

**DISTORTION OF DIMENSION UNDER QUASICONFORMAL
MAPPINGS**

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Academic dissertation

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István Prause

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This doctoral dissertation consists of this summary and the following three articles:

- [A] *A remark on quasiconformal dimension distortion on the line*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 341–352.
- [B] *A note on distortion of Hausdorff measures under quasiconformal mappings*, Preprint 457, Reports in Mathematics, University of Helsinki, 2007.
- [C] *Flatness properties of quasispheeres*, Preprint 458, Reports in Mathematics, University of Helsinki, 2007.

1. INTRODUCTION

Quasiconformal mappings are generalizations of conformal mappings. They constitute a standard tool in a number of areas of complex analysis such as Teichmüller theory, Kleinian groups and complex dynamics. They also appear in various contexts in other parts of mathematics, including connections to elliptic partial differential equations, differential geometry and calculus of variations. As for their role in geometric function theory we refer to [18].

Quasiconformal maps in the plane were introduced by Grötzsch in 1928 and their importance in complex analysis was soon realized by Ahlfors and Teichmüller [1]. Higher dimensional quasiconformal mappings were already considered by Lavrentiev in the 1930's, while their systematic study began with the work of Gehring and Väisälä in the 1960's. Then in the late 1960's, Reshetnyak and the Finnish school, Martio, Rickman and Väisälä initiated the theory of quasiregular mappings, the non-injective counterpart of quasiconformal mappings. This framework offers an extension of complex analysis to \mathbb{R}^n from the viewpoint of real analysis. Recent developments include extension of quasiconformal analysis to general metric measure space setting [14] and the theory of mappings of finite distortion [17]. We refer to the survey of Gehring [12] for an overview of the topic.

Basic pointwise distortion results were established at an early stage of the theory. Much harder is to find precise bounds how quasiconformal maps distort dimension. A complete solution is known only in the plane. In this thesis we are concerned with some aspects of distortion of Hausdorff dimension under quasiconformal mappings both in the two-dimensional and higher dimensional Euclidean setting.

1.1. An example. Quasiconformal mappings constitute a class interpolating between bilipschitz maps and homeomorphisms. Most of the questions we consider

are straightforward for the bilipschitz class; bilipschitz maps preserve dimension and rectifiability. Different phenomena occur in the quasiconformal setting, since quasiconformal curves need not be rectifiable, and moreover, they can have Hausdorff dimension bigger than one. It is a classical fact that both bilipschitz and quasiconformal mappings are differentiable almost everywhere. It is the different nature of singularities at this exceptional set of measure zero that brings out the difference between quasiconformal mappings and bilipschitz mappings. The standard *von Koch snowflake* curve serves as an illustration. It has Hausdorff dimension $\log 4/\log 3$ while being a quasiconformal image of the unit segment. For more examples of quasiconformal circles or spheres, see for instance [27].

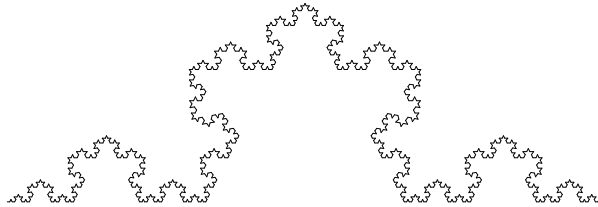


FIGURE 1. The snowflake, a quasiconformal curve

The snowflake is *wiggly* in the following sense: it oscillates around every point and at every scale. Quantitative versions of this property have been studied in [6, 24]. Wiggly or thick sets arise naturally in many parts of analysis, e.g. in connection with Kleinian groups, harmonic measure or bilipschitz extensions. Observe that if we replace the angle of 60 degrees in the snowflake construction by an angle close to 180 degrees then the oscillation becomes very small and the curve will also satisfy an opposite property, a uniform flatness condition, see Section 4.3 for details. Higher dimensional analogous “snowballs” have been constructed in [8].

2. QUASICONFORMAL MAPS AND HAUSDORFF DIMENSION

2.1. Quasiconformal mappings. According to the *analytic definition* a (sense preserving) homeomorphism $f: \Omega \rightarrow \Omega'$ between domains in \mathbb{R}^n , $n \geq 2$, is called *quasiconformal* if $f \in W_{\text{loc}}^{1,n}(\Omega)$ and there exists $1 \leq K < \infty$ such that

$$(2.2) \quad \max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi| \quad \text{a.e. } x \in \Omega.$$

Quantifying this we speak of K -quasiconformal mappings if (2.2) holds. If $K = 1$ we recover conformal maps. According to Liouville’s rigidity theorem it is crucial to allow the dilatation $K > 1$ in order to get an interesting theory in higher dimensions. We refer to [23] for other equivalent definitions and for foundations of quasiconformal mappings. See also [17, 25] for different approaches.

Condition (2.2) expresses that balls are distorted in a uniform manner on the infinitesimal scale. Eventually, this property also leads to global distortion estimates. The following definition from [22] captures a similar phenomenon globally.

2.3. Quasisymmetric maps. Let $\eta: [0, \infty) \rightarrow [0, \infty)$ be an increasing homeomorphism. A homeomorphism $f: X \rightarrow Y$ between metric spaces is η -quasisymmetric if

$$(2.4) \quad \frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq \eta \left(\frac{|a - x|}{|b - x|} \right),$$

for all $a, b, x \in X$ ($b \neq x$). The mapping f is called quasisymmetric if it is η -quasisymmetric with some function η .

Quasisymmetric maps (between domains in \mathbb{R}^n) are always quasiconformal. In the other direction, quasiconformal maps satisfy the quasisymmetry condition semi-globally, in particular, a K -quasiconformal map of the whole space $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \geq 2$, is $\eta_{K,n}$ -quasisymmetric.

In many ways quasiconformal maps interpolate between bilipschitz maps and homeomorphisms. We will see how this is reflected in the way these maps distort Hausdorff dimension.

2.5. Hausdorff dimension. Let $\delta: [0, \infty) \rightarrow [0, \infty)$ be a continuous non-decreasing function with $\delta(0) = 0$. We call δ a measure function and define the Hausdorff δ -measure for a set E as

$$\mathcal{H}^\delta(E) = \liminf_{\varepsilon \rightarrow 0} \sum \delta(\text{diam}(E_i)),$$

where the infimum is taken over all countable coverings of E by sets E_i with $\text{diam}(E_i) < \varepsilon$. If we set $\delta(r) = r^t$ for some $t \in (0, \infty)$, then we obtain the t -dimensional Hausdorff measure and denote it simply by \mathcal{H}^t . The Hausdorff dimension of E is given by

$$\dim E = \inf \{t: \mathcal{H}^t(E) = 0\}.$$

Hausdorff measures and dimension provide a general way to measure metric size; for further details see [19]. The term dimension always refers to Hausdorff dimension in this thesis.

2.6. Higher integrability. It is well known that K -quasiconformal maps are locally Hölder continuous with exponent $1/K$, see [9]. The sharpness of the exponent is seen by considering the radial stretching of the form $f(x) = x|x|^{\frac{1}{K}-1}$. In fact, this example is believed to be extremal for many problems, providing maximal expansion at a point. A remarkable result of Bojarski [7] ($n = 2$) and Gehring [10] ($n \geq 3$) is the higher integrability phenomenon: a K -quasiconformal map f has higher Sobolev regularity than the natural exponent n , that is $f \in W_{\text{loc}}^{1,p}$ for every $p < p_0$ where $p_0 = p_0(K, n) > n$. It is an important problem to identify the precise exponent $p_0(K, n)$.

2.7. Conjecture (Higher integrability conjecture (Gehring)). We may take

$$p_0(K, n) = \frac{nK}{K-1}.$$

Note that the above value of p_0 and the Hölder exponent $1/K$ are related via the Sobolev embedding theorem. This conjecture has been proved in the case $n = 2$ by Astala; for further details see the next section.

Hölder continuity implies that sets of *zero* dimension are preserved, while sets of dimension n are preserved because of the higher integrability phenomenon. However, in general, quasiconformal maps can change the Hausdorff dimension, see [13]. Bishop [5] showed that the dimension of any compact set of positive dimension can, in fact, be raised arbitrarily close to n by a quasiconformal homeomorphism of \mathbb{R}^n .

We are interested in bounds in terms of the dilatation K . Let us note that the Higher integrability conjecture would imply the following (see [13, 15]).

2.8. Conjecture. Let $f: \Omega \rightarrow \Omega'$ be K -quasiconformal in \mathbb{R}^n and suppose $E \subset \Omega$ is compact. Then

$$(2.9) \quad \frac{1}{K} \left(\frac{1}{\dim(E)} - \frac{1}{n} \right) \leq \frac{1}{\dim(f(E))} - \frac{1}{n} \leq K \left(\frac{1}{\dim(E)} - \frac{1}{n} \right).$$

Examples built on Cantor sets via iterations of radial stretchings show that we can have equality on either side.

3. AREA DISTORTION

In this section we confine ourselves to the theory of planar quasiconformal mappings in which case one has an essentially complete understanding of the regularity issues discussed above, due to the work of Astala [2].

In the two dimensional situation there is a strong interaction with elliptic PDE's because of the connection to the *Beltrami equation*

$$(3.1) \quad \bar{\partial}f(z) = \mu(z)\partial f(z) \quad \text{a.e. } z \in \Omega,$$

which is equivalent to (2.2) if we require $\|\mu\|_\infty \leq (K-1)/(K+1) < 1$. One of the cornerstones of the theory is the measurable Riemann mapping theorem which asserts that (3.1) has always (an essentially unique) homeomorphic solution when $\|\mu\|_\infty < 1$.

As we remarked earlier the Higher integrability conjecture has been solved in the plane by Astala. Higher integrability is closely connected with metric distortion properties of quasiconformal maps, and in fact Astala proved the optimal regularity via establishing the Gehring-Reich conjecture on area distortion of quasiconformal maps. Let us record these results.

3.2. Theorem (Area distortion [2]). *Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be a K -quasiconformal mapping in the unit disk $\mathbb{D} \subset \mathbb{C}$ with $f(0) = 0$. Then we have*

$$|fE| \leq C(K)|E|^{1/K},$$

for all Borel measurable sets $E \subset \mathbb{D}$.

3.3. Theorem (Higher integrability [2]). *Let $f: \Omega \rightarrow \Omega'$ be K -quasiconformal in \mathbb{C} . Then*

$$f \in W_{\text{loc}}^{1,p}(\Omega) \quad \text{for all } p < \frac{2K}{K-1}.$$

Higher integrability also controls the change of Hausdorff dimension, thus confirming Conjecture 2.8 for $n = 2$.

3.4. Theorem (Dimension distortion [2]). *Let $f: \Omega \rightarrow \Omega'$ be K -quasiconformal in \mathbb{C} and suppose $E \subset \Omega$ is compact. Then*

$$(3.5) \quad \frac{1}{K} \left(\frac{1}{\dim(E)} - \frac{1}{2} \right) \leq \frac{1}{\dim(f(E))} - \frac{1}{2} \leq K \left(\frac{1}{\dim(E)} - \frac{1}{2} \right).$$

This inequality is best possible.

The previous theorem gives a complete description of dimension distortion under planar quasiconformal mappings. We shall be concerned with two related issues which remain unsettled: (A) *improved distortion* on the line, and (B) *distortion of Hausdorff measures*.

Let us first discuss (B). It is natural to ask, see [2, 3], whether the estimates of (3.5) hold on the level of Hausdorff measures \mathcal{H}^t . That is, if f is a planar K -quasiconformal mapping, $0 < t < 2$ and $d = \frac{2Kt}{2+(K-1)t}$, is it true that

$$(3.6) \quad \mathcal{H}^t(E) = 0 \Rightarrow \mathcal{H}^d(f(E)) = 0?$$

In other words, do we have absolute continuity $f^*\mathcal{H}^d \ll \mathcal{H}^t$? It is classical that quasiconformal mappings are absolutely continuous with respect to the Lebesgue measure, and the Area distortion theorem proves this in a quantitatively optimal form. Very recently, the authors of [3] confirmed (3.6) in the case $d = 1$ and obtained partial results when $d > 1$.

3.7. Dimension of quasicircles. In this paragraph we discuss phenomenon (A). We call a Jordan curve a K -*quasicircle* if it is the image of the unit circle under a global K -quasiconformal map of the plane \mathbb{C} . Quasicircles and domains they bound (quasidisks) have been proved to possess many important function theoretic properties [11]. Here we concentrate on the question on their Hausdorff dimension, and for convenience we fix the notation $k = (K - 1)/(K + 1)$.

From the inequalities (3.5) we see that one can map a 1-dimensional set to a set of $1 + k$ dimension (or $1 - k$ resp.) under a K -quasiconformal map and these bounds are optimal. However, the extremal distortion is achieved for sets of highly irregular character and one can expect better estimates to hold for subsets of rectifiable curves, or more concretely for subsets of the real line. In fact, Becker and Pommerenke showed that the correct asymptotic behavior of the dimension for quasicircles is quadratic in k as $K \rightarrow 1$.

3.8. Theorem ([4]). *For every K -quasicircle Γ , we have*

$$\dim \Gamma \leq 1 + 37k^2.$$

Conversely, for every $K \geq 1$, there exists a K -quasicircle with dimension at least $1 + 0.09k^2$.

Later S. Smirnov improved this to the following.

3.9. Theorem (Smirnov (2000, unpublished)). *For every K -quasicircle Γ , we have*

$$\dim \Gamma \leq 1 + k^2.$$

It would be of particular interest to know whether this estimate is sharp. To date, lower bounds are relatively far from the conjectured value of $1 + k^2$.

3.10. Higher dimensions. Conjectures 2.7 and 2.8 remain widely open in higher dimensions, $n \geq 3$. The solution in the planar case by Astala is largely based on the theory of holomorphic motions. As these planar methods do not carry over to higher dimensions one inevitably needs to find other approaches. See [16, 17] for developments in this direction.

Somewhat similar remarks apply to the arguments in Theorems 3.8 and 3.9, they are analytical and not applicable in higher dimensions. Mattila and Vuorinen in [20] studied related problems from a more geometric point of view and obtained qualitatively the same estimates as in Theorem 3.8. Their idea is to show that quasicircles are flat in a weak sense and this in turn implies a bound on their dimension. For precise definitions, see Subsection 4.3. The advantage is that this approach generalizes to higher dimensions, that is, we can e.g. study the dimension of quasispheres (quasiconformal images of a sphere). The drawback is that one cannot obtain sharp results this way, but nevertheless, it suffices to analyze the asymptotics as $K \rightarrow 1$. *Flatness properties* of quasispheres constitute question (C) of our study.

4. MAIN RESULTS

The papers [A], [B] and [C] contribute to the issues (A), (B) and (C) mentioned above, respectively. We describe the main results in the next three subsections.

4.1. Improved distortion

As we discussed above one expects improved dimension distortion bounds to hold for subsets of the line. The next theorem expresses this in a special case. Recall that $k = (K - 1)/(K + 1)$.

4.1. Theorem ([A, 1.6]). *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal map with $0 < k < 1/\sqrt{8}$ and $E \subset \mathbb{R}$. Then $\dim fE < 1$ provided that $\dim E \leq 1 - 8k^2$. Conversely, if $\dim E = 1$ then $\dim fE > 1 - 8k^2$.*

In view of Stoilow factorization, quasiconformal distortion results have immediate applications to quasiregular removability questions. In fact, the result above (with unspecified constants in place of 8) is due to [3], where the authors studied this problem in connection with their improved version of Painlevé removability for quasiregular mappings. Our approach relies on the area distortion argument from [2] and the quasicircle dimension estimate of Theorem 3.9.

Under the additional assumption that f fixes the real line, we obtain a refined estimate, a dual result to Theorem 3.9. The relevance of the refinement is that it could very well be sharp.

4.2. Theorem ([A, 3.1], Smirnov (unpublished)). *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal map for which $f(\mathbb{R}) = \mathbb{R}$. Then for a 1-dimensional set $E \subset \mathbb{R}$,*

$$\dim fE \geq 1 - k^2.$$

4.2. Distortion of Hausdorff measures

The objective of [B] is to point out that the methods of [2] allow to establish Theorem 3.4 in a slightly stronger form, that is, to show absolute continuity as in (3.6) with respect to some weaker Hausdorff measures.

We consider measure functions $\delta(r) = r^d \varepsilon(r)$ satisfying

$$(4.3) \quad \int_0^\infty \varepsilon(r)^{\frac{Kt}{Kt-d}} \frac{dr}{r} < \infty.$$

We also make the technical assumption that the integrand is decreasing and $\varepsilon(r)$ is increasing in $(0, r_0)$ for some $r_0 > 0$. For instance, we can take $\varepsilon(r) = |\log r|^{-s}$ with $s > 1 - \frac{d}{Kt}$, so that \mathcal{H}^δ has the right dimension d .

4.4. Theorem ([B, 1.9]). *Let $E \subset \mathbb{D}$ be a compact set and let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a K -quasiconformal mapping conformal outside \mathbb{D} , normalized by $f(z) = z + O(1/|z|)$ as $z \rightarrow \infty$. Let $t \in (0, 2)$ and $d = \frac{2Kt}{2+(K-1)t}$. Then we have*

$$\mathcal{H}^\delta(f(E)) \leq C (\mathcal{H}^t(E))^{\frac{d}{Kt}},$$

where the measure function δ satisfies (4.3). The constant C depends only on δ and K .

This is a complementary result to [3, Corollary 2.12] which proves the same result under the assumption that $d > 1$ and $\delta(r) = r^d \varepsilon(r)$ is such that

$$(4.5) \quad \int_0^\infty \varepsilon(r)^{\frac{1}{d-1}} \frac{dr}{r} < \infty.$$

The two results complement each other in the following way: [3, Corollary 2.12] gets sharper as $d \rightarrow 1$, while Theorem 4.4 improves as $K \rightarrow 1$.

4.3. Flatness properties of quasispheres

Although quasispheres need not be rectifiable, they become more and more flat as $K \rightarrow 1$. This flatness property appears uniformly at all scales and locations. We shall work with the following definition due to Mattila and Vuorinen [20].

4.6. LAP property. Let $0 \leq \delta < 1$. We say that a closed set E in \mathbb{R}^n satisfies the d -dimensional δ -linear approximation property (δ -LAP) if there is an $r_0 > 0$ such that for each $x \in E$ and for each $0 < r < r_0$ there exists a d -dimensional affine subspace V through x such that

$$E \cap B^n(x, r) \subset V(\delta r).$$

Here $V(r)$ denotes the r -neighborhood of V ; $V(r) = \{x : d(x, V) < r\}$.

The authors of [20] showed that K -quasispheres satisfy the $(n - 1)$ -dimensional δ -LAP property with $\delta = \delta(K) \rightarrow 0$ as K tends to 1. In [C] we study this and related properties, and in particular, we show the following sharp estimate in terms of the quasisymmetry function η in (2.4).

4.7. Theorem ([C, 5.1]). *Let $1 < K < K_0$ and let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasi-conformal homeomorphism of \mathbb{R}^n . Then the image of a hyperplane H satisfies the $(n - 1)$ -dimensional δ -LAP property with $\delta = \delta(K) = O(\eta_{K,n}(1) - 1)$.*

LAP property implies the following bound on the dimension.

4.8. Theorem ([20]). *There is a positive number δ_0 depending only on d and n such that if a set $E \subset \mathbb{R}^n$ has the d -dimensional δ -LAP property and $0 < \delta < \delta_0$, then*

$$\dim E \leq d + c(d)\delta^2.$$

Combining the two previous theorems and the best-known bounds for $\eta_{K,n}$ [26, 21], we obtain the following.

4.9. Corollary ([C, 5.4]). *For a K -quasisphere E in \mathbb{R}^n with $1 < K < K_0$ we have*

$$\dim E = n - 1 + O((\eta_{K,n}(1) - 1)^2) = n - 1 + O\left((K - 1)^2 \log^2 \frac{1}{K - 1}\right).$$

This result can be considered satisfactory except for the logarithmic term involved, see Questions [27, 1.41 and 1.42]. Nevertheless, it reveals that we have a phenomenon in higher dimensions similar to that of the plane: K -quasispheres have much smaller dimension than K -quasiconformal images of general $(n - 1)$ -dimensional sets (in the case $K \rightarrow 1$). Similar results hold for quasiconformal images of lower dimensional subspaces [C, 5.6 and 5.7].

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A REMARK ON QUASICONFORMAL DIMENSION DISTORTION ON THE LINE

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Abstract. The general dimension distortion result of Astala says that a one dimensional set goes to a set of dimension at least $1 - k$ under a k -quasiconformal mapping. An improved version for rectifiable sets appears in recent work of Astala, Clop, Mateu, Orobitg and Uriarte-Tuero in connection with quasiregular removability problems. We give an alternative proof of their result establishing a bound of the form $1 - ck^2$, provided that either the initial or the target set lies on a straight line. The bound $1 - k^2$ holds under the additional assumption that the line stays fixed.

1. Introduction

A homeomorphism $f: \Omega \rightarrow \Omega'$ between planar domains is called k -quasiconformal if it lies in the Sobolev class $W_{\text{loc}}^{1,2}(\Omega)$ and satisfies the Beltrami equation

$$\bar{\partial}f(z) = \mu(z)\partial f(z) \quad \text{a.e. } z \in \Omega,$$

with a measurable coefficient $\|\mu\|_{\infty} \leq k < 1$.

1.1. Remark. Most commonly, such a map is called a K -quasiconformal map in the literature, with $K = \frac{1+k}{1-k}$. However, we shall work with the definition above, since the L^{∞} -norm of the Beltrami coefficient has a natural role in connection with holomorphic motions. We make an exception in this introductory part and use the traditional form in Corollaries 1.8 and 1.9. In any case, the reader should think of both dilatations, $0 \leq k < 1$ and $K \geq 1$, simultaneously.

Astala [A1], in his seminal paper, gave a complete description of dimension distortion of general sets under planar quasiconformal mappings. Here and in the sequel, dimension always refers to Hausdorff dimension.

1.2. Theorem. ([A1]) *Let $f: \Omega \rightarrow \Omega'$ be k -quasiconformal and suppose $E \subset \Omega$ is compact. Then*

$$(1.3) \quad \frac{1-k}{1+k} \left(\frac{1}{\dim(E)} - \frac{1}{2} \right) \leq \frac{1}{\dim(f(E))} - \frac{1}{2} \leq \frac{1+k}{1-k} \left(\frac{1}{\dim(E)} - \frac{1}{2} \right).$$

This inequality is best possible.

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It is expected that, say, for subsets of the real line the range for dimension distortion should be more restrictive. In fact, this is the case for *quasicircles*, these are quasiconformal images of the unit circle (or a straight line).

1.4. Theorem. ([BP]) *For every k -quasicircle Γ for k close to 0,*

$$\dim \Gamma \leq 1 + 37k^2.$$

1.5. Remark. Note that (1.3) would give the bound $1 + k$. The result above provides a bound of the form $1 + ck^2$, an improvement for small values of k . We could choose $c = 60$ to obtain a valid bound for all values of k . In fact, $\dim \Gamma \leq 1 + k^2$ due to Smirnov's unpublished result. This is conjectured to be sharp. The fact, that the order k^2 is sharp was proven in [BP].

The $(1 + ck^2)$ -type estimate for quasicircles reflects back to the dimension distortion of subsets of the line, as well, allowing us to improve the general estimate (1.3) in the case of the jump to dimension one. Throughout these notes $c \geq 1$ will denote a fixed positive absolute constant, such that $\dim \Gamma \leq 1 + ck^2$ holds, for every k -quasicircle Γ , i.e. we can choose $c = 60$ or even $c = 1$ in view of Smirnov's result.

The following type of result (and in particular Corollary 1.9) is a crucial step in [ACMOU] for their improved version of Painlevé removability for bounded K -quasiregular mappings ($K > 1$): sets of σ -finite Hausdorff measure at the critical dimension are always removable.

1.6. Theorem. *Let $f: \mathbf{C} \rightarrow \mathbf{C}$ be a k -quasiconformal map with $0 < k < 1/\sqrt{8c}$ and $E \subset \mathbf{R}$. Then $\dim fE < 1$ provided that $\dim E \leq 1 - 8ck^2$. Conversely, if $\dim E = 1$ then $\dim fE > 1 - 8ck^2$.*

This result (with unspecified constant) is due to [ACMOU]. Discussing their results with the authors I found a more direct proof to this kind of improved quasiconformal dimension distortion. The purpose of this paper is to present this alternative proof of Theorem 1.6 which has its own interest. Our approach relies on the area distortion argument of Astala, we shall follow the presentation in [A1]. This approach allows for further generalizations and improvements, see [APS].

1.7. Remark. The borderline dimension for the jump to one dimension is $2/(K + 1) = 1 - k$ in the general case. Thus Theorem 1.6 is really an improvement for small values of k and then it is easy to establish some improvement for every k in the sense of Corollary 1.9.

Let us mention two immediate corollaries of Theorem 1.6 from [ACMOU].

1.8. Corollary. ([ACMOU]) *Let $E \subset \mathbf{R}$ be a compact set. For every $1 < K < K_0$ there exists a positive number $\varepsilon(K)$, such that if*

$$\dim E \leq \frac{2}{K + 1} + \varepsilon(K),$$

then E is removable for bounded K -quasiregular mappings.

1.9. Corollary. ([ACMOU]) *Let $E \subset \mathbf{R}$ of dimension 1 and $K > 1$. Then for any K -quasiconformal map $f: \mathbf{C} \rightarrow \mathbf{C}$,*

$$\dim fE \geq \frac{2}{K+1} + \delta(K),$$

where $\delta(K) > 0$ and depends (continuously) only on K .

Section 2 is devoted to the proof of Theorem 1.6, while in Section 3 we discuss related results concerning quasisymmetric maps of the line.

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2. Improved distortion

The key idea in [A1] was to look at quasiconformal mappings as holomorphic motions. Recall that a function $\Phi: \mathbf{D} \times E \rightarrow \overline{\mathbf{C}}$ is a *holomorphic motion* of a set $E \subset \overline{\mathbf{C}}$ if

- for any fixed $z \in E$, the map $\lambda \mapsto \Phi(\lambda, z)$ is holomorphic in \mathbf{D} (the open unit disk),
- for any fixed $\lambda \in \mathbf{D}$, the map $z \mapsto \Phi_\lambda(z) = \Phi(\lambda, z)$ is an injection, and
- the mapping Φ_0 is the identity on E .

A fundamental result about holomorphic motions is the extended version of the λ -lemma by Slodkowski [S], which says that every holomorphic motion extends to a global motion $\Phi: \mathbf{D} \times \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ and $\Phi_\lambda: \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$ is a $|\lambda|$ -quasiconformal mapping.

Let us define the dimension $t(k)$ for $0 < t < 2$ and $k < 1/\sqrt{4c}$ by the formula

$$(2.1) \quad \frac{1}{t(k)} - \frac{1}{2} = \frac{1}{3} \left[\left(\frac{1}{t} - \frac{1}{2} \right) + \frac{1 - 4ck^2}{1 + 4ck^2} \right].$$

Recall that $c > 0$ is an absolute constant. Due to the assumption that $k < 1/\sqrt{4c}$, we see that $0 < t(k) < 2$ as $0 < t < 2$ and that $t(k)$ is continuous and strictly increasing in both t and k . We will see that a t -dimensional set on a line goes to a set of dimension at most $t(k)$ under a k -quasiconformal mapping. In the following theorem we establish the corresponding estimate under *conformality assumption* on finite union of disks.

2.2. Theorem. *Let $f: \mathbf{C} \rightarrow \mathbf{C}$ be a k -quasiconformal homeomorphism of \mathbf{C} ($k < k_0 = 1/\sqrt{4c}$), conformal outside \mathbf{D} , normalized by $f(z) = z + o(1)$ ($z \rightarrow \infty$). Assume that f is conformal on some finite union of disjoint disks $E = \bigcup_{i=1}^n B(z_i, r_i) \subset \mathbf{D}$, where $z_i \in \mathbf{R}$. Then for any $0 < t < 2$,*

$$(2.3) \quad \sum_{i=1}^n (|f'(z_i)|r_i)^{t(k)} \leq C \left(\sum_{i=1}^n r_i^t \right)^{\frac{1}{3} \frac{t(k)}{t}},$$

where C is a positive constant (may be chosen to be 64). The exponent $0 < t(k) < 2$ is determined by formula (2.1). For the value $t = 1 - 8ck^2$, $t(k) < 1$, provided that k is nonzero.

In the other direction, we have

$$(2.4) \quad \sum_{i=1}^n r_i^{t(k)} \leq C \left(\sum_{i=1}^n (|f'(z_i)|r_i)^t \right)^{\frac{1}{3} \frac{t(k)}{t}}.$$

Proof. We closely follow the ideas of [A1]. There, the author considers invariant measures on holomorphically moving Cantor sets and applies the Ruelle–Bowen thermodynamic formalism. The main observation is that we can make use of the quasicircle dimension estimate of Theorem 1.4 in this framework in a natural way. Let us discuss the proof in detail.

Embed the map f into a holomorphic motion in a standard manner. Denote by μ the complex dilatation of f and define $\mu_\lambda = \frac{\lambda\mu}{k}$ for every $\lambda \in \mathbf{D}$. This Beltrami coefficient satisfies $\|\mu_\lambda\|_\infty \leq |\lambda| < 1$ and thus we have a principal solution f_λ by the measurable Riemann mapping theorem. Principal solution refers to the unique homeomorphic solution with asymptotics at infinity $f_\lambda(z) = z + o(1)$. By uniqueness, for $\lambda = k$, we get back our original map, $f_k = f$ and $f_0 = \text{id}$. Since μ and hence μ_λ vanish on E , the complex derivatives $f'_\lambda(z_i)$ exist and are nonzero. We shall use the important fact,

$$(2.5) \quad \text{the function } \lambda \mapsto f'_\lambda(z_i) \text{ is holomorphic [AB, Theorem 3].}$$

By Koebe’s 1/4-theorem

$$D_i(\lambda) = B(f_\lambda(z_i), 1/4|f'_\lambda(z_i)|r_i) \subset f_\lambda(B(z_i, r_i))$$

and $f_\lambda(\mathbf{D}) \subset B(f_\lambda(0), 4)$. Here $D_i(\lambda) - f_\lambda(0) = \psi_{i,\lambda}D_i(0)$, where

$$\psi_{i,\lambda}(z) = f'_\lambda(z_i)(z - z_i) + (f_\lambda(z_i) - f_\lambda(0)).$$

The coefficients of the similarities $\psi_{i,\lambda}$ vary holomorphically in λ , thus $\{D_i(\lambda) - f_\lambda(0)\}_1^n$ is a holomorphic family of disjoint disks contained in $B(0, 4)$. Choosing additional similarities $\phi_i: B(0, 4) \rightarrow D_i(0)$, $\phi_i(z) = \frac{1}{16}r_iz + z_i$, set $\gamma_{i,\lambda} = \psi_{i,\lambda} \circ \phi_i$. These contractions generate a holomorphic family of Cantor sets $C_\lambda \subset B(0, 4)$ as described in [A1]. There is a natural identification of the points of C_λ with sequences of $\{1, \dots, n\}^\mathbf{N}$. This correspondence gives a bijective map $\Phi_\lambda: C_0 \rightarrow C_\lambda$. Here $\Phi_0 = \text{id}$ and $\Phi_\lambda(z)$ depends holomorphically on λ and thus $\Phi_\lambda(z)$ is a holomorphic motion. By the extended λ -lemma of [S], it extends to a global $\Phi_\lambda: \mathbf{C} \rightarrow \mathbf{C}$ $|\lambda|$ -quasiconformal mapping. Observe that $C_0 \subset \mathbf{R}$ since $D_i(0) = B(z_i, 1/4r_i)$ ’s are centered on the real line. This shows that C_λ is contained in a $|\lambda|$ -quasicircle and thus has dimension at most $1 + c|\lambda|^2$ according to Theorem 1.4. On the other hand the dimension s of the self-similar Cantor set C_λ is determined by the formula [H]

$$\sum_{i=1}^n \left(\frac{1}{16}|f'_\lambda(z_i)|r_i \right)^s = 1.$$

We certainly have

$$(2.6) \quad \sum_{i=1}^n \left(\frac{1}{16} |f'_\lambda(z_i)| r_i \right)^{1+c|\lambda|^2} \leq 1.$$

We are going to use this fact to obtain some improvement on the dimension distortion.

For a probability distribution $\{p_i\}_{i=1}^n$ define the function

$$u(\lambda) = 2 \sum p_i \log(a|f'_\lambda(z_i)|r_i) - \sum p_i \log p_i,$$

where we write a for $1/16$ for simplicity. This is a harmonic function by (2.5) and we have the estimate

$$\begin{aligned} u(\lambda) &= \frac{2}{1+c|\lambda|^2} \left[(1+c|\lambda|^2) \sum p_i \log(a|f'_\lambda(z_i)|r_i) - \sum p_i \log p_i \right] \\ &\quad + \frac{1-c|\lambda|^2}{1+c|\lambda|^2} \sum p_i \log p_i \\ &\leq \frac{2}{1+c|\lambda|^2} \log \left(\sum (a|f'_\lambda(z_i)|r_i)^{1+c|\lambda|^2} \right) + \frac{1-c|\lambda|^2}{1+c|\lambda|^2} \sum p_i \log p_i \\ &\leq \frac{1-c|\lambda|^2}{1+c|\lambda|^2} \sum p_i \log p_i \end{aligned}$$

in terms of Jensen's inequality for the concave logarithm function and (2.6).

In order to make use of this estimate for the growth of u , apply Harnack's inequality in the disk $\{|\lambda| < 2k\}$ ($k < 1/2$),

$$(2.7) \quad u(k) \leq \frac{1}{3}u(0) + \frac{2}{3} \frac{1-4ck^2}{1+4ck^2} \sum p_i \log p_i.$$

For dimension estimate, write

$$\begin{aligned} &\sum p_i \log(a|f'_\lambda(z_i)|r_i) - \frac{1}{t(k)} \sum p_i \log p_i \\ &= \frac{1}{2}u(k) + \left(\frac{1}{2} - \frac{1}{t(k)} \right) \sum p_i \log p_i \\ &\stackrel{(2.7)}{\leq} \frac{1}{3} \sum p_i \log(ar_i) + \left[\frac{1}{3} \frac{1-4ck^2}{1+4ck^2} - \frac{1}{6} + \frac{1}{2} - \frac{1}{t(k)} \right] \sum p_i \log p_i \\ &= \frac{1}{3} \left(\sum p_i \log(ar_i) - \frac{1}{t} \sum p_i \log p_i \right) \\ &\quad + \left[\frac{1}{3} \left(\frac{1}{t} - \frac{1}{2} + \frac{1-4ck^2}{1+4ck^2} \right) + \frac{1}{2} - \frac{1}{t(k)} \right] \sum p_i \log p_i \\ &\stackrel{(J)}{\leq} \frac{1}{3t} \log \left(\sum (ar_i)^t \right). \end{aligned}$$

In the last step we see that due to the definition of $t(k)$ in (2.1), the expression in the square brackets is zero, while (J) refers to another application of Jensen’s inequality. With a proper choice of the weights p_i we actually have equality in Jensen’s inequality, namely put $p_i = (|f'(z_i)|r_i)^{t(k)} / \sum (|f'(z_i)|r_i)^{t(k)}$ to arrive at the following form of (2.3)

$$\frac{1}{t(k)} \log \left(\sum (a|f'(z_i)|r_i)^{t(k)} \right) \leq \frac{1}{3t} \log \left(\sum (ar_i)^t \right).$$

Our setting is not symmetric with respect to the inverse mapping, however, invoking Harnack’s inequality the other way around one obtains (2.4) in an analogous way. It remains to observe that in case of $t(k) = 1$, t reads as

$$t = \frac{1 + 4ck^2}{1 + 12ck^2} > 1 - 8ck^2 \quad (k \neq 0). \quad \square$$

Our estimates are only interesting as $k \rightarrow 0$. In particular, we often will make the assumption $k < k_0$ with $k_0 = 1/\sqrt{4c}$, this is the range where $t(k)$ is defined at all. We will need the following standard *deformation lemma* from [A2, Lemma 4.2]. For the sake of completeness we sketch here a short proof based on holomorphic motions.

2.8. Lemma. *Let f be a k -quasiconformal mapping on $\overline{\mathbf{C}}$ fixing $0, 1$ and ∞ . Then for each $\varepsilon > 0$ there is a number $\varrho = \varrho(k, \varepsilon) \in (0, 1)$ and a k_ε -quasiconformal mapping φ on \mathbf{C} such that*

- (a) $\varphi(z) = f(z)$ if $1 \leq |z|$,
- (b) $\varphi(z) = z$ if $|z| \leq \varrho$,

and $k_\varepsilon \rightarrow k$ as $\varepsilon \rightarrow 0$.

Proof. We consider the associated holomorphic motion $\{f_\lambda(z)\}$ as in Theorem 2.2 with the exception that the homeomorphic solution f_λ is now normalized by the condition that it fixes $0, 1$ and ∞ . Consider the following modified motion of the set $\{|z| \leq \varrho\} \cup \{|z| \geq 1\}$ for some $0 < \varrho < 1$,

$$\Phi_\lambda(z) = \begin{cases} f_\lambda(z) & \text{if } |z| \geq 1, \\ z & \text{if } |z| \leq \varrho. \end{cases}$$

Classical distortion properties of quasiconformal mappings assure that the image of the unit circle $f_\lambda(S^1)$ will remain disjoint from the disk $\{|z| \leq \rho\}$ as long as $|\lambda| < \lambda_0 = \lambda_0(\varrho) < 1$, where $\lambda_0(\varrho) \rightarrow 1$ as $\varrho \rightarrow 0$. In other words, $\Phi_\lambda(z)$ is a holomorphic motion parametrized by the disk $\{|\lambda| < \lambda_0\}$. The extension of Φ_k provided by the extended λ -lemma gives a (k/λ_0) -quasiconformal deformation of f described in the statement of the lemma. □

2.9. Lemma. *Assume that $f: \mathbf{C} \rightarrow \mathbf{C}$ is a k -quasiconformal mapping ($k < k_0$) fixing $0, 1$ and ∞ . Let $B_i = B(z_i, r_i)$ ($z_i \in \mathbf{R}$) disjoint disks in \mathbf{D} . Then for every*

sufficiently small $\varepsilon > 0$ we have

$$\sum (\text{diam } fB_i)^{t(k_\varepsilon)} \leq C(k, \varepsilon) \left(\sum r_i^t \right)^{\frac{1}{3} \frac{t(k_\varepsilon)}{t}},$$

with $k_\varepsilon \rightarrow k$ as $\varepsilon \rightarrow 0$. Similarly, in the other direction

$$\sum r_i^{t(k_\varepsilon)} \leq C(k, \varepsilon) \left(\sum (\text{diam } fB_i)^t \right)^{\frac{1}{3} \frac{t(k_\varepsilon)}{t}}.$$

Proof. Apply Lemma 2.8 to deform f in disks B_i and outside \mathbf{D} . We obtain a k_ε -quasiconformal map $\varphi: \mathbf{C} \rightarrow \mathbf{C}$ which agrees with f in $\mathbf{D} \setminus \bigcup B_i$, identity outside $B(0, 1/\varrho)$ and a τ_i similarity inside $B(z_i, \varrho r_i)$. Here τ_i is determined by $\tau_i(z_i) = f(z_i)$ and $\tau_i(z_i + r_i) = f(z_i + r_i)$. Moreover we have a good control on the diameters of the corresponding sets,

$$(2.10) \quad \begin{aligned} |\varphi'(z_i)|r_i &= |\tau_i'|r_i = |f(z_i + r_i) - f(z_i)| \\ &\leq \text{diam } fB_i \lesssim |f(z_i + r_i) - f(z_i)| = |\varphi'(z_i)|r_i, \end{aligned}$$

up to a constant depending only on k , as quasiconformal maps distort circles in a uniform manner.

Conjugating with an additional similarity $u(z) = (1/\varrho)z$, $(u^{-1} \circ \varphi \circ u)$ is identical outside \mathbf{D} and similarity in disks $B(\varrho z_i, \varrho^2 r_i)$. We may apply Theorem 2.2 to find

$$\begin{aligned} \sum (|(u^{-1} \circ \varphi \circ u)'(\varrho z_i)|\varrho^2 r_i)^{t(k_\varepsilon)} &\leq C \left(\sum (\varrho^2 r_i)^t \right)^{\frac{1}{3} \frac{t(k_\varepsilon)}{t}}, \\ \sum (\varrho^2 r_i)^{t(k_\varepsilon)} &\leq C \left(\sum (|(u^{-1} \circ \varphi \circ u)'(\varrho z_i)|\varrho^2 r_i)^t \right)^{\frac{1}{3} \frac{t(k_\varepsilon)}{t}}. \end{aligned}$$

Combining with (2.10), the desired estimates follow. □

Proof of Theorem 1.6. We shall prove the following claim.

Let $f: \mathbf{C} \rightarrow \mathbf{C}$ be a k -quasiconformal map with $k < k_0$ and $E \subset \mathbf{R}$. Then

$$\begin{aligned} \dim E \leq t &\Rightarrow \dim fE \leq t(k), \\ \dim fE \leq t &\Rightarrow \dim E \leq t(k). \end{aligned}$$

Theorem 1.6 follows now from the fact that for $t = 1 - 8ck^2$, $t(k) < 1$. The claim follows from Lemma 2.9 by a standard covering argument. We sketch the proof in the second case, distorting the dimension downwards. The first case is similar.

First of all, we may clearly assume that $E \subset [-1/2, 1/2]$ and f fixes 0, 1 and ∞ . Suppose that $\dim fE = t$, what we need to prove is that $\dim E \leq t(k)$. Choose an exponent $t' > t$. Making use of a basic covering theorem we can find a countable family of disjoint disks $D_i = B(w_i, \varrho_i)$ such that $fE \subset \bigcup 5D_i$, and $\sum \varrho_i^{t'}$ is arbitrary small. Furthermore, we may assume that $w_i \in fE$. Set $z_i = f^{-1}(w_i) \in E$ and $r_i = \text{dist}(z_i, \partial f^{-1}(D_i))$. In this way $B_i = B(z_i, r_i) \subset f^{-1}(D_i)$, so the disks B_i are disjoint, centered on the real line and $\bigcup B_i \subset \mathbf{D}$ may be assumed, as well.

Now the uniform bound of Lemma 2.9 (with a fixed $\varepsilon > 0$) holds for this possibly infinite family of disks, too

$$(2.11) \quad \sum r_i^{t'(k_\varepsilon)} \leq C(k, \varepsilon) \left(\sum (\text{diam } fB_i)^{t'} \right)^{\frac{1}{3} \frac{t'(k_\varepsilon)}{t'}}$$

Observe that $\{f^{-1}(5D_i)\}$ gives a cover of E with sets of size

$$\text{diam } f^{-1}(5D_i) \lesssim r_i,$$

up to a constant depending only on k by distortion properties of quasiconformal maps. While the right-hand side of (2.11) can be made arbitrary small with a proper choice of the family $\{D_i\}$, since $\text{diam } fB_i \leq 2\rho_i$. We conclude that $\dim E \leq t'(k_\varepsilon)$, letting $\varepsilon \rightarrow 0$ and $t' \rightarrow t$, $\dim E \leq t(k)$ follows. \square

3. Distortion of quasymmetric functions

In this section we make the assumption that our map fixes the real line. In other words, we consider *quasisymmetric maps* of \mathbf{R} , where the quasymmetricity is measured by the dilatation of (the best) quasiconformal extension. This assumption allows us to sharpen our estimates and obtain the aesthetically appealing (and possibly sharp) bound $1 - k^2$ for distortion of 1-dimensional sets. This is a dual result to Smirnov's $(1 + k^2)$ -bound on the dimension of quasicircles, apparently known to him. In fact, we rely on some of the ideas of him developed for the quasicircle estimate. We are grateful to him for allowing us to include this result here.

3.1. Theorem. *Let $f: \mathbf{C} \rightarrow \mathbf{C}$ be a k -quasiconformal map for which $f(\mathbf{R}) = \mathbf{R}$. Then for a 1-dimensional set $E \subset \mathbf{R}$,*

$$\dim fE \geq 1 - k^2.$$

Standard covering arguments reduce the theorem to the following statement. We sketch the details after the proof of Lemma 3.2.

3.2. Lemma. *Let there be given a sequence of finite families of disjoint disks $\{B_{i,j} = B(z_{i,j}, r_{i,j})\}_{i=1}^{n_j}$ ($j = 1, 2, \dots$) in the unit disk \mathbf{D} , such that in every collection $z_i \in \mathbf{R}$, for any $t < 1$ $\sum_i r_i^t \rightarrow \infty$, $r_i \leq \delta_j$ and $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. Consider a sequence of k -quasiconformal maps $f_j: \mathbf{C} \rightarrow \mathbf{C}$, $f_j(\bar{z}) = f_j(z)$, f_j conformal outside \mathbf{D} , normalized by $f_j(z) = z + o(1)$ ($z \rightarrow \infty$). Assume that f_j is conformal on the disks $B_{i,j}$ belonging to the level j . Then*

$$\sum_{i=1}^{n_j} \left(\frac{1}{16} |f'_j(z_i)| r_i \right)^{1-k^2-\eta_j} \geq 1.$$

Here $\eta_j \rightarrow 0$ as $j \rightarrow \infty$ for some subsequence.

Proof. For every j embed the map $f = f_j$ into the standard holomorphic motion $f_\lambda(z)$ as in Theorem 2.2. In this way $f_0 = \text{id}$, $f_k = f_j$. Since the level j is fixed for a while we will not explicitly write the dependence on j . As μ the complex

dilatation of f is symmetric with respect to the real axis, we have $\mu_\lambda(\bar{z}) = \overline{\mu_\lambda(z)}$. This inherits to the solutions, $f_\lambda(\bar{z}) = \overline{f_\lambda(z)}$. In particular, for purely imaginary λ

$$(3.3) \quad |f'_{-\lambda}(z_i)| = |f'_\lambda(z_i)|,$$

while for real values of λ the map f_λ is symmetric with respect to the real axis.

Recall from the proof of Theorem 2.2 that the disks

$$D_i(\lambda) = B(f_\lambda(z_i), 1/4|f'_\lambda(z_i)|r_i)$$

are disjoint and included in a disk of radius 4. Hence comparing their area gives (with $a = 1/16$)

$$\sum (a|f'_\lambda(z_i)|r_i)^2 \leq 1.$$

Moreover if λ is real then all the disks $D_i(\lambda)$ are centered on the real line as f_λ preserves the real axis. In this case, we have

$$\sum (a|f'_\lambda(z_i)|r_i) \leq 1.$$

As before, consider the harmonic function for a given probability distribution $\{p_i\}_{i=1}^n$,

$$u(\lambda) = u_j(\lambda) = 2 \sum p_i \log(a|f'_\lambda(z_i)|r_i) - \sum p_i \log p_i.$$

Jensen's inequality and the estimates above tell us that u is negative for every $\lambda \in \mathbf{D}$ and $u(\lambda) \leq \sum p_i \log p_i$ for real valued λ . Due to (3.3) u is even on the imaginary axis, $u(-\lambda) = u(\lambda)$ for $\lambda \in i\mathbf{R}$.

Choose a sequence $t_l \rightarrow 1-$ as $l \rightarrow \infty$. For a fixed l , $\sum_i r_{i,j}^{t_l} \rightarrow \infty$ as $j \rightarrow \infty$ by assumption. So there exists a subsequence j_l such that $\sum_i (ar_{i,j_l})^{t_l} \geq 1$ for every l . For a level $j = j_l$, set the weights

$$p_{i,j} = p_i = \frac{r_i^{t_l}}{\sum r_i^{t_l}}.$$

Then

$$\begin{aligned} u_{j_l}(0) &= 2 \sum p_i \log(ar_i) - \sum p_i \log p_i \\ &= \frac{2}{t_l} \left(\sum p_i \log(ar_i)^{t_l} - \sum p_i \log p_i \right) + \left(\frac{2}{t_l} - 1 \right) \sum p_i \log p_i \\ (3.4) \quad &= \frac{2}{t_l} \log \left(\sum (ar_i)^{t_l} \right) + \left(\frac{2}{t_l} - 1 \right) \sum p_i \log p_i \\ &\geq \left(\frac{2}{t_l} - 1 \right) \sum p_i \log p_i. \end{aligned}$$

Since the family $-\frac{u_{j_l}(\lambda)}{u_{j_l}(0)}$ form a normal family of harmonic functions, there exists a harmonic function u_0 such that $u_{j_l} \rightarrow u_0$ locally uniformly as $j_l \rightarrow \infty$ through a subsequence. For this limit function we have

- $u_0(\lambda) \leq 0$ ($\lambda \in \mathbf{D}$),
- $u_0(-\lambda) = u_0(\lambda)$ for $\lambda \in i\mathbf{R}$,

- $u_0(\lambda) \leq -1$ for $\lambda \in \mathbf{R}$ and $u_0(0) = -1$.

The last one follows from (3.4) and the fact that $u_j(\lambda) \leq \sum p_i \log p_i$ if $\lambda \in \mathbf{R}$.

Now the second item tells us that $\frac{\partial}{\partial y} u_0(0) = 0$ and the third one says $\frac{\partial}{\partial x} u_0(0) = 0$. In this case we have a squared-type Harnack inequality (see Lemma 3.6) of the form

$$u_0(\lambda) \geq \frac{1 + |\lambda|^2}{1 - |\lambda|^2} u_0(0).$$

Put $\lambda = k$, then

$$(3.5) \quad u_j(k) \geq \left(\frac{1 + k^2}{1 - k^2} + \varepsilon_j \right) u_j(0),$$

with $\varepsilon_j \rightarrow 0$ ($j \rightarrow \infty$) for a subsequence.

The usual manipulation with Jensen’s inequality provides the desired estimate (here $j = j_l$ and $j(k)$ denotes an exponent depending on k and j to be chosen later).

$$\begin{aligned} & \frac{1}{j(k)} \log \left(\sum (a|f'_j(z_i)r_i)^{j(k)} \right) \\ & \geq \sum p_i \log(a|f'_j(z_i)r_i) - \frac{1}{j(k)} \sum p_i \log p_i \\ & = \frac{1}{2} u_j(k) + \left(\frac{1}{2} - \frac{1}{j(k)} \right) \sum p_i \log p_i \\ & \stackrel{(3.5)}{\geq} \frac{1}{2} \left(\frac{1 + k^2}{1 - k^2} + \varepsilon_j \right) u_j(0) + \left(\frac{1}{2} - \frac{1}{j(k)} \right) \sum p_i \log p_i \\ & \stackrel{(3.4)}{\geq} \left[\frac{1}{2} \left(\frac{1 + k^2}{1 - k^2} + \varepsilon_j \right) \left(\frac{2}{t_l} - 1 \right) + \left(\frac{1}{2} - \frac{1}{j(k)} \right) \right] \sum p_i \log p_i = 0. \end{aligned}$$

We define $j(k)$ by the expression in the square brackets, so that it will be zero. Since $\varepsilon_j \rightarrow 0$ and $t_l \rightarrow 1$ as $j_l \rightarrow \infty$ (for a subsequence) we see that $j(k) = 1 - k^2 - \eta_j$ where $\eta_j \rightarrow 0$ for some subsequence. □

Proof of Theorem 3.1. Let $E \subset [-1/2, 1/2]$ with $\dim E = 1$. Assume to the contrary that $\dim fE < 1 - k^2$ for some k -quasiconformal map which is, as we may assume, symmetric with respect to the real axis. We can find a sequence of finite families of disjoint disks $\{B_i^j = B(z_i, r_i)\}$ such that $z_i \in \mathbf{R}$, $\sup r_i \rightarrow 0$, for any $t < 1$, $\sum r_i^t \rightarrow \infty$ ($j \rightarrow \infty$) and $\sum (\text{diam } fB_i)^d \rightarrow 0$ ($j \rightarrow \infty$), with a fixed exponent $d < 1 - k^2$. Choose $\varepsilon_0 > 0$ so that also $d < 1 - k_{\varepsilon_0}^2$. Now deform f according to the family on the level j to obtain a k_{ε_0} -quasiconformal map φ_j which is identical outside the unit disk and similarity in $B(\varrho z_i, \varrho^2 r_i)$, here $\varrho = \varrho(\varepsilon_0, k)$. Moreover, $\text{diam } fB_i \asymp |\varphi'_j(\varrho z_i)| r_i$. Apply Lemma 3.2 to the sequence φ_j , we have a contradiction as $j \rightarrow \infty$.

Note that we have skipped one detail, we need to make sure that the deformation preserves the symmetry with respect to the line. This is not a serious problem, e.g. with a little modification in the spirit of the proof of Lemma 2.8 one could omit

the conformality assumption. On the other hand, it is easy to see that the explicit deformation described in [A2, Lemma 4.2] provides a symmetric deformation. \square

3.6. Lemma. (Squared-type Harnack's inequality) *Suppose that the function $u \leq 0$ is harmonic in \mathbf{D} and $\nabla u(0) = 0$. Then we have an improved Harnack's inequality of the form*

$$\frac{1 + |z|^2}{1 - |z|^2} u(0) \leq u(z) \leq \frac{1 - |z|^2}{1 + |z|^2} u(0).$$

Proof. The proof is a slight modification of the complex analytic proof of the standard Harnack's inequality. There is a holomorphic function $f: \mathbf{D} \rightarrow \{w : \Re w < 0\}$ such that $f = u + iv$, where v is real-valued harmonic function. We may assume, that $f(0) = -1$, that is $u(0) = -1$ and $v(0) = 0$. In virtue of the Cauchy–Riemann equations $f'(0) = 0$, since $(\nabla u)(0) = 0$. Map the left half-plane onto the unit disk by the linear fractional transformation $\frac{w+1}{w-1}$, this takes -1 to 0 . The composed function maps the unit disk into the unit disk and vanishes at the origin with double multiplicity. We have a squared-type Schwarz lemma in this situation,

$$\left| \frac{f(z) + 1}{f(z) - 1} \right| \leq |z|^2.$$

Observe the following geometric fact for $u = \Re w$,

$$\frac{u + 1}{u - 1} \leq \left| \frac{w + 1}{w - 1} \right|.$$

Combining the two estimates leads us to

$$u(z) \geq -\frac{1 + |z|^2}{1 - |z|^2}.$$

Noting that $u(0) = -1$, this is the left hand side of the inequality in (3.6). The argument for the right hand side follows similar lines, one just needs to replace the linear fractional transformation $\frac{w+1}{w-1}$ by its negative. \square

3.7. Remark. The order k^2 in Theorem 1.6 and Theorem 3.1 is sharp. Answering a question of Hayman and Hinkkanen, Tukia [T] constructed a k -quasisymmetric map of the unit interval which does not preserve one-dimensional sets. It is actually even more singular, mapping a set of less-than-one dimensional complement to a less-than-one dimensional set. Moreover, the quasisymmetry k can be arbitrary close to 0. An analysis of the example shows that the dimension distortion is of the type $1 - Ck^2$ as $k \rightarrow 0$, with $C > 0$.

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A note on distortion of Hausdorff measures under quasiconformal mappings

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Abstract

Astala [1] gave optimal bounds for the distortion of Hausdorff dimension under planar quasiconformal maps. The corresponding estimates on the level of Hausdorff measures remain open. We show that the techniques of [1] allows to establish absolute continuity for some weaker Hausdorff measures.

1 Introduction

A homeomorphism $\phi: \Omega \rightarrow \Omega'$ between planar domains is called K -quasiconformal if it lies in the Sobolev class $W_{\text{loc}}^{1,2}(\Omega)$ and satisfies the distortion inequality

$$\max_{\alpha} |\partial_{\alpha} \phi| \leq K \min_{\alpha} |\partial_{\alpha} \phi| \quad \text{a.e. in } \Omega.$$

Astala [1], in his seminal work, gave a complete description of dimension distortion of sets under planar quasiconformal mappings.

1.1 Theorem ([1]). *Let $\phi: \Omega \rightarrow \Omega'$ be K -quasiconformal and suppose $E \subset \Omega$ is compact. Then*

$$(1.2) \quad \frac{1}{K} \left(\frac{1}{\dim(E)} - \frac{1}{2} \right) \leq \frac{1}{\dim(\phi(E))} - \frac{1}{2} \leq K \left(\frac{1}{\dim(E)} - \frac{1}{2} \right).$$

This inequality is best possible.

It is natural then to ask, see [1, 3], whether these estimates hold on the level of Hausdorff measures \mathcal{H}^t . That is, if ϕ is a planar K -quasiconformal mapping, $0 < t < 2$ and $d = \frac{2Kt}{2+(K-1)t}$, is it true that

$$(1.3) \quad \mathcal{H}^t(E) = 0 \Rightarrow \mathcal{H}^d(\phi(E)) = 0?$$

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In other words, do we have absolute continuity $\phi^*\mathcal{H}^d \ll \mathcal{H}^t$? It is classical that quasiconformal mappings are absolutely continuous with respect to the Lebesgue measure, and Astala [1] proves this in a quantitatively optimal form,

$$(1.4) \quad |\phi(E)| \leq C|E|^{\frac{1}{K}},$$

with a constant that depends only on K and the normalization.

A convenient normalization we shall work with is the following. We call a quasiconformal map $\phi: \mathbb{C} \rightarrow \mathbb{C}$ *principal* if it is conformal outside a compact set and normalized by $\phi(z) = z + O(1/|z|)$ as $z \rightarrow \infty$.

The question of absolute continuity in (1.3) has recently been studied in [3]. Let us discuss some of their results in this direction. They reduce the absolute continuity question to the following (for the complementary part, see Theorem 2.2).

1.5 Conjecture ([3]). *For any compact set $E \subset \mathbb{C}$ and for any principal K -quasiconformal mapping h which is conformal on $\mathbb{C} \setminus E$, we have for any $d \in (0, 2]$*

$$\mathcal{H}_\infty^d(h(E)) \simeq \mathcal{H}_\infty^d(E)$$

with constants that depend only on K and d . Here \mathcal{H}_∞^d denotes the d -dimensional Hausdorff content.

The authors were able to prove this for $d = 1$, thus confirming absolute continuity for the special case $t = 2/(K + 1)$ and $d = 1$. In the case $d > 1$ they obtained the following partial result, see Corollary 2.12 of [3].

Given $1 < d < 2$ consider a measure function of the form $\delta(r) = r^d \varepsilon(r)$, where

$$(1.6) \quad \int_0^\infty \varepsilon(r)^{\frac{1}{d-1}} \frac{dr}{r} < \infty.$$

1.7 Theorem ([3]). *Let $E \subset \mathbb{D}$ be a compact set and $\phi: \mathbb{C} \rightarrow \mathbb{C}$ be a principal K -quasiconformal mapping conformal outside \mathbb{D} . Let $t \in (\frac{2}{K+1}, 2)$ and $d = \frac{2Kt}{2+(K-1)t}$. Then we have*

$$\mathcal{H}_\infty^\delta(\phi(E)) \leq C (\mathcal{H}_\infty^t(E))^{\frac{d}{Kt}},$$

where the measure function δ satisfies (1.6). The constant C depends only on δ and K .

Our objective is to present a similar complementary result with different Hausdorff measures. We consider measure functions $\delta(r) = r^d \varepsilon(r)$ satisfying

$$(1.8) \quad \int_0^\infty \varepsilon(r)^{\frac{Kt}{Kt-d}} \frac{dr}{r} < \infty.$$

We also make the technical assumption that the integrand is decreasing and $\varepsilon(r)$ is increasing in $(0, r_0)$ for some $r_0 > 0$.

1.9 Theorem. *Let $E \subset \mathbb{D}$ be a compact set and let $\phi: \mathbb{C} \rightarrow \mathbb{C}$ be a principal K -quasiconformal mapping conformal outside \mathbb{D} . Let $t \in (0, 2)$ and $d = \frac{2Kt}{2+(K-1)t}$. Then we have*

$$\mathcal{H}^\delta(\phi(E)) \leq C (\mathcal{H}^t(E))^{\frac{d}{Kt}},$$

where the measure function δ satisfies (1.8). The constant C depends only on δ and K .

1.10 Remark. Note that Theorem 1.7 gets sharper as $d \rightarrow 1$, while Theorem 1.9 improves as $K \rightarrow 1$. Also, Theorem 1.9 is valid for every dimension $0 < t < 2$. In fact, it can be viewed as a result which provides absolute continuity with respect to *some* Hausdorff measure of the right dimension. For instance, we can take $\varepsilon(r) = |\log r|^{-s}$, with $s > 1 - \frac{d}{Kt}$, and then \mathcal{H}^δ has dimension d .

Let us note that a discrete (stronger) variant of Conjecture 1.5 formulated as Question 2.4 in [3] has recently been disproved by Bishop, see [4]. We shall use the usual convention that the constant C may change from line to line, but indicate its dependence on the parameters.

2 Distortion of Hausdorff measures

We will need two complementary results from [1]. The first one is a counterpart of Conjecture 1.5 for the area. See Lemma 3.3 and the remark afterwards in [1].

2.1 Theorem ([1]). *Let $h: \mathbb{C} \rightarrow \mathbb{C}$ be a principal K -quasiconformal mapping which is conformal outside a compact set E . Then we have*

$$|h(E)| \leq K|E|.$$

Another result we need is an optimal discrete version of Theorem 1.1 under *conformality assumption*.

2.2 Theorem. *Let $\phi: \mathbb{C} \rightarrow \mathbb{C}$ be a principal K -quasiconformal mapping which is conformal outside \mathbb{D} . Let \mathcal{B} be a finite family of disjoint disks in \mathbb{D} and assume that ϕ is conformal in $\cup \mathcal{B}$. Then for any $t \in (0, 2]$ and $d = \frac{2Kt}{2+(K-1)t}$ we have*

$$\left(\sum_{B \in \mathcal{B}} \text{diam}(\phi B)^d \right)^{\frac{1}{d}} \leq C(K) \left(\sum_{B \in \mathcal{B}} \text{diam}(B)^t \right)^{\frac{1}{Kt}}.$$

This result is implicit in [1], see Corollary 2.3 and the variational principle on p. 48. It can also be deduced from the improved borderline integrability of the Jacobian under conformality assumption [2]. For this latter approach see [3, (2.6)].

Proof of Theorem 1.9. In [1] the area distortion, cf. (1.4), is proved via a decomposition reducing it to the two complementary cases above (with $t = d = 2$). The difficulty of

establishing absolute continuity for distortion of Hausdorff measures when $0 < t < 2$ is that Conjecture 1.5 is unavailable in general. However, we can still use Theorem 2.1 as a substitute provided that we are concerned with coverings with disks of the same size.

As in the statement of Theorem 1.9 let ϕ be a principal K -quasiconformal mapping, conformal outside \mathbb{D} . Let \mathcal{B} be an arbitrary finite family of disjoint disks in \mathbb{D} . We assume that the disks have diameter less than $a(K)$, a constant specified later. Define a subfamily \mathcal{B}_k for $k \in \mathbb{N}$ by

$$\mathcal{B}_k = \{B \in \mathcal{B} : a2^{-(k+1)K} < \text{diam } B \leq a2^{-kK}\}.$$

The constant $a = a(K) > 0$ is specified as follows. Hölder continuity of quasiconformal mappings, i.e. the area distortion estimate (1.4) for disks, assures that

$$\text{diam } \phi(5B) \leq 2^{-k},$$

provided that $\text{diam } B \leq a(K)2^{-kK}$ with an appropriate constant $a(K)$.

Decompose the map $\phi = \phi_1 \circ h$, where both ϕ_1 and h are principal K -quasiconformal mappings, and h is conformal off $\cup \mathcal{B}_k$ while ϕ_1 is conformal in $h(\cup \mathcal{B}_k)$. This decomposition can be done due to the measurable Riemann mapping theorem. We shall frequently use the fact the quasiconformal maps distort disks in a uniform manner, and thus up to bounded eccentricity we may consider quasidisks as disks in our arguments. We apply Theorem 2.1 to find

$$(2.3) \quad \begin{aligned} \left(\frac{1}{|\mathcal{B}_k|} \sum_{B \in \mathcal{B}_k} \text{diam}(hB)^t \right)^{\frac{1}{t}} &\leq \left(\frac{1}{|\mathcal{B}_k|} \sum_{B \in \mathcal{B}_k} \text{diam}(hB)^2 \right)^{\frac{1}{2}} \\ &\leq \left(C(K) \frac{1}{|\mathcal{B}_k|} \sum_{B \in \mathcal{B}_k} \text{diam}(B)^2 \right)^{\frac{1}{2}} \leq \left(C(K) \frac{1}{|\mathcal{B}_k|} \sum_{B \in \mathcal{B}_k} \text{diam}(B)^t \right)^{\frac{1}{t}}. \end{aligned}$$

The first inequality holds in view of the convexity of the function $r^{2/t}$, while in the last step we used the fact that family \mathcal{B}_k contains disks of essentially the same size (up to a multiplicative constant depending on K).

For the map ϕ_1 we may apply the sharp estimate of Theorem 2.2, as ϕ_1 is conformal in $h(B)$ for every $B \in \mathcal{B}_k$ and outside $h(\mathbb{D})$. Since we have to take care of normalization, note that according to Koebe's 1/4-theorem we have $h(\mathbb{D}) \subset B(h(0), 4)$. We obtain

$$\sum_{B \in \mathcal{B}_k} \text{diam}(\phi B)^d \leq C(K) \left(\sum_{B \in \mathcal{B}_k} \text{diam}(hB)^t \right)^{\frac{1}{K} \frac{d}{t}}.$$

Combining with (2.3), we have

$$(2.4) \quad \sum_{B \in \mathcal{B}_k} \text{diam}(\phi B)^d \leq C(K, t) \left(\sum_{B \in \mathcal{B}_k} \text{diam}(B)^t \right)^{\frac{1}{K} \frac{d}{t}}.$$

We have the same estimate for all \mathcal{B}_k 's with constant independent of the decomposition, and thus we may combine them to find that

$$\begin{aligned}
(2.5) \quad & \sum_{B \in \mathcal{B}} \text{diam}(\phi(5B))^d \varepsilon(\text{diam } \phi(5B)) \leq C(K) \sum_k \varepsilon(2^{-k}) \sum_{B \in \mathcal{B}_k} \text{diam}(\phi B)^d \\
& \leq C(K, t) \sum_k \varepsilon(2^{-k}) \left(\sum_{B \in \mathcal{B}_k} \text{diam}(B)^t \right)^{\frac{1}{K} \frac{d}{t}} \\
& \leq C(K, t) \left(\sum_k \varepsilon(2^{-k})^{\frac{Kt}{Kt-d}} \right)^{1 - \frac{d}{Kt}} \left(\sum_{B \in \mathcal{B}} \text{diam}(B)^t \right)^{\frac{1}{K} \frac{d}{t}}.
\end{aligned}$$

The first sum is finite by the assumption (1.8) on $\varepsilon(r)$. Since we have uniform bounds for all families \mathcal{B} of disjoint disks, a standard $5r$ -covering lemma leads to

$$\mathcal{H}^\delta(\phi(E)) \leq C(K, \delta) \mathcal{H}^t(E)^{\frac{1}{K} \frac{d}{t}}.$$

□

2.6 Remark. The proof shows that (1.3) holds true if the assumption $\mathcal{H}^t(E) = 0$ is replaced by the stronger assumption that E has zero t -dimensional lower Minkowski content.

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FLATNESS PROPERTIES OF QUASISPHERES

ISTVÁN PRAUSE

ABSTRACT. We investigate flatness properties of K -quasiconformal spheres in the euclidean n -dimensional space with the emphasis on the case when K is close to 1. These also lead to bounds for their Hausdorff dimension showing that K -quasispsheres have much smaller dimension than K -quasiconformal images of general $(n - 1)$ -dimensional sets ($K \rightarrow 1$). The corresponding result in the plane is well-known.

1. INTRODUCTION

Quasiconformal homeomorphisms of the Euclidean space \mathbb{R}^n , $n \geq 2$, can distort the Hausdorff dimension of subsets. While the dimensions of sets of Hausdorff dimension zero or n must be preserved, Gehring and Väisälä [GV] construct Cantor sets $E_\alpha \subset \mathbb{R}^n$ with $\dim E_\alpha = \alpha$ ($0 < \alpha < n$) and for any $\alpha, \beta \in (0, n)$, a quasiconformal map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $E_\beta = fE_\alpha$. Bishop [Bi] showed that the dimension of any compact set of positive dimension can, in fact, be raised arbitrarily close to n by a quasiconformal homeomorphism of \mathbb{R}^n . However, the distortion of the Hausdorff dimension is controlled by the maximal dilatation [GV, Ge3]. Optimal dimension distortion bounds are known in the planar case due to Astala's result (see the Appendix).

The above results tell us that a quasisphere, that is, a quasiconformal image of the unit sphere in \mathbb{R}^n need not be rectifiable, and moreover can have Hausdorff dimension arbitrarily close to n . We are interested in bounds of the dimension of a K -quasisphere in the limit case when $K \rightarrow 1$. The local Hölder continuity of a quasiconformal mapping [Ge2] – with exponent $K^{1/(1-n)}$ – implies that for a K -quasisphere E ,

$$(1.1) \quad \dim E \leq (n - 1)K^{1/(n-1)} = n - 1 + (K - 1) + O((K - 1)^2) \quad (K \rightarrow 1).$$

It was observed in [BP] that for quasicircles the correct behavior is quadratic in $K - 1$ as opposed to the linear bound above (see the Appendix for details). Their method is highly analytic and does not generalize to higher dimensions. Our objective is to find a higher dimensional counterpart of this result. The authors of [MV] established qualitatively the same result using geometric considerations. They introduced the so-called linear approximation property (LAP), which expresses some uniform flatness property on every scale and location, which in turn implies a bound for the dimension.

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In this paper, we extend their result by refining some of their arguments, and in particular, we establish LAP property for quasispheeres with improved bounds. This provides an asymptotically nearly sharp bound for their Hausdorff dimension in the limiting case $K \rightarrow 1$.

The organization of the paper is as follows. In the next section we introduce notation and some preliminaries. In Section 3 we present an explicit bound for the quasisymmetry function of quasiconformal maps after [Vu2, Se]. In Section 4 we review flatness and non-flatness conditions and their consequences on the dimension. In Section 5 we show that quasispheeres satisfy the LAP property with essentially sharp bounds in terms of the quasisymmetry constant $\eta_K^*(1)$ (for definition, see Section 3). Finally, in the Appendix we collect some sharp results on distortion of dimension by planar quasiconformal mappings.

2. PRELIMINARIES

We shall follow standard notation and terminology adopted from [Väl1] and [Vu1]. For $x \in \mathbb{R}^n$ and $r > 0$ let $B^n(x, r) = \{z \in \mathbb{R}^n : |z - x| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n = B^n(0, 1)$, $S^{n-1} = S^{n-1}(0, 1)$. Sometimes we also use $B(x, r)$ for $B^n(x, r)$ if the dimension n is clear from the context. The space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is the one-point compactification of \mathbb{R}^n . The standard coordinate unit vectors are denoted by e_1, \dots, e_n . The Lebesgue measure on \mathbb{R}^n is denoted by m . The Hausdorff dimension of a set $E \subset \mathbb{R}^n$ is denoted by $\dim E$. For definition of Hausdorff measures and dimension we refer to [Ma].

2.1. Modulus of a path family. Let Γ be a path family in \mathbb{R}^n , $n \geq 2$. We call a Borel function $\varrho: \mathbb{R}^n \rightarrow [0, \infty]$ *admissible* if

$$\int_{\gamma} \varrho ds \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$. For $1 \leq p \leq \infty$ we define the p -modulus of Γ as

$$M_p(\Gamma) = \inf_{\varrho} \int_{\mathbb{R}^n} \varrho^p dm,$$

where the infimum is taken over all admissible functions. The n -modulus possesses the important property of being conformally invariant, hence it is called *conformal modulus* and is denoted simply by $M(\Gamma)$.

For sets $E, F, G \subset \overline{\mathbb{R}^n}$, let $\Delta(E, F; G)$ denote the path family of all paths joining E and F in G . In the case of $G = \mathbb{R}^n$ we omit G in the notation. A domain R in $\overline{\mathbb{R}^n}$ is called a ring if its complement has two components. If these are E and F , we shall write $R = R(E, F)$. The *capacity of a ring* $R(E, F)$ is defined by $M(\Delta(E, F))$ and denoted $\text{cap}(R)$. For the definitions and basic properties of these notions we refer to [Väl1] and [Vu1].

The *geometric definition* of quasiconformality is the following.

2.2. Definition. Let $K \geq 1$. A homeomorphism $f: D \rightarrow D'$ between domains $D, D' \subset \overline{\mathbb{R}^n}$ is K -quasiconformal if

$$M(\Gamma)/K \leq M(f\Gamma) \leq KM(\Gamma)$$

for every path family Γ in D .

We refer to [Vä1] for other equivalent definitions and for foundations of quasiconformal mappings. Note that the case $K = 1$ recovers the class of conformal mappings.

2.3. Extremal rings. The *Grötzsch ring* in $\overline{\mathbb{R}^n}$ is the ring $R(\overline{B}^n, [se_1, \infty])$ ($s > 1$) and its capacity is denoted by $\gamma_n(s)$. The *Teichmüller ring* is $R([-e_1, 0], [se_1, \infty])$ ($s > 0$) with capacity $\tau_n(s)$. Here and later $[x, \infty]$ means $\{tx : t \geq 1\} \cup \{\infty\}$, $x \in \mathbb{R}^n \setminus \{0\}$. We identify \mathbb{R} with $\mathbb{R}e_1 \subset \mathbb{R}^n$ in such notations later. These capacity functions are decreasing homeomorphisms onto $(0, \infty)$. The relevance of these rings is that they minimize the capacity of certain special ring domains. Their extremality follows from the symmetrization method of Gehring [Ge1].

- Grötzsch extremal problem: $C_0 = \overline{B}^n$, $se_1, \infty \in C_1$ ($s > 1$), minimize $\text{cap}(R(C_0, C_1))$.
- Teichmüller extremal problem: $0, -e_1 \in C_0$, $x, \infty \in C_1$, where $|x| = s$ ($s > 0$), minimize $\text{cap}(R(C_0, C_1))$.

The reflection of the Grötzsch ring in the unit sphere gives a Teichmüller type ring. This leads to a basic functional identity between their capacities [Vu1, Lemma 5.53]

$$(2.4) \quad \gamma_n(s) = 2^{n-1} \tau_n(s^2 - 1) \quad \text{for } s > 1.$$

2.5. Quasispheres. Quasispheres are images of a sphere under a quasiconformal homeomorphism of $\overline{\mathbb{R}^n}$. If the dimension $n = 2$, quasispheres are called quasicircles. One can also consider images of lower dimensional subspaces, for instance, quasiconformal arcs in space.

2.6. Quasisymmetric maps. Let $\eta: [0, \infty) \rightarrow [0, \infty)$ be an increasing homeomorphism which will play a role of a control function. Let D and D' be arbitrary sets in \mathbb{R}^n . A homeomorphism $f: D \rightarrow D'$ is η -quasisymmetric if, for all $a, b, x \in D$ ($b \neq x$),

$$(2.7) \quad \frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq \eta \left(\frac{|a - x|}{|b - x|} \right).$$

The mapping f is called quasisymmetric if it is η -quasisymmetric with some function η .

The definition is purely metric, in fact, quasisymmetry is an important notion in a general metric space. Most of the time, we shall consider quasisymmetric maps between domains in \mathbb{R}^n . It is well-known that quasisymmetric maps (defined on a domain) constitute a subclass of quasiconformal maps – indeed the linear dilatation

(as defined in 3.4) is bounded by $\eta(1)$. For homeomorphisms of the whole space these classes are in fact the same:

2.8. Theorem ([Vä2]). *A K -quasiconformal mapping of the whole space, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, is $\eta_{K,n}$ -quasisymmetric with a control function $\eta_{K,n}$ depending only on K and n .*

We will need explicit estimates with correct limit behavior for the control function $\eta_{K,n}$. In order to study this problem, let us introduce some notation. It is readily seen that the optimal control function is $\eta_{K,n}^*$ as defined by the following extremal problem. For $n \geq 2$, $1 \leq K < \infty$, $t \in [0, \infty)$, let

$$\eta_{K,n}^*(t) = \sup\{|f(x)|: |x| \leq t, f \in QC_K(\mathbb{R}^n), f(0) = 0, f(e_1) = e_1, f(\infty) = \infty\},$$

where $QC_K(\mathbb{R}^n)$ refers to K -quasiconformal homeomorphisms of \mathbb{R}^n .

We shall use the following distortion function,

$$\varphi_K(t) = \varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))}, \quad (0 < t < 1 \text{ and } K > 0).$$

The equality (2.4) tells us that

$$(2.9) \quad \tau_n^{-1}(K\tau_n(s)) = \frac{1}{\varphi_{K,n}\left(\sqrt{\frac{1}{s+1}}\right)^2} - 1 \quad (K > 0).$$

We recall the well-known estimates for $\varphi_{K,n}$.

2.10. Lemma ([Vu1, 7.47]). *For $K \geq 1$, let $\alpha = K^{1/(1-n)}$ and $\beta = 1/\alpha$. Then for all $t \in [0, 1]$*

$$\lambda_n^{1-\beta} t^\beta \leq \varphi_{1/K}(t) \leq t \leq \varphi_K(t) \leq \lambda_n^{1-\alpha} t^\alpha,$$

where λ_n is the so-called Grötzsch ring constant, $4 \leq \lambda_n \leq 2e^{n-1}$.

The constant λ_n appears in the limit behavior of the capacity of the Grötzsch ring $\gamma_n(s)$ as $s \rightarrow \infty$; for details see p. 169 of [AVV].

Vuorinen has shown that one can choose the function of quasisymmetry so that $\eta_{K,n}(t) \rightarrow t$ as $K \rightarrow 1$, with explicit estimates. The following theorem is a refined version of [Vu2].

2.11. Theorem. *For $K < K_0 = 4/3$ we have the following estimates*

$$\eta_{K,n}^*(t) \leq \begin{cases} 1 + c_0 \left((K-1) \log \frac{1}{K-1} \right), & t = 1, \\ \eta_{K,n}^*(1) \varphi_{K,n}(t), & 0 < t < 1, \\ \eta_{K,n}^*(1) / \varphi_{1/K,n}(1/t), & t > 1. \end{cases}$$

Here c_0 is a positive constant, e.g. $c_0 = 600$ will do.

The main part of the previous theorem is to estimate $\eta_{K,n}^*(1)$ which we will carry out in the next section.

3. QUASISYMMETRY

The purpose of this section is to present an explicit bound for the quasimetricity of a K -quasiconformal map with the correct limit behavior as $K \rightarrow 1$.

The following result is essentially the main result of [Vu2] with the improved estimates of [Se]. However, our approach is somewhat further improved and simplified compared to [Se], see the remarks following the proof. The form of $\eta_{K,n}(1)$ implies the estimate for $\eta_{K,n}^*(1)$ as stated in Theorem 2.11.

3.1. Theorem.

$$(3.2) \quad \eta_{K,n}^*(1) \leq \eta_{K,n}(1) = \begin{cases} \exp((4\sqrt{2} - \log(K-1))(K^2 - 1)), & 1 < K \leq 2, \\ \exp(4\sqrt{K}(K^2 - 1)), & K > 2. \end{cases}$$

Proof. Assume that $f \in QC_K(\mathbb{R}^n)$ with 0 and ∞ as fixed points. We have to estimate the ratio $|f(x)|/|f(y)|$ for arbitrary $|x| = |y| = 1$. Choose $z = -t\frac{y}{|y|}$, where $0 < t < 1$. Consider the ring domains $R = R([0, z], [y, \infty])$ and $R' = R'([0, f(z)], [f(x), \infty])$. The first ring is a Teichmüller type ring with capacity $\text{cap}(R) = \tau_n(|y|/|z|) = \tau_n(1/t)$. The capacity of its image $f(R)$ can be estimated by Teichmüller's extremal problem, $\text{cap}(f(R)) \geq \tau_n(|f(y)|/|f(z)|)$. Comparing these, we have

$$\frac{|f(y)|}{|f(z)|} \geq \tau_n^{-1}(K\tau_n(1/t)).$$

For the ring R' we estimate as in [Vu2], namely $\text{cap}(R') \leq \tau_n(|f(x)|/|f(z)| - 1)$ by a result of Gehring, while $\text{cap}(f^{-1}(R')) \geq \tau_n(|x|/|z|) = \tau_n(1/t)$ by Teichmüller's extremal problem. This gives

$$\frac{|f(x)|}{|f(z)|} \leq 1 + \tau_n^{-1}\left(\frac{1}{K}\tau_n(1/t)\right).$$

Combining with the previous bound we have

$$\frac{|f(x)|}{|f(y)|} \leq \frac{1 + \tau_n^{-1}\left(\frac{1}{K}\tau_n\left(\frac{1}{t}\right)\right)}{\tau_n^{-1}\left(K\tau_n\left(\frac{1}{t}\right)\right)} = \frac{\varphi_{K,n}(\sqrt{r})^2}{\varphi_{1/K,n}(\sqrt{r})^2(1 - \varphi_{K,n}(\sqrt{r})^2)},$$

where $r = t/(1+t)$ and we have used (2.9). The estimate is valid for all $0 < t < 1$, or equivalently, for all $0 < r < 1/2$. We use Lemma 2.10 to conclude that

$$\eta_{K,n}^*(1) \leq \lambda_n^{2(\beta-\alpha)} r^{\alpha-\beta} \frac{1}{1 - \lambda_n^{2(1-\alpha)} r^\alpha}.$$

We choose $r = (\lambda_n^{2(\alpha-1)}(K-1)/K)^\beta$, and then

$$\eta_{K,n}^*(1) \leq K\lambda_n^{2(\beta-\alpha)} \left(\lambda_n^{2(1-\beta)} \frac{(K-1)^\beta}{K^\beta} \right)^{\alpha-\beta} = \lambda_n^{2(\beta^2-1)} K^{\beta^2} (K-1)^{1-\beta^2}.$$

Use the inequality $\lambda_n^{1-\beta} \geq 2^{1-K} K^{-K}$ from [Vu1, Lemma 7.50] and the trivial $\beta \leq K$ to obtain

$$\begin{aligned} \eta_{K,n}^*(1) &\leq 4^{K^2-1} K^{3K^2+2K} (K-1)^{1-\beta^2} \\ &= \exp\left((K^2-1)\log 4 + (3K^2+2K)\log K + (1-\beta^2)\log(K-1)\right). \end{aligned}$$

To simplify the expression use $\log K \leq \frac{K-1}{\sqrt{K}}$ and $2K \leq 3K$

$$\begin{aligned} \eta_{K,n}^*(1) &\leq \exp\left((3K+1)\sqrt{K}(K-1) + 2\sqrt{K}(K-1)\right. \\ &\quad \left.+ (K^2-1)\log 4 + (1-\beta^2)\log(K-1)\right) \\ &= \exp\left((K^2-1)(3\sqrt{K} + \log 4) + (1-\beta^2)\log(K-1)\right). \end{aligned}$$

The desired inequality follows now, since $1 \leq \beta \leq K$ and $\log 4 < \sqrt{2}$.

3.3. Remark. We could further improve the estimates, if we choose r in the optimal way, so that $(K-1)\log(1/r) = r$. However, we do not include this here, since it provides only a slight improvement. On the other hand, it is expected that $\eta_{K,n}^*(1) = 1 + O(K-1)$ as $K \rightarrow 1$. For other open problems on the special function $\eta_{K,n}^*$, see [AVV].

For the rest of the paper, let us fix the notation for $\eta_{K,n}(1)$ as in (3.2). Observe that the bound does not depend on n , and thus we may use the notation $\eta_K(1)$, as well.

Seittenranta in [Se] adopted the technique above to estimate the linear dilatation of a quasiconformal map between subdomains of \mathbb{R}^n . We show here directly that it does not exceed $\eta_{K,n}^*(1)$, so automatically the same bound holds for the linear dilatation, as well. We will follow a normal family argument, closely related to [Se, Remark 5.2].

3.4. Definition. Let f be a homeomorphism between domains $D, D' \subset \mathbb{R}^n$. The *linear dilatation* of f at a point $x \in D$ is

$$H_f(x) = \limsup_{r \rightarrow 0^+} \frac{L_f(x, r)}{l_f(x, r)}.$$

Where

$$\begin{aligned} L_f(x, r) &= \sup_z \{|f(z) - f(x)| : |z - x| \leq r\}, \\ l_f(x, r) &= \inf_z \{|f(z) - f(x)| : |z - x| \geq r\}, \end{aligned}$$

$x \in D$, and $\bar{B}(x, r) \subset D$.

We introduce the following notation

$$H(K, D) = \sup\{H_f(x) : x \in D, f: D \rightarrow D' \subset \mathbb{R}^n \text{ is } K\text{-quasiconformal}\}.$$

3.5. Theorem. *With the notation above*

$$H(K, D) \leq \eta_{K,n}^*(1).$$

Proof. Let $f: D \rightarrow D'$ be a K -quasiconformal homeomorphism and $x_0 \in D$ be arbitrary. We need to show that $H_f(x_0) \leq \eta_{K,n}^*(1)$. By definition of $H_f(x_0)$, there exists a sequence $r_i \in (0, d(x_0, \partial D)/i)$ and points y_i and z_i such that $|y_i - x_0| =$

$|z_i - x_0| = r_i$ and $|f(y_i) - f(x_0)|/|f(z_i) - f(x_0)| \rightarrow H_f(x_0)$. There exist similarities α_i and β_i such that $\beta_i(0) = x_0$, $\beta_i(e_1) = z_i$ and $\alpha_i(f(x_0)) = 0$, $\alpha_i(f(z_i)) = e_1$. Then the mapping $g_i = \alpha_i \circ f \circ \beta_i: D_i \rightarrow g_i(D_i) \subset \mathbb{R}^n$, where $D_i = \beta_i^{-1}(D)$, is K -quasiconformal, $\bar{B}^n(0, i) \subset D_i$, $g_i(0) = 0$, $g_i(e_1) = e_1$, and for $w_i = \beta_i^{-1}(y_i)$, $|w_i| = 1$ and $|g_i(w_i)| = |\alpha_i(f(y_i))| \rightarrow H_f(x_0)$. The functions $\{g_i|_{B^n(0, k)}\}_{i=k}^\infty$ form a normal family (see [Vä1, 19.4, 20.5]) for every $k \in \mathbb{N}$, so we can find subsequences $\{g_i^k\}$ such that $\{g_i^{k+1}\}$ is a subsequence of $\{g_i^k\}$ and $\{g_i^k\}$ converges to a K -quasiconformal map h_k in $B^n(0, k)$ (see [Vä1, 21.3, 37.4]). From the construction we see that $h_i|_{B^n(0, k)} = h_k$ for $i \geq k$. Thus we have a globally defined K -quasiconformal map $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and there is a subsequence of $\{g_i\}$ which converges to h uniformly in $\bar{B}^n(0, 1)$. For this subsequence, we conclude that

$$H_f(x_0) = \limsup_{i \rightarrow \infty} |g_i(w_i)| \leq \limsup_{i \rightarrow \infty} |h(w_i)| \leq \eta_{K,n}^*(1),$$

since $h(0) = 0$, $h(e_1) = e_1$ and $|w_i| = 1$.

3.6. Remark. The *metric definition* of quasiconformality (see eg. [Vä1]) requires a uniform bound for the linear dilatation, i.e.

$$H_f(x) \leq H < \infty.$$

Theorem 3.5 provides an explicit, asymptotically sharp bound as $K \rightarrow 1$ for this H in terms of the geometric dilatation K .

4. FLATNESS PROPERTIES

Although quasicircles need not be rectifiable, they become more and more flat as $K \rightarrow 1$. This flatness property appears uniformly at all (small) scales and locations. There are several concepts in the literature to express this intuitive notion.

4.1. LAP property. Mattila and Vuorinen introduced the *linear approximation property* [MV].

Let $0 \leq \delta < 1$. We say that a closed set E in \mathbb{R}^n satisfies the d -dimensional δ -linear approximation property (δ -LAP) if there is an $r_0 > 0$ such that for each $x \in E$ and for each $0 < r < r_0$ there exists a d -dimensional affine subspace V through x such that

$$E \cap B^n(x, r) \subset V(\delta r).$$

Here $V(r)$ denotes the r -neighborhood of V ; $V(r) = \{x : d(x, V) < r\}$.

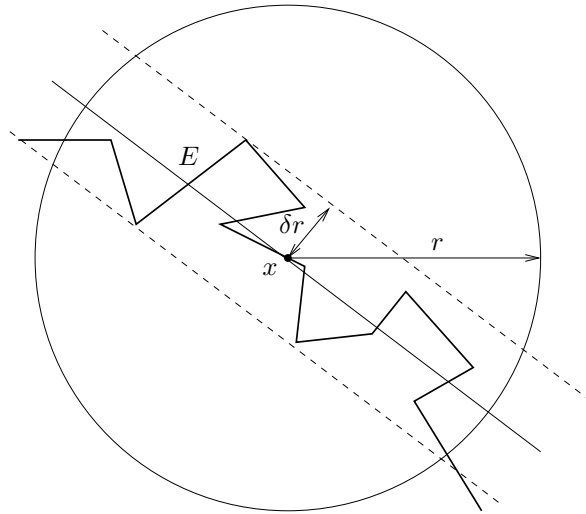


FIGURE 1. LAP property.

We will see that K -quasispheres satisfy $(n - 1)$ -dimensional δ -LAP property with parameter $\delta = \delta(K) \rightarrow 0$ as K tends to 1. On the other hand, LAP property implies a bound on the Hausdorff dimension of the set.

4.2. Theorem ([MV]). *There is a positive number δ_0 depending only on d and n such that if a set $E \subset \mathbb{R}^n$ has the d -dimensional δ -LAP property and $0 < \delta < \delta_0$, then*

$$\dim E \leq d + c(d)\delta^2.$$

Examples of snowflake curves in [MV] show the sharpness of the inequality above for $d = 1$.

4.3. Remark. Actually, Theorem 4.2 holds true with Hausdorff dimension replaced by (upper) Minkowski dimension, where only coverings with sets of equal size are taken into account. However, we shall formulate our results only for Hausdorff dimension.

4.4. Thickness. An opposite property to the linear approximation property is the thickness property of [VVW].

Let $q > 0$ and $1 \leq d \leq n$ an integer. A closed set $E \subset \mathbb{R}^n$ is (q, d) -thick if there exists an $r_0 > 0$ such that, for all $x \in E$ and $0 < r < r_0$ one can find a d -simplex with vertices in $E \cap B^n(x, r)$ and with d -volume $\geq qr^d$.

As mentioned above, this expresses an opposite property to the LAP property; the set E can be nowhere well approximated by $(d - 1)$ -dimensional planes.

4.5. Jones' β 's. In the same year as [MV] appeared, P. Jones [Jo] introduced the so-called β -parameters for the investigation of the "traveling salesman problem". We are going to relate the above properties to these β 's. The definition given here is from [Da].

Let an integer dimension $d < n$ be given, and let \mathcal{V} be the set of affine planes of dimension d . We introduce the following numbers to measure the uniform flatness of E :

$$\beta_{E,d}(x, r) = \inf_{V \in \mathcal{V}} \sup \left\{ \frac{1}{r} d(y, V) : y \in E \cap B(x, r) \right\}$$

for $x \in \mathbb{R}^n$ and $r > 0$.

LAP property says that the set is uniformly flat – the β 's are uniformly small, while thickness says that the set is uniformly non-flat – the β 's are uniformly big. More precisely, one can relate the corresponding parameters in the following way.

$$\begin{array}{l} d\text{-dim } \delta\text{-LAP} \\ \delta(\beta_0) = 2\beta_0 \end{array} \iff \begin{array}{l} d\text{-dim } \beta\text{'s } \leq \beta_0 \text{ (} x \in E, 0 < r < r_0 \text{)} \\ \beta_0(\delta) = \delta \end{array}$$

$$\begin{array}{l} (q, d)\text{-thickness} \\ q(\beta_0) = c(d)\beta_0^d \end{array} \iff \begin{array}{l} (d-1)\text{-dim } \beta\text{'s } \geq \beta_0 \text{ (} x \in E, 0 < r < r_0 \text{)} \\ \beta_0(q) = c(d)q \end{array}$$

As one would naturally expect, uniform non-flatness implies a bound on the dimension of connected sets. Based on Jones' "traveling salesman theorem" Bishop and Jones [BJ] proved the following.

4.6. Theorem ([BJ]). *If for a closed, connected set $E \subset \mathbb{R}^2$ there exists an $r_0 > 0$ such that $\beta_{E,1}(x, r) \geq \beta_0 > 0$ for every $x \in E$ and $0 < r < r_0$, then $\dim E \geq 1 + c\beta_0^2$, where $c > 0$ is an absolute constant.*

In 2004, David [Da] generalized this result to dimension d in \mathbb{R}^n . One also needs a d -dimensional topological nondegeneracy condition, such as connectivity in Bishop and Jones' theorem for $d = 1$. For instance the following Condition B of Semmes implies such a nondegeneracy condition for the one codimensional case.

4.7. Definition. Let E be a closed set in \mathbb{R}^n . We say that E satisfies Condition B (locally) if there are constants $0 < \alpha < 1/2$ and $r_0 > 0$ such that, for all choices of $x \in E$ and $0 < r < r_0$, we can find two balls B_1 and B_2 contained in $B(x, r) \setminus E$, of radius αr , and such that B_1 and B_2 lie in different components of $B(x, r) \setminus E$.

In our terminology David's theorem reads as follows (for $d = n - 1$).

4.8. Theorem ([Da]). *If a closed set $E \subset \mathbb{R}^n$ is (q, n) -thick and satisfies Condition B, then*

$$\dim E \geq d^* > n - 1,$$

for some d^* that depends only on n, q , and α .

It is well-known that quasispsheres satisfy Condition B of Semmes, for a quantitative statement see [MV, Lemma 5.8]. In fact, Condition B and the $(n - 1)$ -dimensional LAP property are closely related, when α is close to its maximum value $1/2$. The proof of the LAP property in [MV] is based on this relation, and the result is the following.

4.9. Theorem ([MV]). *A K -quasisphere satisfies the $(n - 1)$ -dimensional linear approximation property with parameter $\delta = 4\sqrt{1 - 2t}$, where $2t = \eta_{K,n}(1)^{-2}$, for all $1 < K < K_0$. Here K_0 must be chosen sufficiently close to 1, namely to satisfy $\eta_{K_0,n}(1) < \sqrt{16/15}$.*

4.10. Remark. Since quasispheres satisfy Condition B with $\alpha = \alpha(K)$, a (q, n) -thick K -quasisphere has Hausdorff dimension $\geq d^*(q, n, K) > n - 1$ by Theorem 4.8. This result provides an answer to a question by Vuorinen [Vu3, Problem 1.52]. The existence of thick quasispheres is highly nontrivial, it follows from the work of [DT].

5. QUASISPHERES SATISFY THE LAP PROPERTY

The authors of [MV] proved that K -quasispheres satisfy $(n - 1)$ -dimensional δ -LAP property with $\delta = \delta(K) \rightarrow 0$ as K tends to 1 (Theorem 4.9). However, in higher dimensional case ($n \geq 3$), this parameter δ tends to zero rather slowly, and cannot be considered satisfactory as it does not allow proving better bounds on the dimension than the linear one which immediately follows from the Hölder continuity [GV] (cf. (1.1)).

We prove a stronger LAP property in terms of quasiasymmetry of f , rather than its dilatation K . The best-known bound on the quasiasymmetry function from Section 3 then gives an almost quadratic asymptotic bound on the dimension of quasispheres. Our estimates make sense in the limiting case $K \rightarrow 1$, and therefore we often will assume that K is sufficiently close to 1, $K < K_0$. One can choose for instance $K_0 = 4/3$ in the rest of this section.

5.1. Theorem. *Let $1 < K < K_0$ be a small dilatation and let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal homeomorphism of \mathbb{R}^n . Then the image of a hyperplane H satisfies the $(n - 1)$ -dimensional δ -LAP property with $\delta = \delta(K) = O(\eta_K(1) - 1)$.*

Proof. Consider a normalized K -quasiconformal map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f(0) = 0$, $f(e_1) = e_1$. Let V be the hyperplane orthogonal to $[0, e_1]$ through $e_1/2$. Let $A = B(e_1/2, \sqrt{3}/2) \cap V$, so $A = B(0, 1) \cap V$. We can see from the quasiasymmetry property that f cannot move A far away from V . A point $x \in A$ satisfies $1/2 \leq |x| = t = |x - e_1| \leq 1$. Thus for the image $y = f(x)$ one has by (2.7)

$$1/\eta_K(1/t) \leq |y| \leq \eta_K(t), \quad 1/\eta_K(1/t) \leq |y - e_1| \leq \eta_K(t).$$

Then by Theorem 2.11 we have

$$(5.2) \quad \begin{aligned} 2d(y, V) &= \left| |y|^2 - |y - e_1|^2 \right| \leq \eta_K^2(t) - 1/\eta_K^2(1/t) \\ &\leq \eta_K^2(1)\varphi_K^2(t) - \varphi_{1/K}^2(t)/\eta_K^2(1). \end{aligned}$$

The estimates of Lemma 2.10 provide $\varphi_K(t) = t + O(K - 1)$, $\varphi_{1/K}(t) = t + O(K - 1)$ uniformly in $t \in [1/2, 1]$. Continuing (5.2), we obtain

$$(5.3) \quad 2d(y, V) \leq t^2 (\eta_K^2(1) - 1/\eta_K^2(1)) + O(K - 1) = O(\eta_K(1) - 1).$$

Now we show that the translated hyperplane $V' = V + (f(e_1/2) - e_1/2)$, which passes through $f(e_1/2)$, approximates the set fV well in the ball $B(f(e_1/2), \sqrt{3}/8)$. Assuming $f(x) \in B(f(e_1/2), \sqrt{3}/8)$ for $x \in V$, we have

$$|f(x) - f(e_1/2)| \leq \sqrt{3}/8 \leq \sqrt{3}/2 |f(e_1/2) - f(0)|,$$

since $|f(e_1/2) - f(0)| \geq 1/4$ provided that K is close enough to 1. By quasisymmetry, $|x - e_1/2| \leq \eta_K(\sqrt{3}/2)/2 < \sqrt{3}/2$ for $K < K_0$. This means that $x \in A$ and from (5.3) it follows that

$$d(f(x), V') \leq d(f(x), V) + d(f(e_1/2), V) = \delta(K)/(\sqrt{3}/8) = O(\eta_K(1) - 1).$$

We obtain the δ -LAP property of fH by rescaling f at an arbitrary point x of H for all scales, so that x corresponds to $e_1/2$ and H to V . This yields to a scale-free LAP property, that is, we can take $r_0 = +\infty$ in the definition.

5.4. Corollary. *For a K -quasisphere E in \mathbb{R}^n with $1 < K < K_0$ we have*

$$\dim E = n - 1 + O((\eta_K(1) - 1)^2) = n - 1 + O\left((K - 1)^2 \log^2 \frac{1}{K - 1}\right).$$

Proof. Combine theorems 2.11, 5.1 and 4.2.

5.5. Remarks. Theorem 5.1 and Corollary 5.4 provide partial answers to questions [Vu3, 1.41, 1.42]. Note that for $n = 2$, the optimal quasisymmetry function satisfies $\eta_{K,2}^*(1) - 1 = O(K - 1)$, thus Corollary 5.4 gives the correct, quadratic order for quasicircles in the plane (c.f. Theorem A.4). See [AVV, Theorem 10.33] for an explicit estimate: $\eta_{K,2}^*(1) \leq \exp(a(K - 1))$, with a constant $a \leq 4.38$.

As the above shows, a K -quasiarc in the plane satisfies the δ -LAP property with $\delta = c(K - 1)$, and thus for the thickness parameter the same bound has to hold. For a $(q, 2)$ -thick K -quasiarc in \mathbb{R}^2 , $q \leq c(K - 1)$. This is an improvement of [VWV, Theorem 5.9], and essentially sharp as it is shown there.

One can prove a similar result for quasiconformal images of lower dimensional subspaces.

5.6. Theorem. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal homeomorphism of \mathbb{R}^n with small dilatation ($1 < K < K_0$). Then the image of a d -dimensional plane H satisfies the d -dimensional δ -LAP property with $\delta = \delta(K) = O(\eta_K(1) - 1)$.*

Proof. Represent H as the intersection of $n - d$ pairwise orthogonal hyperplanes, $H = \bigcap_{i=1}^{n-d} V_i$. We use the one codimensional linear approximation property (Theorem 5.1) for the sets fV_i , $i = 1, \dots, n - d$, where the hyperplane V'_i approximates the set fV_i in a neighborhood of $x \in H$. The approximating d -plane will be the intersection $\bigcap_{i=1}^{n-d} V'_i$. To assure that this is indeed a d -plane and that the parameter $\delta(K)$ can be chosen $O(\eta_K(1) - 1)$ note that the hyperplanes V'_i are almost orthogonal to each other as $K \rightarrow 1$. This is an immediate consequence of the quasisymmetry property in Theorem 2.11.

5.7. Corollary. *For a d -dimensional K -quasisphere E in \mathbb{R}^n ($1 < K < K_0$)*

$$\dim E = d + O\left((K-1)^2 \log^2 \frac{1}{K-1}\right).$$

5.8. Remarks. Following an idea in [VVW, p. 141] one can construct a K -quasi-conformal self-homeomorphism of \mathbb{R}^n which maps a line segment to an arc of Hausdorff dimension $\geq 1 + c_n(K-1)^2$. The construction is given by a polyhedral building block via an iterative process.

We expect that the optimal quasismetry function satisfies $\eta_{K,n}^*(1) = 1 + O(K-1)$ (as this is the case for $n = 2$). This would allow us to remove the \log^2 terms in the dimension estimates of Corollary 5.4 and 5.7.

Although LAP property is not bilipschitz invariant, we do have invariance even for quasiconformal maps provided the dilatation is close to 1. This kind of quasiconformal invariance was studied in [VVW] for the thickness property. In the following we concentrate on the one codimensional case, $d = n - 1$.

5.9. Theorem. *Assume that a closed set E has the $(n-1)$ -dimensional δ -LAP property. Then the image fE under a K -quasiconformal map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $1 \leq K < K_0$, satisfies the δ_1 -LAP property, where $\delta_1 = \delta_1(\delta, K) \rightarrow \delta$ as $K \rightarrow 1$.*

Proof. The proof will be similar to the proof of Theorem 5.1. Consider a normalized K -quasiconformal mapping and fix the same notation for the hyperplane V as in the proof of Theorem 5.1. Assume that for a point x we have $\max(|x|, |x - e_1|) \leq 1$, $d(x, V) \leq \varepsilon$. We want to show that $y = f(x)$ is also close to V . We may assume that $|x| \leq |x - e_1|$. Then by the η_K -quasismetry of f ,

$$\begin{aligned} 2d(y, V) &= \left| |y|^2 - |y - e_1|^2 \right| \leq \eta_K^2(|x - e_1|) - 1/\eta_K^2(1/|x|) \\ &\leq \eta^2 \varphi_K^2(|x - e_1|) - \varphi_{1/K}^2(|x|)/\eta^2, \end{aligned}$$

where $\eta = \eta_K(1)$. Continuing this by the estimates of φ_K , we have

$$\begin{aligned} 2d(y, V) &\leq \eta^2|x - e_1|^2 - 1/\eta^2|x|^2 + O(K-1) \\ &= 1/\eta^2(|x - e_1|^2 - |x|^2) + (\eta^2 - 1/\eta^2)|x - e_1|^2 + O(K-1) \\ &\leq 2d(x, V) + O(\eta - 1) \leq 2\varepsilon + O(\eta - 1). \end{aligned}$$

We obtained that $d(y, V) \leq \varepsilon_1$, with $\varepsilon_1 \rightarrow \varepsilon$ as $K \rightarrow 1$. Renormalizing to all scales as in Theorem 5.1 we get δ_1 -LAP property in a ball with center $e_1/2$. Here $\delta_1 = \delta + O(\eta - 1)$.

APPENDIX A. PLANAR RESULTS

The Beltrami equation and the measurable Riemann mapping theorem lends a special flavor to the planar theory. Optimal regularity and distortion results are known in this case. In this section, we record some of these, in order to provide a comparison to our results discussed earlier.

The celebrated result of Astala provides the optimal higher integrability.

A.1. Theorem ([As]). *If f is a K -quasiconformal mapping in a domain $D \subset \mathbb{C}$ then $f \in W_{loc}^{1,p}(D)$ for all $p < \frac{2K}{K-1}$.*

Closely related to the above, he also gave a description of the distortion of Hausdorff dimension.

A.2. Theorem ([As]). *Let $f: D \rightarrow D'$ be K -quasiconformal in the plane and suppose $E \subset D$ is compact. Then*

$$(A.3) \quad \frac{1}{K} \left(\frac{1}{\dim(E)} - \frac{1}{2} \right) \leq \frac{1}{\dim(f(E))} - \frac{1}{2} \leq K \left(\frac{1}{\dim(E)} - \frac{1}{2} \right).$$

This inequality is best possible.

For dimension of quasicircles the following improved bounds hold.

A.4. Theorem ([BP]). *For a K -quasicircle E ,*

$$\dim E \leq 1 + 37(K-1)^2/(K+1)^2 \leq 1 + 10(K-1)^2.$$

If K is close to one, then there is a K -quasicircle E with

$$\dim E \geq 1 + 0.09(K-1)^2/(K+1)^2.$$

As we noted in Remark 5.5, the qualitative upper bound also follows from our considerations. Exact bounds for the dimension of quasicircles is of particular importance, and in this direction S. Smirnov proved (unpublished) that the dimension of a K -quasicircle is less than or equal to $1 + k^2$, where $k = (K-1)/(K+1)$.

Let us mention that the formula A.3 is expected to generalize to higher dimensions in the following way.

A.5. Conjecture ([IM, 17.4.1]). *Let $f: D \rightarrow \mathbb{R}^n$ be a quasiconformal mapping and E a compact subset of D . Then*

$$\frac{1}{K_O} \left(\frac{1}{\dim(E)} - \frac{1}{n} \right) \leq \frac{1}{\dim(f(E))} - \frac{1}{n} \leq K_I \left(\frac{1}{\dim(E)} - \frac{1}{n} \right),$$

where K_O (K_I) is the outer (inner) dilatation of f .

Higher dimensional counterparts of the examples of extremal distortion in (A.3) shows that this would be optimal. This means that, generally, we can distort sets in linear order of $K-1$. Thus Corollaries 5.4 and 5.7 reveal that we have a significant improvement for distortion of spheres, a similar phenomenon to that of the planar theory.

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