb{D}} \sum_{n=1}^{\infty} |\xi_n h_n(z) g_n(z)| \cdot \|x_n\|_X v(z) \le C\|\xi\|_\infty,$$

$$\|Uf\|_\infty \le \sup_{k\in\mathbb{N}} \frac{4\|f(z_k)\|_Y \tilde{v}(\varphi(z_k))}{\|\psi(z_k)\|_{L(X,Y)}} \le \frac{4}{c} \sup_{k\in\mathbb{N}} \|f\|_{H_w^\infty(Y)}.$$

Finally, a direct calculation yields that $(UW_{\psi,\varphi}T)\xi = \xi$ for $\xi \in \ell^\infty$, so that the restriction of $W_{\psi,\varphi}$ to $M = T(\ell_\infty)$ is an isomorphism, where M is linearly isomorphic to ℓ^∞. This completes the proof of Lemma 3.2 and thus of the necessity part of Theorem 3.1. □

Let

$$\|U\|_e = dist(U, K(E,F)), \quad \|U\|_w = dist(U, W(E,F)),$$

be the essential, respectively the weak essential, norm of the bounded operator $U \in L(E,F)$ for Banach spaces E and F.

Proof of sufficiency in Theorem 3.1. This part follows from the estimates (3.3) and (3.4) below, which we include here since they require no extra work compared to the desired qualitative results. Note that (3.3) and (3.4) yield somewhat specialized estimates for $\|W_{\psi,\varphi}\|_e$ and $\|W_{\psi,\varphi}\|_w$ because of the compactness assumptions on T_ψ.

Claim 3.3. *(a) If $T_\psi\colon X \to H_w^\infty(Y)$ is compact, then*

$$\|W_{\psi,\varphi}\|_e \le 2 \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)}. \tag{3.3}$$

(b) If $T_\psi\colon X \to H_w^\infty(Y)$ is weakly compact, then

$$\|W_{\psi,\varphi}\|_w \le 2 \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)}. \tag{3.4}$$

We require the elementary estimate

$$\|z^k x_k\|_X v(z) \le \|f\|_{H_v^\infty(X)}, \tag{3.5}$$

for $z \in \mathbb{D}$ and $f \in H_v^\infty(X)$, where f has the Taylor expansion $f(z) = \sum_{k=0}^\infty z^k x_k$. This is immediate from $z^k x_k = \int_0^{2\pi} f(ze^{i\theta}) e^{-ik\theta} \frac{d\theta}{2\pi i}$ in the case of a radial weight v.

Towards (3.3) and (3.4) we first establish a useful auxiliary result. Put $r_n = \frac{n}{n+1}$ for $n \in \mathbf{N}$.

Lemma 3.4. *Suppose that $W_{\psi,\varphi}\colon H_v^\infty(X) \to H_w^\infty(Y)$ is bounded, and define the maps $K_n\colon H_v^\infty(X) \to H_v^\infty(X)$ by $(K_n f)(z) = f(r_n z)$ for $n \in \mathbf{N}$. If $T_\psi\colon X \to H_w^\infty(Y)$ is a compact (respectively, weakly compact) operator, then $W_{\psi,\varphi} \circ K_n\colon H_v^\infty(X) \to H_w^\infty(Y)$ is compact (respectively, weakly compact) for $n \in \mathbb{N}$.*

Proof. We first observe that

$$\|K_n : H_v^\infty(X) \to H_v^\infty(X)\| \le 1, \quad n \in \mathbb{N}. \tag{3.6}$$

Towards (3.6) it will be enough to check that $\|f\|_{\widehat{v}} = \|f\|_v$ for $f \in H_v^\infty(X)$, where the weight $\widehat{v}$ is the radial non-increasing majorant of v defined by

$$\widehat{v}(z) = \sup\{v(w) : |z| \le |w|\}, \quad z \in \mathbb{D}.$$

Clearly $\|f\|_v \le \|f\|_{\widetilde{v}}$. For the converse inequality let $z \in \mathbb{D}$ and $|z| \le r < 1$. The maximum principle and the radiality of v imply that

$$|f(z)|v(r) \le \sup_{|w|=r} |f(w)|v(w).$$

By taking the supremum over $|z| \le r$ we get that $\|f\|_{\widetilde{v}} \le \|f\|_v$ for $f \in H_v^\infty(X)$.

Assume that T_ψ is compact and fix $n \in \mathbb{N}$. For $m \in \mathbb{N}$ let $P_m \colon H_v^\infty(X) \to H_v^\infty(X)$ be the truncation operators

$$P_m(f) = \sum_{k=0}^{m} z^k x_k,$$

where $f \in H_v^\infty(X)$ has the Taylor series expansion $f(z) = \sum_{k=0}^\infty z^k x_k$. We first check that $W_{\psi,\varphi}P_m$ is compact for all m. In fact, consider the linear maps $q_k \colon H_v^\infty(X) \to X$ for $k \in \mathbb{N}$, where $q_k(f) = x_k$ for $f(z) = \sum_{k=0}^\infty z^k x_k$. Here q_k are bounded operators, since $\|q_k\| \le 2^k/c$ by (3.5), where $c = \inf_{|z|=\frac{1}{2}} v(z) > 0$. Hence $T_\psi q_k$ are compact $H_v^\infty(X) \to H_w^\infty(Y)$ for each k by our assumption. Since

$$(W_{\psi,\varphi}P_m f)(z) = \sum_{k=0}^{m} \varphi(z)^k \cdot \psi(z) x_k = \sum_{k=0}^{m} \varphi(z)^k \cdot (T_\psi q_k f)(z),$$

it follows that $W_{\psi,\varphi}P_m$ are compact operators $H_v^\infty(X) \to H_w^\infty(Y)$ for all m, because multiplication by $\varphi(\cdot)^k$ defines bounded operators on $H_w^\infty(Y)$. Furthermore, from (3.5) we obtain that

$$\sup_{z\in\mathbb{D}} \|(K_n f - P_m K_n f)(z)\|_X v(z) \le \sum_{k=m+1}^{\infty} r_n^k \|f\|_{H_v^\infty(X)} \to 0$$

as $m \to \infty$. Hence $W_{\psi,\varphi}K_n = \lim_m W_{\psi,\varphi}P_m K_n$ is also a compact operator by approximation.

The weakly compact case is similar, and hence we omit the details. $\square$

Proof of (3.3) *and* (3.4). We only prove (3.4) here, since the other case is entirely analogous.

Suppose that $T_\psi \colon X \to H_w^\infty(Y)$ is weakly compact. Lemma 3.4 yields that $W_{\psi,\varphi}K_n \colon H_v^\infty(X) \to H_w^\infty(Y)$ are weakly compact operators for any n. Hence $\|W_{\psi,\varphi}\|_w \le \|W_{\psi,\varphi} - W_{\psi,\varphi}K_n\|$, so it will be enough to verify that

$$\limsup_{n\to\infty} \|W_{\psi,\varphi} - W_{\psi,\varphi}K_n\| \le 2 \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)}. \tag{3.7}$$

The argument is a fairly straightforward vector-valued modification of that of [10, Thm. 4.2], but we include a sketch for completeness. Fix $r \in (0,1)$ and $n \in \mathbb{N}$. If $f \in H_v^\infty(X)$, then we split

$$\|(W_{\psi,\varphi}f - W_{\psi,\varphi}K_n f)(z)\|_Y w(z) \le \sup_{|\varphi(z)|>r} \|(W_{\psi,\varphi}f - W_{\psi,\varphi}K_n f)(z)\|_Y w(z)$$
$$+ \sup_{|\varphi(z)|\le r} \|(W_{\psi,\varphi}f - W_{\psi,\varphi}K_n f)(z)\|_Y w(z) =: A_{n,r} + B_{n,r}.$$

We obtain from (2.1) and (3.6) that

$$\begin{aligned} A_{n,r} &\le \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)} \|f - K_n f\|_{H^\infty_{\tilde{v}}(X)} \\ &\le 2 \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)} \|f\|_{H^\infty_v(X)}, \end{aligned}$$

and

$$B_{n,r} \le \sup_{|\varphi(z)|\le r} \|(f - K_n f)(\varphi(z))\|_X \|\psi\|_{H^\infty_w(L(X,Y))}. \tag{3.8}$$

There is a constant $M(r) < \infty$ so that

$$\sup_{\|f\|_{H^\infty_v(X)}\le 1} \sup_{|w|\le r} \|(f - K_n f)(w)\|_X \le \frac{M(r)}{n},$$

and thus the right-hand side of (3.8) tends to 0 as $n \to \infty$ (for this use the Cauchy integral formula or apply the corresponding scalar-valued estimate from [2, p. 144]). Consequently

$$\limsup_{n\to\infty} \|W_{\psi,\varphi} - W_{\psi,\varphi} K_n\| \le 2 \sup_{|\varphi(z)|>r} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)},$$

whence we obtain (3.7) by letting $r \to 1$.

This completes the proof of Claim 3.3 and thus of Theorem 3.1. □

For completeness we record below the special cases of Theorem 3.1 that concern operator-valued multipliers M_ψ and composition operators C_φ, some of which were known earlier. In fact, (i) can be deduced from [20, Prop. 3.1], (iii) and (iv) are contained in [19, Thms. 6 and 7] for $v = w = 1$, and the case $v = w$ follows from [4, Cor. 15 and 16] or [16]. The corresponding results for $X = Y = \mathbb{C}$ are found e.g. in [3], [9], [21], and [10].

Corollary 3.5. *Let X, Y be Banach spaces, $v, w : \mathbb{D} \to (0,\infty)$ be weight functions, and $\psi : \mathbb{D} \to L(X,Y)$, $\varphi: \mathbb{D} \to \mathbb{D}$ be analytic maps.*

(i) M_ψ is bounded $H^\infty_v(X) \to H^\infty_w(Y)$ if and only if

$$\sup_{z\in\mathbb{D}} \|\psi(z)\|_{L(X,Y)} w(z)/\tilde{v}(z) < \infty. \tag{3.9}$$

(ii) Let v and w be radial weights, and suppose that (3.9) *holds. Then M_ψ is compact (resp., weakly compact) $H^\infty_v(X) \to H^\infty_w(Y)$ if and only if T_ψ is compact (resp., weakly compact) $X \to H^\infty_w(Y)$ and*

$$\lim_{|z|\to 1} \|\psi(z)\|_{L(X,Y)} w(z)/\tilde{v}(z) = 0.$$

(iii) C_φ is bounded from $H^\infty_v(X) \to H^\infty_w(X)$ if and only if

$$\sup_{z\in\mathbb{D}} w(z)/\tilde{v}(\varphi(z)) < \infty. \tag{3.10}$$

(iv) Let v and w be radial weights, and suppose that (3.10) *holds. Then C_φ is weakly compact $H^\infty_v(X) \to H^\infty_w(X)$ if and only if the identity operator $X \to X$ is weakly compact (that is, X is reflexive) and*

$$\lim_{|z|\to 1} w(z)/\tilde{v}(\varphi(z)) = 0.$$

Remark 3.6. The simplifying assumption about the radiality of the weights v and w was used in the proof Lemma 3.4.

There are analogues of Theorem 3.1 for certain operator-norm closed operator ideals I in the sense of Pietsch [22]. In fact, one may apply Lemma 3.2 provided $id_{\ell^\infty} \notin I(\ell^\infty)$, while the approximation argument for (3.4) extends in a straightforward fashion. In order to state a result of this type, recall that $U : E \to F$ is a weakly conditionally compact operator if (Ux_n) contains a weakly Cauchy subsequence for any sequence $(x_n) \subset B_E$. Clearly the class of weakly conditionally compact operators contains the (weakly) compact ones.

Theorem 3.7. *Let v and w be radial weight functions $\mathbb{D} \to (0, \infty)$. Then $W_{\psi,\varphi}\colon H_v^\infty(X) \to H_w^\infty(Y)$ is weakly conditionally compact if and only if $T_\psi\colon X \to H_w^\infty(Y)$ is weakly conditionally compact and (3.1) is satisfied.*

Analogous results can be obtained for the complete continuity, or the strict singularity or cosingularity of $W_{\psi,\varphi}$.

4. Compactness properties of T_ψ

The compactness properties of $W_{\psi,\varphi}\colon H_v^\infty(X) \to H_w^\infty(Y)$ in Theorem 3.1 involve the auxiliary operator $T_\psi : X \to H_w^\infty(Y)$, where $x \mapsto \psi(\cdot)x$. In this section we look more closely at some properties of T_ψ.

Observe first that if $T_\psi \in K(X, H_w^\infty(Y))$, then all individual operators $\psi(z) \in K(X, Y)$ for $z \in \mathbb{D}$, since the point evaluations $\delta_z : f \mapsto f(z)$ define bounded operators $H_w^\infty(Y) \to Y$ for any $z \in \mathbb{D}$. An analogous fact holds for weak compactness. The following example demonstrates that the converse does not hold in general.

Example 4.1. There is an analytic operator-valued map $\psi \in H^\infty(L(\ell^1))$, so that $\psi(z) \in K(\ell^1)$ for all $z \in \mathbb{D}$, but $T_\psi\colon \ell^1 \to H^\infty(\ell^1)$ is not even weakly conditionally compact.

Proof. Define the bounded operator-valued analytic map $\psi : \mathbb{D} \to L(\ell^1)$ by $\psi(z) = \sum_{k=1}^{\infty} z^k e_k^* \otimes e_k$, where (e_k) denotes the standard unit vector basis of ℓ^1 and $(e_k^*) \subset c_0$ its biorthogonal sequence. In other words,

$$\psi(z)x = \sum_{k=1}^{\infty} z^k x_k e_k, \quad x = (x_k) \in \ell^1,\ z \in \mathbb{D},$$

that is, $\psi(z)$ is the compact diagonal operator on ℓ^1 determined by the sequence (z^k) for $z \in \mathbb{D}$.

We claim that T_ψ is not weakly conditionally compact as an operator $\ell^1 \to H^\infty(\ell^1)$. Suppose to the contrary that T_ψ is weakly conditionally compact. Hence $(T_\psi(e_n))$ admits a weakly Cauchy subsequence $(T_\psi(e_{n_j}))$, whence the difference sequence $(T_\psi(e_{n_{2j+1}} - e_{n_{2j}}))$ is weak-null in $H^\infty(\ell^1)$. By Mazur's theorem

$$\| \sum_{j=1}^{s} c_j T_\psi(e_{n_{2j+1}} - e_{n_{2j}}) \|_{H^\infty(\ell^1)} < \frac{1}{2}$$

for a suitable convex combination, where $\sum_{j=1}^{s} c_j = 1$ and $c_j \geq 0$ for $j = 1, \dots, s$. On the other hand, a direct evaluation reveals that

$$\|\sum_{j=1}^{s} c_j T_\psi(e_{n_{2j+1}} - e_{n_{2j}})\|_{H^\infty(\ell^1)} = \sup_{z\in\mathbb{D}} \|\sum_{j=1}^{s} c_j (z^{n_{2j+1}} e_{n_{2j+1}} - z^{n_{2j}} e_{n_{2j}})\|_{\ell^1}$$
$$= \sup_{z\in\mathbb{D}} \sum_{j=1}^{s} c_j (|z|^{n_{2j+1}} + |z|^{n_{2j}}) \geq \sum_{j=1}^{s} c_j = 1,$$

which is a contradiction. □

Remark 4.2. For simplicity Example 4.1 was formulated for the constant weight $v = 1$, but it can be modified without difficulties to apply to $H_v^\infty(\ell^1)$ for any radial weight v on $\mathbb{D}$.

According to the preceding example the compactness (respectively, weak compactness) of $T_\psi \colon X \to H_w^\infty(Y)$ cannot be replaced in Theorem 3.1 by the condition that $\psi(\mathbb{D}) \subset K(X,Y)$ (respectively, $\psi(\mathbb{D}) \subset W(X,Y)$). However, Theorem 4.4 below shows that the above pointwise condition does imply the compactness of T_ψ for a large class of operator-valued maps ψ and for radial weights.

Let E be a Banach space and v a radial weight function. It will be convenient to define $H_v^0(E)$ as the closure of the analytic E-valued polynomials in $H_v^\infty(E)$. If $v(z) \equiv 1$, then $H_v^0(E)$ is a vector-valued analogue of the disk algebra. If $\lim_{t\to 1} v(t) = 0$, then one may show (analogously to [26, pp. 83–84]) that

$$H_v^0(E) = \{f \in H_v^\infty(E) : \lim_{|z|\to 1} \|f(z)\|_E v(z) = 0\}. \tag{4.1}$$

Thus H_v^0 is in this case the familiar small version of H_v^∞. We will require the following technical fact, which is obvious from (4.1) when $\lim_{t\to 1} v(t) = 0$. However, we will not separately justify (4.1), because it is not needed here.

Lemma 4.3. *Suppose that v is any radial weight, let $E_0 \subset E$ be a closed subspace and suppose that $f \in H_v^\infty(E)$. Then $f \in H_v^0(E_0)$ if and only if $f \in H_v^0(E)$ and $f(\mathbb{D}) \subset E_0$.*

Proof. Assume that $f \in H_v^0(E)$ and $f(\mathbb{D}) \subset E_0$. Let $f_r(z) = f(rz)$ for $r \in (0,1)$, so that $f_r \in H_v^\infty(E_0)$ by (3.6). Pick E-valued polynomials $p_n(z) = \sum_{j=0}^{N_n} z^j x_j^{(n)}$ for $n \in \mathbb{N}$ so that $p_n \to f$ in $H_v^\infty(E)$ as $n \to \infty$. Since the weight v is radial, we get again from (3.6) that $(p_n)_r \to f_r$ as $r \to 1$. Moreover,

$$\|p_n - (p_n)_r\|_{H_v^\infty(E)} \leq (N_n + 1) \sup_{0\leq j\leq N_n} \|x_j^{(n)}\|_E (1 - r^{N_n}) \sup_{z\in\mathbb{D}} v(z) \to 0,$$

as $r \to 1$. Since

$$\|f - f_r\|_{H_v^\infty(E_0)} = \|f - f_r\|_{H_v^\infty(E)} \leq \|f - p_n\|_{H_v^\infty(E)} +$$
$$+ \|p_n - (p_n)_r\|_{H_v^\infty(E)} + \|(p_n)_r - f_r\|_{H_v^\infty(E)},$$

we deduce that $f_r \to f$ in $H_v^\infty(E_0)$ as $r \to 1$.

Fix next $r \in (0,1)$ and consider the truncations $(P_n f)_r(z) = \sum_{j=0}^{n} r^j z^j y_j$, where $y_j \in E_0$ is the j:th Taylor coefficient of f and $P_n f = \sum_{k=0}^{n} z^k y_k$ as in the proof of Lemma 3.4. From the radiality of v and (3.5) we get that

$$\|f_r - (P_n f)_r\|_{H_v^\infty(E_0)} \le \|f\|_{H_v^\infty(E_0)} \sum_{j=n+1}^{\infty} r^j \to 0, \quad n \to \infty,$$

so that $(P_n f)_r \to f_r$ in $H_v^\infty(E_0)$ as $n \to \infty$. Consequently there are subsequences (r_m) and (n_m) for which $(P_{n_m} f)_{r_m} \to f$ in $H_v^\infty(E_0)$ as $m \to \infty$. □

Theorem 4.4. *Let X and Y be complex Banach spaces, w a radial weight function, and assume that $\psi \in H_w^0(L(X,Y))$. Then $T_\psi \colon X \to H_w^\infty(Y)$ is compact (respectively, weakly compact) if and only if $\psi(\mathbb{D}) \subset K(X,Y)$ (respectively, $\psi(\mathbb{D}) \subset W(X,Y)$).*

Proof. We will only include the details in the compact case, since the weakly compact one is very similar.

Assume that $\psi(\mathbb{D}) \subset K(X,Y)$, whence $\psi \in H_w^0(K(X,Y))$ by Lemma 4.3. Let $\psi_n(z) = \sum_{k=0}^{n} z^k U_k^{(n)}$ be analytic $K(X,Y)$-valued polynomials for $n \in \mathbb{N}$ so that $\psi_n \to \psi$ in $H_w^\infty(K(X,Y))$. It follows from (2.2) that $\|T_\psi - T_{\psi_n}\| \to 0$, so it will suffice to verify that $T_{\psi_n} \colon X \to H_w^\infty(Y)$ is compact for each $n \in \mathbb{N}$.

For this observe that the maps $\theta_k \colon Y \to H_w^\infty(Y)$ defined by $(\theta_k y)(z) = z^k y$ for $y \in Y$ and $z \in \mathbb{D}$ are bounded for all $k \in \mathbb{N}$. Hence the compactness of T_{ψ_n} follows from the factorization $T_{\psi_n} = \sum_{k=0}^{n} \theta_k \circ U_k^{(n)}$. □

We conclude by stating a couple of examples about the simplest operator-valued multipliers as applications of Corollary 3.5 and Theorem 4.4.

Example 4.5. Put $v_p(z) = (1 - |z|^2)^p$ for $z \in \mathbb{D}$ and $p > 0$.
(i) Let X be any Banach space and $\psi(z) \equiv U$ for $z \in \mathbb{D}$, where $U \in K(X)$ is a fixed operator. Then M_ψ is compact $H^\infty(X) \to H_{v_p}^\infty(X)$.
(ii) Let X be a reflexive Banach space and $\psi(z) \equiv V$ for $z \in \mathbb{D}$, where $V \notin K(X)$ is a fixed operator. Then M_ψ is weakly compact, but not compact $H^\infty(X) \to H_{v_p}^\infty(X)$. In particular, the formal inclusion $H^\infty(X) \to H_{v_p}^\infty(X)$ is weakly compact (for this choose $V = I_X$, the identity map on X).

In fact, in case (i) the multiplier M_ψ is compact according to Corollary 3.5.(ii), since T_ψ is compact $X \to H_{v_p}^\infty(X)$ by e.g. Theorem 4.4, and

$$\lim_{r\to 1} \sup_{|z|>r} \|U\| v_p(z) = \|U\| \lim_{r\to 1} (1 - r^2)^p = 0.$$

Similarly $M_\psi \in W(H^\infty(X), H_{v_p}^\infty(X))$ in case (ii), since V is weakly compact in view of the reflexivity of X. Moreover, M_ψ cannot be compact because $\psi(z) = V \notin K(X)$.

Our results suggest the following question.

Problem. Characterize boundedness and compactness properties of the general operator-weighted composition operators $W_{\psi,\varphi}$ on the vector-valued Hardy spaces $H^p(X)$, or on the analogous X-valued Bergman spaces, in the cases $1 \le p < \infty$. Would an operator analogous to T_ψ play any role here? The references [11] and [13] contain the basic results in the scalar-valued setting.

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