

EXTENSION OF LEHTO'S EXISTENCE THEOREM AND ITS CONSEQUENCES

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Abstract

We show that many recent and new results on the existence of ACL homeomorphic solutions for the Beltrami equation follow from our extension of the well-known Lehto existence theorem.

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1 Introduction

The present preprint is a continuation of our last preprint [RSY₁] in the subject where we have obtained one extension of the Lehto theorem on the existence of homeomorphic solutions for the Beltrami equation, see Section 2. The goal here is to show that the extension has as corollaries the main known existence theorems as well as a series of more advanced theorems for the Beltrami equation, see Sections 4 and 5. The base for these advances is some lemmas on measure and integral in Section 3. The corresponding historic comments and some comparisons can be found in Section 6.

Let D be a domain in the complex plane \mathbb{C} , i.e., open and connected subset of \mathbb{C} , and let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The **Beltrami equation** is

$$(1.1) \quad f_{\bar{z}} = \mu(z) \cdot f_z$$

where $f_{\bar{z}} = \bar{\partial}f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y , correspondingly. The Beltrami equation plays a great role in the mapping theory and has an independent interest itself.

The function μ is called the **complex coefficient** and

$$(1.2) \quad K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

the **maximal dilatation** or in short the **dilatation** of the equation (1.1). The Beltrami equation (1.1) is said to be degenerate if $\text{ess sup } K_\mu(z) = 1$.

Use will be made also the **tangential dilatation** with respect to a point $z_0 \in D$ which is defined by

$$(1.3) \quad K_\mu^T(z, z_0) = \frac{\left| 1 - \frac{\overline{z-z_0}}{z-z_0} \mu(z) \right|^2}{1 - |\mu(z)|^2}.$$

Recall that a mapping $f : D \rightarrow \mathbb{C}$ is **absolutely continuous on lines**, abbr. $f \in \mathbf{ACL}$, if, for every closed rectangle R in D whose sides are parallel to the coordinate axes, $f|_R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R . In particular, f is ACL if it belongs to the Sobolev class $W_{loc}^{1,1}$, see e.g. [Ma], p. 8. Note that, if $f \in \mathbf{ACL}$, then f has partial derivatives f_x and f_y a.e. and, thus, by the well-known Gehring-Lehto theorem every ACL homeomorphism $f : D \rightarrow \mathbb{C}$ is differentiable a.e., see [GL] or [LV], p. 128. For a sense-preserving ACL homeomorphism $f : D \rightarrow \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$ is nonnegative a.e., see [LV], p. 10. In this case, the **complex dilatation** of f is the ratio $\mu(z) = f_{\bar{z}}/f_z$, and $|\mu(z)| \leq 1$ a.e., and the **dilatation** of f is $K_\mu(z)$ from (1.2) and $K_\mu(z) \geq 1$ a.e. Here we set by definition $\mu(z) = 0$ and, correspondingly, $K_\mu(z) = 1$ if $f_z = 0$. The complex dilatation and the dilatation of f will also be denoted by μ_f and K_f , respectively.

Given a measurable function $Q : D \rightarrow [1, \infty]$, we say that a homeomorphism $f : D \rightarrow \overline{\mathbb{C}}$ is a **Q-homeomorphism** if

$$(1.4) \quad M(f\Gamma) \leq \int_D Q(z) \cdot \rho^2(z) \, dx dy$$

holds for every path family Γ in D and each $\rho \in \mathit{adm} \Gamma$. This term was introduced in [MRSY₁], see also [MRSY₂] and [MRSY₃], and the inequality was used in [RSY₂] and [RSY₃] for studying the so-called BMO-quasiconformal mappings.

Recall that, given a family of paths Γ in $\overline{\mathbb{C}}$, a Borel function $\rho : \overline{\mathbb{C}} \rightarrow [0, \infty]$ is called **admissible** for Γ , abbr. $\rho \in \mathit{adm} \Gamma$, if

$$(1.5) \quad \int_\gamma \rho(z) |dz| \geq 1$$

for each $\gamma \in \Gamma$. The **modulus** of Γ is defined by

$$(1.6) \quad M(\Gamma) = \inf_{\rho \in \mathit{adm} \Gamma} \int_{\mathbb{C}} \rho^2(z) \, dx dy.$$

We say that a property P holds for **almost every (a.e.)** path γ in a family Γ if the subfamily of all paths in Γ for which P fails has modulus zero. In particular, almost all paths in \mathbb{C} are rectifiable.

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \rightarrow \overline{\mathbb{C}}$ which join E and F in D , i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E, F) = \Gamma(E, F, \overline{\mathbb{C}})$ if $D = \overline{\mathbb{C}}$. A **ring domain**, or shortly a **ring** in $\overline{\mathbb{C}}$ is a doubly connected domain R in $\overline{\mathbb{C}}$. Let R be a ring in

$\bar{\mathbb{C}}$. If C_1 and C_2 are the connected components of $\bar{\mathbb{C}} \setminus R$, we write $R = R(C_1, C_2)$. The **capacity** of R can be defined by

$$(1.7) \quad \text{cap } R(C_1, C_2) = M(\Gamma(C_1, C_2, R)) ,$$

see e.g. [Ge₁]. Note also, see e.g. Theorem 11.3 in [Va], that

$$(1.8) \quad M(\Gamma(C_1, C_2, R)) = M(\Gamma(C_1, C_2)) .$$

Motivated by the ring definition of quasiconformality in [Ge₂], we introduced in [RSY₄] the following notion that localizes and extends the notion of a Q -homeomorphism. Let D be a domain in \mathbb{C} , $z_0 \in D$, $r_0 \leq \text{dist}(z_0, \partial D)$ and $Q : D(z_0, r_0) \rightarrow [0, \infty]$ a measurable function in the disk

$$(1.9) \quad D(z_0, r_0) = \{z \in \mathbb{C} : |z - z_0| < r_0\} .$$

Set

$$(1.10) \quad A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\} ,$$

$$(1.11) \quad C(z_0, r_i) = \{z \in \mathbb{C} : |z - z_0| = r_i\}, \quad i = 1, 2.$$

We say that a homeomorphism $f : D \rightarrow \bar{\mathbb{C}}$ is a **ring Q -homeomorphism** at the point $z_0 \in D$ if

$$(1.12) \quad M(\Gamma(fC_1, fC_2)) \leq \int_A Q(z) \cdot \eta^2(|z - z_0|) \, dx dy$$

for every annulus $A = A(z_0, r_1, r_2)$, $0 < r_1 < r_2 < r_0$, and for every measurable function $\eta : (r_1, r_2) \rightarrow [0, \infty]$ such that

$$(1.13) \quad \int_{r_1}^{r_2} \eta(r) \, dr = 1 .$$

Note that every Q -homeomorphism $f : D \rightarrow \bar{\mathbb{C}}$ is a ring Q -homeomorphism at each point $z_0 \in D$.

An ACL homeomorphism $f_\mu : D \rightarrow \mathbb{C}$ is called a **ring solution** of the Beltrami equation (1.1) if f satisfies (1.1) a.e., $f_\mu^{-1} \in W_{loc}^{1,2}$ and f is a ring Q -homeomorphism at every point $z_0 \in D$ with $Q_{z_0}(z) = K_\mu^T(z, z_0)$.

The condition $f^{-1} \in W_{loc}^{1,2}$ given in the definition of a ring solution implies that a.e. point z is a **regular point** for the mapping f , i.e., f is differentiable at z and $J_f(z) \neq 0$. Note that the condition $K_\mu \in L_{loc}^1$ is necessary for a homeomorphic ACL solution f of (1.1) to have the property $g = f^{-1} \in W_{loc}^{1,2}$ because this property implies that

$$\int_C K_\mu(z) \, dx dy \leq 4 \int_C \frac{dx dy}{1 - |\mu(z)|^2} = 4 \int_{f(C)} |\partial g|^2 \, dudv < \infty$$

for every compact set $C \subset D$. Note also that every homeomorphic ACL solution f of the Beltrami equation with $K_\mu \in L^1_{loc}$ belongs to the class $W^{1,1}_{loc}$ as in all our theorems. Note also that if, in addition, $K_\mu \in L^p_{loc}$, $p \in [1, \infty]$, then $f_\mu \in W^{1,s}_{loc}$ where $s = 2p/(1+p) \in [1, 2]$. In the classical case when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty$, every ACL homeomorphic solution f of the Beltrami equation (1.1) is in the class $W^{1,2}_{loc}$ together with its inverse mapping f^{-1} . In the case $\|\mu\|_\infty = 1$ and when $K_\mu \leq Q \in \text{BMO}$, again $f^{-1} \in W^{1,2}_{loc}$ and f belongs to $W^{1,s}_{loc}$ for all $1 \leq s < 2$ but not necessarily to $W^{1,2}_{loc}$, see [RSY₂] and [RSY₃]. The inequality (1.12), which ring solutions satisfy, is an important tool in deriving various properties of the solutions.

2 Extension of Lehto's existence theorem

Lehto considers in [Le] degenerate Beltrami equations in the special case where the **singular set** S_μ

$$(2.1) \quad S_\mu = \{z \in \mathbb{C} : \lim_{\varepsilon \rightarrow 0} \|K_\mu\|_{L^\infty(D(z,\varepsilon))} = \infty\}$$

of the complex coefficient μ in (1.1) is of measure zero and shows that, if for every $z_0 \in \mathbb{C}$ and every r_1 and $r_2 \in (0, \infty)$ the integral

$$(2.2) \quad \int_{r_1}^{r_2} \frac{dr}{r(1+q_{z_0}^T(r))}, \quad r_2 > r_1,$$

where

$$(2.3) \quad q_{z_0}^T(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{|1 - e^{-2i\vartheta} \mu(z_0 + re^{i\vartheta})|^2}{1 - |\mu(z_0 + re^{i\vartheta})|^2} d\vartheta,$$

is positive and tends to ∞ as either $r_1 \rightarrow 0$ or $r_2 \rightarrow \infty$, then there exists a homeomorphism $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ which is ACL in $\mathbb{C} \setminus S_\mu$ and satisfies (1.1) a.e. Note that the integrand in (2.3) is just the tangential dilatation $K_\mu^T(z, z_0)$, see (1.3).

In this section we reformulate the extension of Lehto's existence theorem first obtained in [RSY₁], cf. alternative formulations and proofs of this important result in [RSY₄]–[RSY₆], and then, in Sections 4 and 5, derive from it some new existence theorems. In this extension we state the existence of a ring solution in a domain $D \subset \mathbb{C}$. By the definition of a ring solution, see the introduction, our solution is, in particular, ACL in D and not only in $D \setminus S_\mu$. Note that the situation where $S_\mu = D$ is possible in the following theorem. Note also that the condition (2.5) in our theorem is weaker than the condition in Lehto's existence theorem. See Remark 5.33 for the case where $\infty \in D$.

2.4. Theorem. *Let D be a domain in \mathbb{C} and let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$ such that at every point $z_0 \in D$*

$$(2.5) \quad \int_0^{\delta(z_0)} \frac{dr}{rq_{z_0}^T(r)} = \infty$$

where $\delta(z_0) < \text{dist}(z_0, \partial D)$ and $q_{z_0}^T(r)$ is the mean of $K_\mu^T(z, z_0)$ over $|z - z_0| = r$. Then the Beltrami equation (1.1) has a ring solution f_μ .

As a simplest consequence of Theorem 2.4, we have the following statement.

2.6. Corollary. *If $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$ and at every point $z_0 \in D$*

$$(2.7) \quad q_{z_0}^T(r) = O\left(\log \frac{1}{r}\right) \quad \text{as } r \rightarrow 0,$$

then (1.1) has a ring solution.

As one more consequence of Theorem 2.4 we obtain the following result which is due to Miklyukov and Suvorov [MS] for the case $K_\mu \in L_{loc}^p$, $p > 1$.

2.8. Corollary. *If $K_\mu \in L_{loc}^p$ for $p \geq 1$ and (2.5) holds for $K_\mu(z)$ instead of $K_\mu^T(z, z_0)$ for every point $z_0 \in D$, then (1.1) has a $W_{loc}^{1,s}$ homeomorphic solution where $s = 2p/(p+1)$.*

Other corollaries will be formulated in Sections 4 and 5 after some measure and integral considerations in Section 3 related to the condition (2.5).

3 Integral and measure conditions

One may use integral constraints and measure constraints on dilatations in existence theorems for the Beltrami equation (1.1). Below we show that integral constraints are reduced to the corresponding constraints in measure. Hence we consider first of all the constraints in measure which have a more general nature.

If $Q(z) : U \rightarrow [0, \infty]$ is a measurable function in $U \subset \mathbb{C}$ and $\Phi : [0, \infty] \rightarrow [0, \infty]$ is non-decreasing and

$$(3.1) \quad \int_U \Phi(Q(z)) \, dx dy \leq C < \infty,$$

then, arguing by contradiction, we obtain that

$$(3.2) \quad M(t) \leq \frac{C}{\Phi(t)} \quad \forall t \in [0, \infty)$$

where

$$(3.3) \quad M(t) = M_{Q,U}(t) = |\{z \in U : Q(z) > t\}|, \quad t \geq 0.$$

Thus, every integral condition implies the corresponding measure constraint and we may restrict ourselves to measure constraints.

Let U be an open set in \mathbb{C} , $Q : U \rightarrow [0, \infty]$ a measurable function, and let

$$(3.4) \quad M(t) \leq \varphi(t)$$

for some function $\varphi : [0, \infty] \rightarrow [0, \infty]$ where $M(t)$ is defined by (3.3).

We may assume here with no loss of generality that φ is non-increasing and continuous from the right. Indeed, if φ_L is the **lower envelope** of φ , i.e.,

$$(3.5) \quad \varphi_L(t) = \lim_{b \rightarrow t+0} \inf_{0 \leq a \leq b} \varphi(a),$$

in other words, the greatest function which is non-increasing and continuous from the right such that $\varphi_L(t) \leq \varphi(t)$ for all $t \in [0, \infty]$, then (3.4) is equivalent to

$$(3.6) \quad M(t) \leq \varphi_L(t).$$

For every non-increasing function $\varphi : [0, \infty] \rightarrow [0, \infty]$, the **inverse function** $\varphi^{-1} : [0, \infty] \rightarrow [0, \infty]$ can be well defined by setting

$$(3.7) \quad \varphi^{-1}(\tau) = \inf_{\varphi(t) \leq \tau} t.$$

As usual, here inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\varphi(t) \leq \tau$ is empty. Note that the function φ^{-1} is also non-increasing.

Similarly, for every non-decreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$, we set

$$(3.8) \quad \Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t.$$

Again, here inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

3.9. Proposition. *Let $\psi : [0, \infty] \rightarrow [0, \infty]$ be a sense-reversing homeomorphism and $\varphi : [0, \infty] \rightarrow [0, \infty]$ a monotone function. Then*

$$(3.10) \quad [\psi \circ \varphi]^{-1}(\tau) = \varphi^{-1} \circ \psi^{-1}(\tau) \quad \forall \tau \in [0, \infty]$$

and

$$(3.11) \quad [\varphi \circ \psi]^{-1}(\tau) \leq \psi^{-1} \circ \varphi^{-1}(\tau) \quad \forall \tau \in [0, \infty]$$

and, except a countable collection of $\tau \in [0, \infty]$,

$$(3.12) \quad [\varphi \circ \psi]^{-1}(\tau) = \psi^{-1} \circ \varphi^{-1}(\tau).$$

The equality (3.12) holds for all $\tau \in [0, \infty]$ iff the function $\varphi : [0, \infty] \rightarrow [0, \infty]$ is strictly monotone.

3.13. Remark. If ψ is a sense-preserving homeomorphism, then (3.10) and (3.12) are obvious for every monotone function φ .

Proof of Proposition 3.9. Let us first prove (3.10). If φ is non-increasing, then

$$[\psi \circ \varphi]^{-1}(\tau) = \inf_{\psi(\varphi(t)) \geq \tau} t = \inf_{\varphi(t) \leq \psi^{-1}(\tau)} t = \varphi^{-1} \circ \psi^{-1}(\tau).$$

Similarly, if φ is non-decreasing, then

$$[\psi \circ \varphi]^{-1}(\tau) = \inf_{\psi(\varphi(t)) \leq \tau} t = \inf_{\varphi(t) \geq \psi^{-1}(\tau)} t = \varphi^{-1} \circ \psi^{-1}(\tau).$$

Now, let us prove (3.11) and (3.12). If φ is non-increasing, then applying the substitution $\eta = \psi(t)$ we have

$$\begin{aligned} [\varphi \circ \psi]^{-1}(\tau) &= \inf_{\varphi(\psi(t)) \geq \tau} t = \inf_{\varphi(\eta) \geq \tau} \psi^{-1}(\eta) = \psi^{-1} \left(\sup_{\varphi(\eta) \geq \tau} \eta \right) \leq \\ &\leq \psi^{-1} \left(\inf_{\varphi(\eta) \leq \tau} \eta \right) = \psi^{-1} \circ \varphi^{-1}(\tau), \end{aligned}$$

i.e., (3.11) holds for all $\tau \in [0, \infty]$. Evidently that here the strict inequality is possible only for a countable collection of $\tau \in [0, \infty]$ because an interval of constancy of φ corresponds to every such τ . Hence (3.12) holds for all $\tau \in [0, \infty]$ if and only if φ is decreasing.

Similarly, if φ is non-decreasing, then

$$\begin{aligned} [\varphi \circ \psi]^{-1}(\tau) &= \inf_{\varphi(\psi(t)) \leq \tau} t = \inf_{\varphi(\eta) \leq \tau} \psi^{-1}(\eta) = \psi^{-1} \left(\sup_{\varphi(\eta) \leq \tau} \eta \right) \leq \\ &\leq \psi^{-1} \left(\inf_{\varphi(\eta) \geq \tau} \eta \right) = \psi^{-1} \circ \varphi^{-1}(\tau), \end{aligned}$$

i.e., (3.11) holds for all $\tau \in [0, \infty]$ and again the strict inequality is possible only for a countable collection of $\tau \in [0, \infty]$. In the case, (3.12) holds for all $\tau \in [0, \infty]$ if and only if φ is increasing.

3.14. Corollary. *In particular, if $\varphi : [0, \infty] \rightarrow [0, \infty]$ is a monotone function and $\psi = j$ where $j(t) = 1/t$, then $j^{-1} = j$ and*

$$(3.15) \quad [j \circ \varphi]^{-1}(\tau) = \varphi^{-1} \circ j(\tau) \quad \forall \tau \in [0, \infty]$$

i.e.,

$$(3.16) \quad \varphi^{-1}(\tau) = \Phi^{-1}(1/\tau) \quad \forall \tau \in [0, \infty]$$

where $\Phi = 1/\varphi$,

$$(3.17) \quad [\varphi \circ j]^{-1}(\tau) \leq j \circ \varphi^{-1}(\tau) \quad \forall \tau \in [0, \infty]$$

i.e., the inverse function of $\varphi(1/t)$ is dominated by $1/\varphi^{-1}$, and except a countable collection of $\tau \in [0, \infty]$

$$(3.18) \quad [\varphi \circ j]^{-1}(\tau) = j \circ \varphi^{-1}(\tau).$$

$1/\varphi^{-1}$ is the inverse function of $\varphi(1/t)$ if and only if the function φ is strictly monotone.

Now, let (X, Σ) and (X', Σ') be measurable spaces and let $g : X \rightarrow X'$ be a measurable transformation, i.e., $S = g^{-1}S' \in \Sigma$ for all $S' \in \Sigma'$. If m is a measure on the σ -algebra Σ , then the measure $m' = mg^{-1}$ on the σ -algebra Σ' is given by the equality

$$(3.19) \quad m'(S') = mg^{-1}(S') = m(g^{-1}(S')) \quad \forall S' \in \Sigma'$$

Below we use the following statement on the change of measure, see Theorem 39C in [Ha]. $\overline{\mathbb{R}}$ denotes the extended real axes $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

3.20. Proposition. *Let $g : X \rightarrow X'$ be a measurable transformation from the measurable space (X, Σ) with a measure m into a measurable space (X', Σ') . Then a function $f : X' \rightarrow \overline{\mathbb{R}}$ is integrable with respect to the measure $m' = mg^{-1}$ if and only if the function $f \circ g$ is integrable with respect to the measure m and, in the case of integrability, the equality*

$$(3.21) \quad \int_{g^{-1}(S')} f(g(x)) \, dm = \int_{S'} f(x') \, dm'$$

holds for every set $S' \in \Sigma'$.

Let (X, Σ) be a measurable space and let m be a measure on the σ -algebra Σ . Below we also use the **measure function** $M(t)$ for $g : X \rightarrow \overline{\mathbb{R}}$

$$(3.22) \quad M(t) = M_g(t) = m(\{x \in X : g(x) > t\})$$

for which the following statement holds, see e.g. [MZ], p. 3.

3.23. Proposition. *Let (X, Σ) be a measurable space and let $g : X \rightarrow [0, \infty]$ be a measurable function with respect to a measure m on Σ . Then*

$$(3.24) \quad \int_X g \, dm = \int_0^{+\infty} M(t) \, dt .$$

By the geometric sense of integral as a consequence of Proposition 3.23 we have the following statement.

3.25. Proposition. *Let (X, Σ) be a measurable space and let $g : X \rightarrow [0, \infty]$ be a measurable function with respect to a measure m on Σ . Then*

$$(3.26) \quad \int_Z g \, dm \leq \int_0^{m(Z)} M^{-1}(\tau) \, d\tau$$

for every measurable set $Z \subset X$ and, moreover,

$$(3.27) \quad \int_X g \, dm = \int_0^{m(X)} M^{-1}(\tau) \, d\tau .$$

Proof. Indeed, fix $\tau_0 \leq M(0)$ and set $t_0 = M^{-1}(\tau_0)$. Note that $M(t_0) = \tau_0$ because $M(t)$ is continuous from the right. Hence among all measurable sets $Z \subseteq X$ with $m(Z) = \tau_0$, the integral from the left hand side in (3.26) attains the greatest possible value on the set

$$Z_0 = \{x \in X : g(x) > t_0\}$$

for which we have

$$M_{Z_0}(t) := m(\{x \in Z_0 : g(x) > t\}) = M(t) \quad \forall t \geq t_0$$

and $M_{Z_0}(t) \equiv m(Z_0) = \tau_0$ for all $t < t_0$. Consider the function $g_0(x) = g(x) - t_0 \geq 0$ given on the set Z_0 . Then

$$M_0(t) := m(\{x \in Z_0 : g_0(x) > t\}) = M_Z(t + t_0) \quad \forall t \geq 0$$

and by Proposition 3.23

$$\int_Z g \, dm \leq \int_{Z_0} g \, dm = t_0 \cdot \tau_0 + \int_0^\infty M_0(t) \, dt = t_0 \cdot \tau_0 + \int_{t_0}^\infty M(t) \, dt .$$

Finally, by the geometric sense of integral from the right hand side, we obtain (3.26). The above arguments imply also the second relation (3.27).

For integrable functions f and $g : [0, 1] \rightarrow [0, \infty]$, it is said that g **majorizes** f or f is **majorized** by g and written $f \prec g$ if

$$(3.28) \quad \int_0^x M_f^{-1}(t) \, dt \leq \int_0^x M_g^{-1}(t) \, dt \quad \forall x \in [0, 1]$$

and

$$(3.29) \quad \int_0^1 M_f^{-1}(t) \, dt = \int_0^1 M_g^{-1}(t) \, dt .$$

Note that M_f^{-1} and M_g^{-1} are non-increasing (equi-measurable) rearrangements of f and g , cf. [HLP₁], [HLP₂] and [MO].

The continuity of functions in the following lemma is understood in the sense of the topology of the extended positive real axes $[0, \infty]$.

3.30. Lemma. *Let $\kappa : [0, \infty] \rightarrow [0, \infty]$ be a continuous non-increasing convex function and $\eta : [0, 1] \rightarrow [0, \infty]$ a continuous non-increasing function. Then the inequality*

$$(3.31) \quad \int_0^1 \eta(t) \, \kappa(f(t)) \, dt \geq \int_0^1 \eta(t) \, \kappa(g(t)) \, dt$$

holds for all integrable functions f and $g : [0, 1] \rightarrow [0, \infty]$ such that g is non-increasing and $f \prec g$.

3.32. Remark. The discrete version of the inequality (3.31) can be found as H.2.b in [MO], p. 92, that is applied below under the proof of Lemma 3.30.

Proof. Without loss of generality we may assume that the left hand side in (3.31) is finite.

1) First consider the case where the functions κ and η are bounded. Let f and $g : [0, 1] \rightarrow [0, \infty]$ be integrable, g non-increasing and $f \prec g$. Set f_n and g_n the functions which are equal to the averages of f and g on the intervals $(k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$, $k = 0, 1, \dots, 2^n - 1$, $n = 1, 2, \dots$, correspondingly. By the well-known Lebesgue theorem on differentiability of indefinite integral, see e.g. IV(6.3) in [Sa], $f_n \rightarrow f$ and $g_n \rightarrow g$ a.e. as $n \rightarrow \infty$. Note that by the construction f_n and g_n are measurable and bounded in view of integrability of f and g and, moreover, $f_n \prec g_n$ for every $n = 1, 2, \dots$. The latter is true because by (3.27) and (3.29)

$$\int_0^1 f_n(t) dt = \int_0^1 f(t) dt = \int_0^1 g(t) dt = \int_0^1 g_n(t) dt$$

and by (3.26) and (3.28)

$$\int_0^{c_{k,n}} f_n(t) dt = \int_0^{c_{k,n}} f(t) dt \leq \int_0^{c_{k,n}} g(t) dt = \int_0^{c_{k,n}} g_n(t) dt$$

for all $c_{k,n} = k2^{-n}$, $k = 1, \dots, 2^n$, and between the points $c_{k,n}$ the indefinite integrals of f_n and g_n are linear functions in the variable x . The discrete variant of the inequality (3.31), see H.2.b in [MO], p. 92, gives

$$\int_0^1 \eta_n(t) \kappa(f_n(t)) dt \geq \int_0^1 \eta_n(t) \kappa(g_n(t)) dt$$

where $\eta_n(t) = \eta((k+1) \cdot 2^{-n})$ on the intervals $(k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$, $k = 0, 1, \dots, 2^n - 1$, $n = 1, 2, \dots$, correspondingly. Note that $\eta_n(t) \leq \eta(t)$ $n = 1, 2, \dots$ and $\eta_n(t) \rightarrow \eta(t)$ a.e. as $n \rightarrow \infty$. Moreover, by the Jensen inequality with weights, see e.g. Theorem 2.6.2 in [Ra], applied on the intervals $(k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$, $k = 0, 1, \dots, 2^n - 1$, $n = 1, 2, \dots$, we obtain that

$$\int_0^1 \eta_n(t) \kappa(f_n(t)) dt \leq \int_0^1 \eta(t) \kappa(f(t)) dt < \infty$$

and hence also

$$\int_0^1 \eta_n(t) \kappa(g_n(t)) dt \leq \int_0^1 \eta(t) \kappa(f(t)) dt < \infty.$$

Thus, by the Fatou lemma, see e.g. [Ha], p. 113,

$$\int_0^1 \eta(t) \kappa(f(t)) dt = \lim_{n \rightarrow \infty} \int_0^1 \eta_n(t) \kappa(f_n(t)) dt$$

and

$$\int_0^1 \eta(t) \kappa(g(t)) dt = \lim_{n \rightarrow \infty} \int_0^1 \eta_n(t) \kappa(g_n(t)) dt$$

and hence (3.31) holds.

2) Now, if either κ or η is not bounded, then consider the functions κ_n or η_n given by the relations

$$\eta_n(t) = \begin{cases} \eta(t), & \text{if } \eta(t) \leq n, \\ n, & \text{if } \eta(t) > n \end{cases}$$

and

$$\kappa_n(t) = \begin{cases} \kappa(u), & \text{if } u \geq u_n, \\ a_n + b_n(u - u_n), & \text{if } u \leq u_n \end{cases}$$

where $u_n = \kappa^{-1}(n)$, $a_n = \kappa(u_n)$ and $b_n = \lim_{u \rightarrow u_n+0} \kappa'(u)$. Note that by the construction η_n is continuous, bounded and non-increasing and $\kappa_n(t)$ is continuous, bounded, convex and non-increasing, see e.g. Proposition 1.4.8 in [Bo]. Thus, by the point 1) of the proof we obtain that

$$(3.33) \quad \int_0^1 \eta_n(t) \kappa_n(f(t)) dt \geq \int_0^1 \eta_n(t) \kappa_n(g(t)) dt$$

for all integrable functions f and $g : [0, 1] \rightarrow [0, \infty]$ such that g is non-increasing and $f \prec g$. Note also that by the construction $\eta_n(t) \leq \eta(t)$ and $\kappa_n(u) \leq \kappa(u)$, $n = 1, 2, \dots$ and $\eta_n(t) \rightarrow \eta(t)$ and $\kappa_n(u) \rightarrow \kappa(u)$ everywhere as $n \rightarrow \infty$. Hence

$$\int_0^1 \eta_n(t) \kappa_n(f(t)) dt \leq \int_0^1 \eta(t) \kappa(f(t)) dt$$

and by the Lebesgue bounded convergence theorem, see e.g. [Ha], p. 110,

$$\lim_{n \rightarrow \infty} \int_0^1 \eta_n(t) \kappa_n(f(t)) dt = \int_0^1 \eta(t) \kappa(f(t)) dt .$$

By (3.33) we also have that

$$\int_0^1 \eta_n(t) \kappa_n(g(t)) dt \leq \int_0^1 \eta(t) \kappa(f(t)) dt < \infty .$$

Hence by the Fatou lemma

$$\lim_{n \rightarrow \infty} \int_0^1 \eta_n(t) \kappa_n(g(t)) dt = \int_0^1 \eta(t) \kappa(g(t)) dt$$

and, finally, by (3.33) we obtain (3.31).

Below we use the notation of the **unit disk**

$$(3.34) \quad \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} .$$

3.35. Lemma. Let $Q : \mathbb{D} \rightarrow [0, \infty]$ be an integrable function and $\varphi : [0, \infty] \rightarrow [0, \infty]$ non-increasing. If

$$(3.36) \quad M(t) := |\{z \in \mathbb{D} : Q(z) > t\}| \leq \varphi(t) \quad \forall t \in [0, \infty],$$

then

$$(3.37) \quad \int_0^1 \frac{dr}{rq(r)} \geq \frac{1}{2} \int_0^\pi \frac{d\tau}{\tau\varphi^{-1}(\tau)}$$

where $q(r)$ is the mean value of $Q(z)$ over the circle $|z| = r$.

Proof. Set $\alpha(0) = 0$ and $\alpha(a) = q\left(\sqrt{\frac{a}{\pi}}\right)$ for $a \in (0, \pi]$. Note that

$$\int_0^{\pi\rho^2} \alpha(a) da = \int_{|z|<\rho} Q(z) dx dy \quad \forall \rho \in (0, 1)$$

Hence by Proposition 3.25

$$\int_0^A \alpha(a) da \leq \int_0^A M^{-1}(\tau) d\tau \quad \forall A \in [0, \pi]$$

and

$$\int_0^\pi \alpha(a) da = \int_0^\pi M^{-1}(\tau) d\tau < \infty,$$

i.e. $\alpha \prec M^{-1}$, and by Lemma 3.30 we have that

$$\int_0^1 \frac{da}{a\alpha(a)} \geq \int_0^\pi \frac{d\tau}{\tau M^{-1}(\tau)}$$

and after the change of variables $a = \pi r^2$ we come to the inequality (3.37) because $M^{-1}(\tau) \leq \varphi^{-1}(\tau)$.

3.38. Theorem. Let $\varphi : [0, \infty] \rightarrow [0, \infty]$ be a non-increasing function such that

$$(3.39) \quad \int_0^\pi \frac{d\tau}{\tau\varphi^{-1}(\tau)} = \infty.$$

Then

$$(3.40) \quad \int_0^1 \frac{dr}{\int_{|z|=r} Q(z)} = \infty,$$

for every integrable function $Q : \mathbb{D} \rightarrow [0, \infty]$ such that

$$(3.41) \quad |\{z \in \mathbb{D} : Q(z) > t\}| \leq \varphi(t) \quad \forall t \in [0, \infty].$$

Proof. The statement follows immediately from Lemma 3.35.

3.42. Remark. Note that

$$(3.43) \quad K_\mu^T(z, z_0) \leq K_\mu(z) \quad \text{a.e.}$$

and, thus, $K_\mu^T(z, z_0)$ is locally integrable in D if $K_\mu(z)$ is so. This is a base for the following applications of Theorem 3.38 to the existence problem for the Beltrami equation below.

4 Existence theorems with measure constraints

The following existence theorem is a direct consequence of Theorems 2.4 and 3.38.

4.1. Theorem. *Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} where*

$$(4.2) \quad |\{z \in U_{z_0} : K_\mu^T(z, z_0) > t\}| \leq \varphi_{z_0}(t) \quad \forall t \in [0, \infty)$$

for a non-increasing function $\varphi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ such that

$$(4.3) \quad \int_0^{\delta(z_0)} \frac{d\tau}{\tau \varphi_{z_0}^{-1}(\tau)} = \infty$$

for some $\delta(z_0) < \varphi_{z_0}(+0)$. Then the Beltrami equation (1.1) has a ring solution.

Note that the additional condition $\delta(z_0) < \varphi_{z_0}(+0)$ is necessary because by the definition $\varphi_{z_0}^{-1}(\tau) = 0$ for all $\tau \geq \varphi_{z_0}(+0)$. Really it is important only degree of convergence $\varphi_{z_0}^{-1}(\tau) \rightarrow \infty$ as $\tau \rightarrow 0$ or, the same, degree of convergence $\varphi_{z_0}(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. Indeed, if $\delta(z_0) < \varphi_{z_0}(+0)$, then $\varphi_{z_0}^{-1}(\delta(z_0)) > 0$ and hence, for every $\delta \in (0, \delta(z_0))$,

$$(4.4) \quad \int_\delta^{\delta(z_0)} \frac{d\tau}{\tau \varphi_{z_0}^{-1}(\tau)} \leq \frac{1}{\varphi_{z_0}^{-1}(\delta(z_0))} \log \frac{\delta(z_0)}{\delta} < \infty.$$

Thus, the condition (4.3) implies that

$$(4.5) \quad \int_0^\delta \frac{d\tau}{\tau \varphi_{z_0}^{-1}(\tau)} = \infty$$

for every $\delta \in (0, \delta(z_0))$. Let

$$(4.6) \quad D(z_0, \rho) = \{z \in D : |z - z_0| \leq \rho\} \subset U_{z_0}$$

such that $\delta = \pi\rho^2 \in (0, \delta(z_0))$ and set $\zeta = (z - z_0)/\rho$, $Q(\zeta) = K_\mu^T(z_0 + \rho\zeta, z_0)$, $\zeta \in \mathbb{D}$. Then

$$(4.7) \quad M(t) := |\{\zeta \in \mathbb{D} : Q(\zeta) > t\}| \leq \varphi(t) \quad \forall t \in [0, \infty]$$

where $\varphi(t) = \varphi_{z_0}(t)/\rho^2$. By Proposition 3.9 $\varphi^{-1}(\tau) = \varphi_{z_0}^{-1}(\rho^2\tau)$, $\varphi_{z_0}^{-1}(\tau) = \varphi^{-1}(\tau/\rho^2)$, and then by (4.5)

$$(4.8) \quad \int_0^\pi \frac{d\tau}{\tau\varphi^{-1}(\tau)} = \infty.$$

Thus, by Theorem 3.38

$$(4.9) \quad \int_0^\rho \frac{dr}{rq_{z_0}^T(r)} = \infty$$

where $q_{z_0}^T(r)$ is the mean value of $K_\mu^T(z, z_0)$ over the circle $|z - z_0| = r$. Finally, by Theorem 2.4 we have the conclusion of Theorem 4.1.

Since $K_\mu^T(z, z_0) \leq K_\mu(z)$ we have, in particular, the following consequence of Theorem 4.1 in terms of the maximal dilatation $K_\mu(z)$.

4.10. Corollary. *Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$. Suppose that*

$$(4.11) \quad |\{z \in D : K_\mu(z) > t\}| \leq \varphi(t) \quad \forall t \in [1, \infty)$$

for a non-increasing function $\varphi : [0, \infty] \rightarrow [0, \infty]$ such that

$$(4.12) \quad \int_0^\delta \frac{d\tau}{\tau\varphi^{-1}(\tau)} = \infty$$

for some $\delta < \varphi(+0)$. Then the Beltrami equation (1.1) has a ring solution.

Here we may assume without loss of generality that $\varphi(t) = \varphi(1+0)$ for all $t \in [0, 1]$ and, thus, $\varphi(+0) = \varphi(1+0)$. Furthermore, we may assume that $\varphi(t) = \infty$ for all $t \in [0, T]$ with any prescribed $T > 1$.

The following statement gives a series of equivalent and sufficient conditions for (4.3) and (4.12) that follows immediately from Proposition 3.9 and Corollary 3.14. Below, in (4.15) and (4.16), we complete the definition of integrals by ∞ if $\varphi(t) = 0$, i.e., $H(t) = \infty$, for all $t \geq T < \infty$.

4.13. Proposition. *Let $\varphi : [0, \infty] \rightarrow [0, \infty]$ be a non-increasing function and set*

$$(4.14) \quad H(t) = \log \frac{1}{\varphi(t)}.$$

Then the equality

$$(4.15) \quad \int_\Delta^\infty H'(t) \frac{dt}{t} = \infty$$

implies the equality

$$(4.16) \quad \int_\Delta^\infty \frac{dH(t)}{t} = \infty$$

and (4.16) is equivalent to

$$(4.17) \quad \int_{\Delta}^{\infty} H(t) \frac{dt}{t^2} = \infty$$

for some $\Delta > 0$, and (4.17) is equivalent to every of the following equalities:

$$(4.18) \quad \int_0^{\delta} H\left(\frac{1}{t}\right) dt = \infty$$

for some $\delta > 0$,

$$(4.19) \quad \int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty$$

for some $\Delta_* > H(+0)$,

$$(4.20) \quad \int_0^{\delta_*} \frac{d\tau}{\tau\varphi^{-1}(\tau)} = \infty$$

for some $\delta_* < \varphi(+0)$.

Moreover, (4.15) is equivalent to (4.16) and hence (4.15)–(4.20) are equivalent each to other if φ is absolutely continuous and non-increasing. In particular, all the conditions (4.15)–(4.20) are equivalent if φ is convex and non-increasing.

Proof. The equality (4.15) implies (4.16) because

$$\int_{\Delta}^{\infty} \frac{dH(t)}{t} \geq \int_{\Delta}^{\infty} H'(t) \frac{dt}{t},$$

see e.g. Theorem 7.4 of Chapter IV in [Sa] p. 119. The equality (4.16) is equivalent to (4.17) by integration by parts, see e.g. Theorem 14.1 of Chapter III in [Sa], p. 102.

Now, (4.17) is equivalent to (4.18) by the change of variables $t \rightarrow 1/t$, and (4.18) is equivalent to (4.19) by the geometric sense of integral and by Corollary 3.14.

By Corollary 3.14 the inverse function of $H(1/t)$ coincides with $1/H^{-1}(t)$ at all points except a countable collection and, thus, by geometric sense of integral (4.19) is equivalent to (4.18).

Further, set $\psi(\xi) = \log(1/\xi)$. Then $H = \psi \circ \varphi$ and by Proposition 3.9 $H^{-1} = \varphi^{-1} \circ \psi^{-1}$, i.e., $H^{-1}(\eta) = \varphi^{-1}(e^{-\eta})$, and by the substitutions $\tau = e^{-\eta}$, $\eta = \log(1/\tau)$ we have the equivalence of (4.19) and (4.20).

Finally, (4.15) and (4.16) are equivalent if φ is absolutely continuous, see e.g. Theorem 7.4 of Chapter IV in [Sa] p. 119.

In particular, Proposition 4.13 makes possible to formulate the following consequences of Theorem 4.1.

4.21. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (4.2) holds at every point $z_0 \in D$ with a non-increasing function $\varphi_{z_0} : [0, \infty) \rightarrow [0, \infty)$ such that

$$(4.22) \quad \int_{\Delta(z_0)}^{\infty} \log \frac{1}{\varphi_{z_0}(t)} \frac{dt}{t^2} = \infty$$

for some $\Delta(z_0) > 0$, then (1.1) has a ring solution.

4.23. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (4.2) holds at every point $z_0 \in D$ for a non-increasing function $\varphi_{z_0} : [0, \infty) \rightarrow [0, \infty)$ such that

$$(4.24) \quad \int_{\Delta(z_0)}^{\infty} \left(\log \frac{1}{\varphi_{z_0}(t)} \right)' \frac{dt}{t} = \infty$$

for some $\Delta(z_0) > 0$, then (1.1) has a ring solution.

The following theorem gives the most general form of criteria in terms of measure for the existence of ACL homeomorphic solutions of the Beltrami equation.

4.25. Theorem. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} where

$$(4.26) \quad |\{z \in U_{z_0} : K_\mu^T(z, z_0) > t\}| \leq \varphi_{z_0}(t) \quad \forall t \in [0, \infty)$$

for a function $\varphi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ such that

$$(4.27) \quad \int_0^{\delta(z_0)} \frac{d\tau}{\tau \psi_{z_0}(\tau)} = \infty$$

for some $\delta(z_0) < \liminf_{t \rightarrow 0} \varphi_{z_0}(t)$ where

$$(4.28) \quad \psi_{z_0}(\tau) = \inf_{\varphi_{z_0}(t) \leq \tau} t.$$

Then the Beltrami equation (1.1) has a ring solution.

Proof. Indeed, the condition (4.26) can be rewritten in terms of the lower envelope of the function φ_{z_0} , see (3.5) for the definition,

$$(4.29) \quad |\{z \in U_{z_0} : K_\mu^T(z, z_0) > t\}| \leq (\varphi_{z_0})_L(t), \quad \forall t \in [0, \infty).$$

Moreover, it is clear that $(\varphi_{z_0})_L^{-1} \leq \psi_{z_0}$ because $(\varphi_{z_0})_L \leq \varphi_{z_0}$ and hence (4.27) implies the condition

$$(4.30) \quad \int_0^{\delta(z_0)} \frac{d\tau}{\tau (\varphi_{z_0})_L^{-1}(\tau)} = \infty$$

for $\delta(z_0) < (\varphi_{z_0})_L(+0)$. Thus, we may apply Theorem 4.1 to $(\varphi_{z_0})_L$.

4.31. Corollary. *Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that*

$$(4.32) \quad |\{z \in D : K_\mu(z) > t\}| \leq \varphi(t) \quad \forall t \in [1, \infty)$$

for a function $\varphi : [0, \infty] \rightarrow [0, \infty]$ such that

$$(4.33) \quad \int_0^\delta \frac{d\tau}{\tau\psi(\tau)} = \infty$$

for some $\delta < \liminf_{t \rightarrow 0} \varphi(t)$ where

$$(4.34) \quad \psi(\tau) = \inf_{\varphi(t) \leq \tau} t.$$

Then the Beltrami equation (1.1) has a ring solution.

4.35. Remark. Note, here we do not assume that the function φ is non-increasing and hence it has no sense to say on the criterion ψ as on the inverse function of φ . However, the ψ -criterion effectively works in the general case.

5 Existence theorems with integral constraints

By (3.2) and (3.16) we have the following consequence of Theorem 4.1.

5.1. Theorem. *Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} where*

$$(5.2) \quad \int_{U_{z_0}} \Phi_{z_0}(K_\mu^T(z, z_0)) \, dx dy < \infty$$

for a non-decreasing function $\Phi_{z_0} : [0, \infty) \rightarrow [0, \infty]$ such that

$$(5.3) \quad \int_{\Delta(z_0)}^\infty \frac{d\tau}{\tau\Phi_{z_0}^{-1}(\tau)} = \infty$$

for some $\Delta(z_0) > \Phi_{z_0}(+0)$. Then the Beltrami equation (1.1) has a ring solution.

Note that the additional condition $\Delta(z_0) > \Phi_{z_0}(+0)$ is necessary because by the definition $\Phi_{z_0}^{-1}(\tau) = 0$ for all $\tau \leq \Phi_{z_0}(+0)$. Really it is important only degree of convergence $\Phi_{z_0}^{-1}(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$ or, the same, degree of convergence $\Phi_{z_0}(t) \rightarrow \infty$ as $t \rightarrow \infty$.

5.4. Corollary. *Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that*

$$(5.5) \quad \int_D \Phi(K_\mu(z)) \, dx dy < \infty$$

for a non-decreasing function $\Phi : [0, \infty) \rightarrow [0, \infty]$ such that

$$(5.6) \quad \int_{\Delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\Delta > \Phi(+0)$. Then the Beltrami equation (1.1) has a ring solution.

Here we may assume without loss of generality that $\Phi(t) = \Phi(1+0) = 0$ for all $t \in [0, 1]$ and, thus, $\Phi(+0) = \Phi(1+0) = 0$. Furthermore, we may assume that $\Phi(t) = 0$ for all $t \in [0, T]$ under any prescribed $T > 1$.

By Proposition 4.13 we obtain a series of conditions which are sufficient and equivalent for (5.3) and (5.6), see also (3.16) in Corollary 3.14. Below, in (5.9) and (5.10), we complete the definition of integrals by ∞ if $\Phi(t) = \infty$, i.e., $H(t) = \infty$, for all $t \geq T < \infty$. More precisely, the following statement holds.

5.7. Proposition. *Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a non-decreasing function and let*

$$(5.8) \quad H(t) = \log \Phi(t) .$$

Then the equality

$$(5.9) \quad \int_{\Delta}^{\infty} H'(t) \frac{dt}{t} = \infty$$

implies the equality

$$(5.10) \quad \int_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty$$

and (5.10) is equivalent to

$$(5.11) \quad \int_{\Delta}^{\infty} H(t) \frac{dt}{t^2} = \infty$$

for some $\Delta > 0$, and (5.11) is equivalent to every of the equalities:

$$(5.12) \quad \int_0^{\delta} H\left(\frac{1}{t}\right) dt = \infty$$

for some $\delta > 0$,

$$(5.13) \quad \int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty$$

for some $\Delta_ > H(+0)$,*

$$(5.14) \quad \int_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\delta_ > \Phi(+0)$.*

Moreover, (5.9) is equivalent to (5.10) and hence (5.9)–(5.14) are equivalent each to other if Φ is absolutely continuous and non-decreasing. In particular, all the conditions (5.9)–(5.14) are equivalent if Φ is convex and non-decreasing.

5.15. Remark. In Theorem 5.1 and its corollaries below, the condition that Φ_{z_0} is non-decreasing can be omitted, however, then in (5.3) and in the relations of type (5.9)–(5.14) it is necessary to use instead of Φ_{z_0} the function

$$(5.16) \quad \Psi_{z_0}(t) = \inf_{\eta \geq t} \Phi_{z_0}(\eta) \leq \Phi_{z_0}(t)$$

because (5.2) implies that

$$(5.17) \quad \int_{U_{z_0}} \Psi_{z_0}(K_\mu^T(z, z_0)) \, dx dy < \infty .$$

5.18. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$. If the condition (5.2) holds at every point $z_0 \in D$ with a non-decreasing function $\Phi_{z_0} : [0, \infty) \rightarrow [0, \infty)$ such that

$$(5.19) \quad \int_{\Delta(z_0)}^{\infty} \log \Phi_{z_0}(t) \frac{dt}{t^2} = \infty$$

for some $\Delta(z_0) > 0$, then (1.1) has a ring solution.

5.20. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$. If the condition (5.2) holds at every point $z_0 \in D$ for a continuous non-decreasing function $\Phi_{z_0} : [0, \infty) \rightarrow [0, \infty)$ such that

$$(5.21) \quad \int_{\Delta(z_0)}^{\infty} (\log \Phi_{z_0}(t))' \frac{dt}{t} = \infty$$

for some $\Delta(z_0) > 0$, then (1.1) has a ring solution.

5.22. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$. If the condition (5.2) holds at every point $z_0 \in D$ for $\Phi_{z_0} = \exp H_{z_0}$ where H_{z_0} is non-constant, non-decreasing and convex, then (1.1) has a ring solution.

5.23. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{loc}^1$. If the condition (5.2) holds at every point $z_0 \in D$ for $\Phi_{z_0} = \exp H_{z_0}$ with a twice continuously differentiable increasing function H_{z_0} such that

$$(5.24) \quad H_{z_0}''(t) \geq 0, \quad t \geq t(z_0),$$

then (1.1) has a ring solution.

5.25. Theorem. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$ such that

$$(5.26) \quad \int_D \Phi(K_\mu(z)) \, dx dy < \infty$$

where $\Phi : [0, \infty) \rightarrow [0, \infty]$ is non-decreasing such that

$$(5.27) \quad \int_\Delta^\infty \log \Phi(t) \frac{dt}{t^2} = \infty$$

for some $\Delta > 0$. Then the Beltrami equation (1.1) has a ring solution f_μ .

5.28. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (5.26) holds with a non-decreasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ such that

$$(5.29) \quad \int_{t_0}^\infty (\log \Phi(t))' \frac{dt}{t} = \infty$$

for some $t_0 > 0$, then (1.1) has a ring solution.

5.30. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If the condition (5.26) holds for $\Phi = e^H$ where H is non-constant, non-decreasing and convex, then (1.1) has a ring solution.

5.31. Corollary. Let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If the condition (5.26) holds for $\Phi = e^H$ where H is twice continuously differentiable, increasing and

$$(5.32) \quad H''(t) \geq 0, \quad t \geq t_0,$$

then (1.1) has a ring solution.

Note that among twice continuously differentiable functions, the condition (5.32) is equivalent to the convexity of $H(t)$, $t \geq t_0$. Of course, the convexity of $H(t)$ implies the convexity of $\Phi(t) = e^{H(t)}$, $t \geq t_0$, because the function $\exp x$ is convex. However, in general, the convexity of Φ does not imply the convexity of $H(t) = \log \Phi(t)$ and it is known that the convexity of $\Phi(t)$ in (5.26) is not sufficient for the existence of ACL homeomorphic solutions of the Beltrami equation. There exist examples of the complex coefficients μ such that $K_\mu \in L^p$ with an arbitrarily large $p \geq 1$ for which the Beltrami equation (1.1) has no ACL homeomorphic solutions, see e.g. [RSY₂].

5.33. Remark. Theorem 2.4 and its consequences can be extended by Theorem 4.23 in [RSY₁] to the case where $\infty \in D \subset \overline{\mathbb{C}}$ in the standard way if one replaces (2.5) by the following condition at ∞

$$(5.34) \quad \int_\delta^\infty \frac{dr}{rq(r)} = \infty$$

where $\delta > 0$ and

$$(5.35) \quad q(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{|1 - e^{-2i\vartheta} \mu(re^{i\vartheta})|^2}{1 - |\mu(re^{i\vartheta})|^2} d\vartheta .$$

In this case, there exists a homeomorphic $W_{loc}^{1,1}$ solution $f = f_\mu$ in D with $f(\infty) = \infty$ and $f_\mu^{-1} \in W_{loc}^{1,2}$. Here $f \in W_{loc}^{1,1}$ in D means that $f \in W_{loc}^{1,1}$ in $D \setminus \{\infty\}$ and that $f^*(z) = 1/\overline{f(1/\bar{z})}$ belongs to $W^{1,1}$ in a neighborhood of 0. The statement $f^{-1} \in W_{loc}^{1,2}$ has a similar meaning.

Similarly, the integral condition (5.2) can be replaced by the following condition at ∞

$$(5.36) \quad \int_{|z|>\delta} \Phi_\infty(K_\mu^T(z, \infty)) \frac{dxdy}{|z|^4} < \infty$$

where $\delta > 0$, Φ_∞ satisfies the conditions of either Theorem 5.1 or equivalent conditions in Proposition 5.7 and

$$(5.37) \quad K_\mu^T(z, \infty) = \frac{|1 - \frac{\bar{z}}{z} \mu(z)|^2}{1 - |\mu(z)|^2} .$$

The measure condition (4.2) is replaced at ∞ by the condition

$$(5.38) \quad S(\{|z| > R : K_\mu^T(z, \infty) > t\}) \leq \varphi_\infty(t) , \quad t \in [t_0, \infty) .$$

Here $S(E)$ for $E \subset \bar{\mathbb{C}}$ notes the spherical area of E , i.e.,

$$(5.39) \quad S(E) = \int_E \frac{dxdy}{(1 + |z|^2)^2} ,$$

and $\varphi_\infty(t)$ satisfies the condition of either Theorem 4.1 or equivalent conditions in Proposition 4.13. For points $z_0 \in D \cap \mathbb{C}$, one may use the spherical area instead of the Euclidean one because the spherical and Euclidean metrics are equivalent on compact subsets in \mathbb{C} .

6 Historic comments and comparisons

The existence problem for degenerate Beltrami equations is currently an active area of research. It has been studied extensively and many contributions have been made, see e.g. [BJ₁]-[BJ₂], [Da], [GMSV], [IM], [Kr], [Le], [MM], [MMV], [MS], [Pe], [Tu], [RSY₁]-[RSY₈] and [Ya], see also the survey [SY]. Many of those and new theorems can be derived from Theorem 2.4 as it was shown above.

The first investigation of the existence problem for degenerate Beltrami equations with integral constraints (5.26) as in Theorem 5.25 has been made by Pesin [Pe] who studied the special case where $\Phi(t) = e^{t^\alpha} - 1$ with $\alpha > 1$. Basically, Corollary 5.31 is due to Kruglikov [Kr]. The first contribution in the existence problem with measure constraints (4.2) as in Theorem 4.1 but with $K_\mu(z)$ instead

of $K_\mu^T(z, z_0)$ is due to David [Da] and Tukia [Tu] who considered the special case where $\varphi_{z_0}(t) \equiv \varphi(t) = \alpha \cdot e^{-\beta t}$.

The next step has been made by Brakalova and Jenkins [BJ₁] who proved the existence of ACL homeomorphic solutions for the case of the integral constraints (5.2) as in Theorem 5.1 with $K_\mu(z)$ instead of $K_\mu^T(z, z_0)$ and with

$$(6.1) \quad \Phi_{z_0}(t) \equiv \Phi(t) = \exp \left(\frac{\frac{t+1}{2}}{1 + \log \frac{t+1}{2}} \right).$$

Note that, in the case [BJ₁], the condition (5.9) in Proposition 5.7 can be easily verified by the calculations

$$(6.2) \quad (\log \Phi(t))' = \frac{1}{2} \frac{\log \frac{t+1}{2}}{(1 + \log \frac{t+1}{2})^2} \sim \frac{1}{2} \frac{1}{\log t}.$$

Later on Iwaniec and Martin [IM] proved the existence of solutions in the Orlicz–Sobolev classes in the case where

$$(6.3) \quad \Phi_{z_0}(t) \equiv \Phi(t) = \exp \left(\frac{pt}{1 + \log t} \right)$$

for some $p > 0$.

Corollary 5.18 is due to Gutlyanskii, Martio, Sugawa and Vuorinen in [GMSV] where they have established the existence of ACL homeomorphic solutions of (1.1) under $K_\mu \in L_{loc}^p$ with $p > 1$ for

$$(6.4) \quad \Phi_{z_0}(t) \equiv \Phi(t) := \exp H(t)$$

with $H(t)$ being a continuous increasing function such that $\Phi(t)$ is convex and

$$(6.5) \quad \int_{\delta}^{\infty} H(t) \frac{dt}{t^2} = \infty$$

for some $\delta > 0$.

Recently Brakalova and Jenkins have proved the existence of ACL homeomorphic solutions under (5.2) again with $K_\mu(z)$ instead of $K_\mu^T(z, z_0)$ and with

$$(6.6) \quad \Phi_{z_0}(t) \equiv \Phi(t) = \exp h \left(\frac{t+1}{2} \right)$$

where they assumed that h is increasing and convex and $h(x) \geq C_\lambda x^\lambda$ for any $\lambda > 1$ with some $C_\lambda > 0$ and

$$(6.7) \quad \int_{\Delta(z_0)}^{\infty} \frac{d\tau}{\tau h^{-1}(\tau)} = \infty.$$

Note, the condition $h(x) \geq C_\lambda x^\lambda$ for any $\lambda > 1$ as well as the convexity and sub-exponential conditions imply, in particular, that K_μ is locally integrable with any degree $p \in [1, \infty)$.

Many of the given conditions are not necessary as it is clear from Theorem 5.1, Proposition 5.7 and the following lemma.

6.8. Lemma. *There exist continuous increasing convex functions $\Phi : [1, \infty) \rightarrow [1, \infty)$ such that*

$$(6.9) \quad \int_1^{\infty} \log \Phi(t) \frac{dt}{t^2} = \infty ,$$

$$(6.10) \quad \liminf_{t \rightarrow \infty} \frac{\log \Phi(t)}{\log t} = 1$$

and, in addition,

$$(6.11) \quad \Phi(t) \geq t \quad \forall t \in [1, \infty) .$$

Moreover, there exist non-decreasing functions $\Phi : [1, \infty) \rightarrow [1, \infty)$ with the properties (6.9)–(6.10) which are neither continuous, nor strictly monotone and nor convex in any neighborhood of ∞ . Furthermore, $\Phi(t) = \alpha_{n+1}$ for all $t \in (\gamma_n, \alpha_{n+1}]$ and has a jump at γ_n where $\gamma_n < \alpha_{n+1} < \gamma_{n+1}$, $n = 1, 2, \dots$, and $\gamma_n \rightarrow \infty$ as $n \rightarrow \infty$.

6.12. Remark. The condition (6.10) implies, in particular, that there exist non $\lambda > 1, C > 0$ and $T \in (1, \infty)$ such that

$$(6.13) \quad \Phi(t) \geq C t^\lambda \quad \forall t \geq T .$$

In addition, for the example of Φ given in the proof of Lemma 6.8,

$$(6.14) \quad \limsup_{t \rightarrow \infty} \frac{\log \Phi(t)}{\log t} = \infty .$$

Proof of Lemma 6.8. Below we use the known criterion which says that a function Φ is convex on an open interval I if and only if Φ is continuous and its derivative Φ' exists and is non-decreasing in I except a countable set of points in I , see e.g. Proposition 1.4.8 in [Bo]. We construct Φ by induction sewing together pairs of functions of the two types $\varphi(t) = \alpha + \beta t$ and $\psi(t) = a e^{bt}$ with suitable positive parameters a, b and β and possibly negative α .

More precisely, set $\Phi(t) = \varphi_1(t)$ for $t \in [1, \gamma_1^*]$ and $\Phi(t) = \psi_1(t)$ for $t \in [\gamma_1^*, \gamma_1]$ where $\varphi_1(t) = t$, $\gamma_1^* = e$, $\psi_1(t) = e^{-(e-1)t}$, $\gamma_1 = e + 1$. Let us assume that we already constructed $\Phi(t)$ on the segment $[1, \gamma_n]$ and hence that $\Phi(t) = a_n e^{b_n t}$ on the last subsegment $[\gamma_n^*, \gamma_n]$ of the segment $[\gamma_{n-1}, \gamma_n]$. Then we set $\varphi_{n+1}(t) = \alpha_{n+1} + \beta_{n+1} t$ where the parameters α_{n+1} and β_{n+1} are found from the conditions $\varphi_{n+1}(\gamma_n) = \Phi(\gamma_n)$ and $\varphi'_{n+1}(\gamma_n) \geq \Phi'(\gamma_n - 0)$, i.e., $\alpha_{n+1} + \beta_{n+1} \gamma_n = a_n e^{b_n \gamma_n}$ and $\beta_{n+1} \geq a_n b_n e^{b_n \gamma_n}$. Let $\beta_{n+1} = a_n b_n e^{b_n \gamma_n}$ and $\alpha_{n+1} = a_n e^{b_n \gamma_n} (1 - b_n \gamma_n)$, choose a large enough $\gamma_{n+1}^* > \gamma_n$ from the condition

$$(6.15) \quad \log (\alpha_{n+1} + \beta_{n+1} \gamma_{n+1}^*) \leq \left(1 + \frac{1}{n}\right) \log \gamma_{n+1}^*$$

and, finally, set $\Phi(t) \equiv \varphi_{n+1}(t)$ on $[\gamma_n, \gamma_{n+1}^*]$.

Next, we set $\psi_{n+1}(t) = a_{n+1}e^{b_{n+1}t}$ where parameters a_{n+1} and b_{n+1} are found from the conditions that $\psi_{n+1}(\gamma_{n+1}^*) = \varphi_{n+1}(\gamma_{n+1}^*)$ and $\psi'_{n+1}(\gamma_{n+1}^*) \geq \varphi'_{n+1}(\gamma_{n+1}^*)$, i.e.,

$$(6.16) \quad b_{n+1} = \frac{1}{\gamma_{n+1}^*} \log \frac{\alpha_{n+1} + \beta_{n+1}\gamma_{n+1}^*}{a_{n+1}}$$

and, taking into account (6.16),

$$(6.17) \quad b_{n+1} \geq \frac{\beta_{n+1}}{\alpha_{n+1} + \beta_{n+1}\gamma_{n+1}^*}.$$

Note that (6.17) holds if we take small enough $a_{n+1} > 0$ in (6.16). In addition, we may choose here $b_{n+1} > 1$.

Now, let us choose a large enough γ_{n+1} with $e^{-1}\gamma_{n+1} \geq \gamma_{n+1}^*$ from the condition that

$$(6.18) \quad \log \psi_{n+1}(e^{-1}\gamma_{n+1}) \geq e^{-1}\gamma_{n+1},$$

i.e.,

$$(6.19) \quad \log a_{n+1} + b_{n+1}e^{-1}\gamma_{n+1} \geq e^{-1}\gamma_{n+1}.$$

Note that (6.19) holds for all large enough γ_{n+1} because $b_{n+1} > 1$ although $\log a_{n+1}$ can be negative.

Setting $\Phi(t) = \psi_{n+1}(t)$ on the segment $[\gamma_{n+1}^*, \gamma_{n+1}]$, we have that

$$(6.20) \quad \log \Phi(t) \geq t \quad \forall t \in [e^{-1}\gamma_{n+1}, \gamma_{n+1}]$$

where the subsegment $[e^{-1}\gamma_{n+1}, \gamma_{n+1}] \subseteq [\gamma_{n+1}^*, \gamma_{n+1}]$ has the logarithmic length 1.

Thus, (6.11) holds because by the construction $\Phi(t)$ is absolutely continuous, $\Phi(1) = 1$ and $\Phi'(t) \geq 1$ for all $t \in [1, \infty)$; the equality (6.9) holds by (6.20); (6.10) by (6.11) and (6.15); (6.14) by (6.20).

Finally, the corresponding example of a non-decreasing function Φ which is neither continuous, nor strictly monotone and nor convex in any neighborhood of ∞ is obtained in the above construction if we take $\beta_{n+1} = 0$ and $\alpha_{n+1} > \gamma_n$ such that $\alpha_{n+1} > \Phi(\gamma_n)$ and $\Phi(t) = \alpha_{n+1}$ for all $t \in (\gamma_n, \gamma_{n+1}^*]$, $\gamma_{n+1}^* = \alpha_{n+1}$.

6.21. Proposition. *Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a measurable function such that*

$$(6.22) \quad \int_{\delta}^{\infty} \log \Phi(t) \frac{dt}{t^2} = \infty$$

for some $\delta > 0$. Then

$$(6.23) \quad \limsup_{t \rightarrow \infty} \frac{\Phi(t)}{t^\lambda} = \infty \quad \forall \lambda \in \mathbb{R}.$$

6.24. Remark. In particular, (6.23) itself implies the relation (6.14). Indeed, we have from (6.23) that there exists a monotone sequence $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$(6.25) \quad \Phi(t_n) \geq t_n^n, \quad n = 1, 2, \dots,$$

i.e.,

$$(6.26) \quad \frac{\log \Phi(t_n)}{\log t_n} \geq n, \quad n = 1, 2, \dots$$

Proof of Proposition 6.21. It is sufficient to consider the case $\lambda > 0$. Set $H(t) = \log \Phi(t)$, i.e., $\Phi(t) = e^{H(t)}$. Note that $e^x \geq x^n/n!$ for all $x \geq 0$ and $n = 1, 2, \dots$, because $e^x = \sum_{n=0}^{\infty} x^n/n!$. Fix $\lambda > 0$ and $n > \lambda$. Then $q := \lambda/n$ belongs to $(0, 1)$ and

$$\frac{H(t)}{t^q} \leq \left(\frac{\Phi(t)}{t^\lambda} \right)^{\frac{1}{n}} \cdot \sqrt[n]{n!}.$$

Let us assume that

$$(6.27) \quad C := \limsup_{t \rightarrow \infty} \frac{\Phi(t)}{t^\lambda} < \infty.$$

Then

$$\begin{aligned} \int_{\Delta}^{\infty} H(t) \frac{dt}{t^2} &< 2 \sqrt[n]{Cn!} \int_{\Delta}^{\infty} \frac{dt}{t^{2-q}} = -\frac{2}{1-q} \frac{\sqrt[n]{Cn!}}{t^{1-q}} \Big|_{\Delta}^{\infty} = \\ &= \frac{2}{1-q} \frac{\sqrt[n]{Cn!}}{\Delta^{1-q}} < \infty \end{aligned}$$

for large enough $\Delta > \delta > 0$. The latter contradicts (6.22). Hence the assumption (6.27) was not true and, thus, (6.23) holds for all $\lambda \in \mathbb{R}$.

6.28. Remark. Lemma 6.8 shows that, generally speaking, \limsup in (6.23) cannot be changed by \lim for an arbitrary $\lambda > 1$ under the condition (6.22) even if Φ is continuous, increasing and convex.

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