DEFINITION AND PROPERTIES OF SUPERSOLUTIONS TO THE POROUS MEDIUM EQUATION

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ABSTRACT. We study a wide class of supersolutions of the porous medium equation. These supersolutions are defined as lower semicontinuous functions obeying the comparison principle. We show that they have a spatial Sobolev gradient and give sharp summability exponents. We also study pointwise behaviour.

1. INTRODUCTION

The porous medium equation

$$\Delta(u^m) = \frac{\partial u}{\partial t} \tag{1.1}$$

has been studied intensively during the last decades and the theory for its solutions is rather complete by now. Especially the slow diffusion case $m > 1$ has attracted the interest of many mathematicians, because disturbances propagate with finite speed and interfaces may appear. We refer to [12] and [11] for the theory of this fascinating equation. The objective of our work is to study a class of supersolutions, defined in an analogous way as in classical potential theory. The leading example with a singularity is the so-called Barenblatt solution, which is the fundamental solution of the porous medium equation.

The supersolutions that we have in mind are defined as lower semicontinuous functions obeying the parabolic comparison principle with respect to solutions. For lack of a better name, we have taken ourselves the liberty to call these pointwise defined functions viscosity supersolutions, thus distinguishing them from the ordinary supersolutions. In the stationary case the viscosity supersolutions v are exactly characterized by the property that the power v^m is a superharmonic function, defined as in classical potential theory. In the case $m = 1$ the equation reduces to the heat equation and we have the supercaloric functions.

Ordinary supersolutions are weak solutions of the inequality

$$\Delta(u^m) \le \frac{\partial u}{\partial t}$$

defined in the usual way with the test functions under the integral sign. Thus they are called weak supersolutions. Belonging, by definition, to a parabolic Sobolev space, they are more tractable, when it comes to a priori estimates. The reader should carefully distinguish between *viscosity supersolutions* (Definition 3.1) and *weak supersolutions* (Definition 2.1). To this we may add that an even more restricted class of supersolutions has been treated in [9]. Among those it is the viscosity supersolutions that form a good class, closed under monotone convergence.

The most important example is a celebrated function found by Barenblatt [3] and Zel'dovich and Kompaneets [13]. It has the formula

$$\mathcal{B}_m(x,t) = \begin{cases} t^{-\lambda}\left(C - \frac{\lambda(m-1)}{2mN}\frac{|x|^2}{t^{2\lambda/N}}\right)_+^{1/(m-1)}, & t > 0, \\ 0, & t \le 0, \end{cases} \qquad (1.2)$$

where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_N^2$ and

$$\lambda = \frac{N}{N(m-1)+2}.$$

The constant $C > 0$ is at our disposal and $m > 1$. Here $f_+ = \max(f, 0)$ is the positive part of f. As $m \to 1+$ we can obtain the heat kernel. Notice the interface (free boundary), having the equation

$$t = c|x|^{N(m-1)+2}.$$

The function $\mathcal{B}_m$ is, indeed, a weak solution when $t > 0$, but the singularity at the origin prevents $\mathcal{B}_m$ from being a solution in $\mathbb{R}^N \times \mathbb{R}$. Strictly speaking, it is not even a weak supersolution because

$$\int_{-1}^{1} \int_{|x|<1} |\nabla \mathcal{B}_m^m(x,t)|^2 \, dx \, dt = \infty,$$

violating the a priori summability in the definition. However, the function $\mathcal{B}_m$ is a viscosity supersolution in the whole space $\mathbb{R}^N \times \mathbb{R}$. Needless to say, a definition that would exclude the Barenblatt solution cannot be regarded as satisfactory. As a matter of fact, the Barenblatt solution is extrem in many ways. We will utilize it to show that some results are sharp.

Our first result is that locally bounded viscosity supersolutions are weak supersolutions. Also the converse statement is true, provided the issue of semicontinuity is properly handled. We establish the existence of the spatial gradient $\nabla(|v|^{m-1}v)$ in Sobolev's sense. Nothing like this holds

for the time derivative, which is merely a distribution. For example, the function

$$v(x,t) = \begin{cases} 1, & t > 0, \\ 0, & t \le 0, \end{cases}$$

is a viscosity supersolution. Dirac's delta function appears in the time derivative. More generally, all functions of the type $v(x,t) = g(t)$ are viscosity supersolutions, if $g(t)$ is a lower semicontinuous increasing function. -We have the following theorem.

Theorem 1.3. *Let $m \ge 1$. Suppose that v is a locally bounded viscosity supersolution in $\Omega \subset \mathbb{R}^{N+1}$. Then the Sobolev derivatives*

$$\frac{\partial(|v|^{m-1}v)}{\partial x_i}, \quad i = 1, 2, \dots, N,$$

exist and the local summability

$$\int_{t_1}^{t_2} \int_D |\nabla(|v|^{m-1}v)|^2 \, dx \, dt < \infty$$

holds for each $D \times (t_1, t_2) \Subset \Omega$. Moreover, we have

$$\int_{t_1}^{t_2} \int_D \left(\nabla(|v|^{m-1}v) \cdot \nabla\varphi - v\frac{\partial\varphi}{\partial t} \right) dx \, dt \ge 0$$

whenever $\varphi \in C_0^\infty(D \times (t_1, t_2))$ and $\varphi \ge 0$.

The proof is based on a delicate approximation procedure. The approximants are constructed as solutions of auxiliary variational inequalities coming from a sequence of obstacle problems. The obstacles are smooth functions approximating the original function pointwise from below.

For unbounded viscosity supersolutions we can extract information by applying the theorem to the truncated functions $\min(v(x,t), j)$, $j = 1, 2, \dots$, which are weak supersolutions. By an iterative procedure we obtain estimates which are independent of the level of truncation. The result is the theorem below.

Theorem 1.4. *Let $m \ge 1$. Suppose that v is a viscosity supersolution in $\Omega \subset \mathbb{R}^{N+1}$. Then $v \in L_{loc}^q(\Omega)$, whenever $0 < q < m + 2/N$. Moreover, the Sobolev derivatives*

$$\frac{\partial(|v|^{m-1}v)}{\partial x_i}, \quad i = 1, 2, \dots, N,$$

exist and the local summability

$$\int_{t_1}^{t_2} \int_D |\nabla(|v|^{m-1}v)|^q \, dx \, dt < \infty$$

holds for each $D \times (t_1, t_2) \Subset \Omega$, whenever $0 < q < 1 + 1/(1 + mN)$.

The Barenblatt solution shows that *the bounds for the exponents q are sharp* in the theorem above. There is a reason for using the mth power in the theorems. It appears even for solutions. For a nonnegative solution u it may happen that the derivative $\nabla(u^\alpha)$ does not exist in Sobolev's sense, if $0 < \alpha < (m-1)/2$. This is the case for the Barenblatt solution. For a viscosity supersolution we have the restrictions

$$\frac{m-1}{2} < \alpha < m + \frac{1}{N}$$

on the power α to guarantee that $\nabla(u^\alpha)$ exists in Sobolev's sense. This phenomenon is studied in the final section of the paper, where we give a Caccioppoli estimate in the above range of powers.

This phenomenon also causes a technical difficulty for the mollifications with respect to the time variable. We have to mollify $|v|^{m-1}v$ instead of v itself. We have found the convolution

$$f^*(x,t) = \frac{1}{\sigma} \int_0^t e^{(s-t)/\sigma} f(x,s)\, ds, \qquad \sigma > 0,$$

to be very useful, in particular because its time deivative $(f^*)_t$ has a convenient form. Some technical difficulties can be concentrated in an excess term which disappears from the final estimate. The excess term is zero for smooth functions. So far, we have not been able to find any other practical way to compensate for the missing time derivative.

We have also included a section about the fine properties. While weak supersolutions are defined only almost everywhere, a distict feature of the viscosity supersolutions is that they are defined at every point in their domain. Thus the pointwise behaviour can be investigated. A central result is that, at each point in its domain, a viscosity supersolution takes the value

$$v(x,t) = \operatorname*{ess\,lim\,inf}_{(y,\tau) \to (x,t), \tau < t} v(y,\tau).$$

This is the content of our theorem in section 6, which is an extension of Brelot's classical theorem for superharmonic functions. Here the notion of the essential limes inferior means that any set of $(N+1)$-dimensional Lebesgue measure zero can be neglected in the calculation of the lower limit.

Let us finally remark that we have deliberately decided to exclude the fast diffusion case $m < 1$. In the linear case $m = 1$ our results can be read off from linear representation formulas for the heat equation. In our case the principle of superposition is not available.

2. Preliminaries

This section contains some notation, definitions, and basic estimates. Also an interesting convolution is described.

In what follows, Q will always stand for a parallelepiped

$$Q = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N),$$

$a_i < b_i$, $i = 1, 2, \ldots, N$, in $\mathbb{R}^N$ and the abbreviations

$$Q_T = Q \times (0, T), \qquad Q_{t_1, t_2} = Q \times (t_1, t_2),$$

where $T > 0$ and $t_1 < t_2$, are used for the space-time boxes in $\mathbb{R}^{N+1}$. The *parabolic boundary* of Q_T is

$$\Gamma_T = (\overline{Q} \times \{0\}) \cup (\partial Q \times [0, T]).$$

Observe that the interior of the top $\overline{Q} \times \{T\}$ is not included. Similarly, Γ_{t_1, t_2} is the parabolic boundary of Q_{t_1, t_2}. The parabolic boundary of a space-time cylinder $D_{t_1, t_2} = D \times (t_1, t_2)$, where $D \subset \mathbb{R}^N$ is an open set, has a similar definition. In order to describe the appropriate function spaces, we recall that $H^1(Q)$ denotes the Sobolev space of functions $u \in L^2(Q)$ whose first distributional partial derivatives belong to $L^2(Q)$. The norm in $H^1(Q)$ is

$$\|u\|_{H^1(Q)} = \|u\|_{L^2(Q)} + \|\nabla u\|_{L^2(Q)}.$$

The Sobolev space with zero boundary values, denoted by $H_0^1(Q)$, is the completion of $C_0^\infty(Q)$ with respect to the norm $\|u\|_{H^1(Q)}$. We denote by $L^2(t_1, t_2; H^1(Q))$ the space of functions such that for almost every t, $t_1 \le t \le t_2$, the function $x \mapsto u(x, t)$ belongs to $H^2(Q)$ and

$$\int_{t_1}^{t_2} \int_Q \left(|u(x, t)|^2 + |\nabla u(x, t)|^2\right) dx\, dt < \infty.$$

Notice that the time derivative u_t is deliberately avoided. The definition for the space $L^2(t_1, t_2; H_0^1(Q))$ is analogous.

To be on the safe side we give the definition of the (super)solutions of the porous medium equation, interpreted in the weak sense.

Definition 2.1. Let Ω be an open set in $\mathbb{R}^{N+1}$ and suppose that $|u|^{m-1}u \in L^2(t_1, t_2; H^1(Q))$ whenever $Q_{t_1, t_2} \Subset \Omega$. Then u is called a *weak solution*, if

$$\int_{t_1}^{t_2} \int_Q \left(\nabla(|u|^{m-1}u) \cdot \nabla \varphi - u\frac{\partial \varphi}{\partial t}\right) dx\, dt = 0 \qquad (2.2)$$

whenever $Q_{t_1, t_2} \Subset \Omega$ and $\varphi \in C_0^\infty(Q_{t_1, t_2})$. Further, we say that u is a *weak supersolution*, if the integral (2.2) is non-negative for all $\varphi \in C_0^\infty(Q_{t_1, t_2})$ with $\varphi \ge 0$. If this integral is non-positive instead, we say that u is a *weak subsolution*.

Several remarks are related to the definition. The function $|u|^{m-1}u \in L^1_{\text{loc}}(\Omega)$ is called a *distributional solution*, if

$$\int_{t_1}^{t_2} \int_Q \left(|u|^{m-1}u\,\Delta\varphi + u\frac{\partial\varphi}{\partial t} \right) dx\,dt = 0$$

whenever $Q_{t_1,t_2} \Subset \Omega$ and $\varphi \in C_0^\infty(Q_{t_1,t_2})$. It is clear that every weak solution is a distributional solution. On the other hand, it can be shown that the converse is true as well, but the proof is more involved. The reverse heat equation is evoked in the proof. See [6] and [2].

By parabolic regularity theory the weak solutions are locally Hölder continuous, after a possible redefinition on a set of measure zero. *A continuous weak solution is called a solution.* Even the spatial gradient ∇u of a weak solution is locally Hölder continuous. See [4] and [11] for the regularity theory. We will not use the Hölder continuity of the gradient, but an intrinsic Harnack inequality proved by DiBenedetto is needed, see [4]. The time derivative u_t has to be avoided to a certain extent, because it does not necessarily exist in Sobolev's sense. A regularization will be used to overcome this default.

If the test function φ is required to vanish only on the lateral boundary $\partial Q \times [t_1, t_2]$, then the boundary terms

$$\int_Q u(x,t_1)\varphi(x,t_1)\,dx = \lim_{\sigma\to 0} \frac{1}{\sigma} \int_{t_1}^{t_1+\sigma} \int_Q u(x,t)\varphi(x,t)\,dx\,dt$$

and

$$\int_Q u(x,t_2)\varphi(x,t_2)\,dx = \lim_{\sigma\to 0} \frac{1}{\sigma} \int_{t_2-\sigma}^{t_2} \int_Q u(x,t)\varphi(x,t)\,dx\,dt$$

have to be included. A direct evaluation of the integrals on the left-hand side, without the limit procedure, may occasionally yield wrong values. In the presence of discontinuities we have to pay due attention to this notation. In the case of a weak supersolution to the porous medium equation the condition becomes

$$\int_{t_1}^{t_2} \int_Q \left(\nabla(|u|^{m-1}u) \cdot \nabla\varphi - u\frac{\partial\varphi}{\partial t} \right) dx\,dt$$
$$+ \int_Q u(x,t_2)\varphi(x,t_2)\,dx - \int_Q u(x,t_1)\varphi(x,t_1)\,dx \geq 0 \tag{2.3}$$

for almost all $t_1 < t_2$ with $Q_{t_1,t_2} \Subset \Omega$.

The following existence result for the local Cauchy problem will be useful for us later.

Theorem 2.4. *Let ψ be a continuous function on the parabolic boundary Γ_T of Q_T. Then there is a weak solution $u \in C(\overline{Q_T})$ of the porous medium equation in Q_T such that $u = \psi$ on Γ_T.*

The uniqueness of the solution of the Cauchy problem follows from the comparison principle below, see [2], [6] and [11].

Theorem 2.5 (Comparison Principle). *Let ψ_1 and ψ_2 be continuous functions on the parabolic boundary Γ_T of Q_T such that $0 \le \psi_1 \le \psi_2$. If $u \in C(\overline{Q_T})$ is a weak subsolution with $u = \psi_1$ on Γ_T and $v \in C(\overline{Q_T})$ is a weak supersolution with $v = \psi_2$ on Γ_T, then $u \le v$ in Q_T.*

There is a principal, well-recognized difficulty with the definition. Namely, in proving estimates we usually need a test function φ that depends on the solution itself, for example $\varphi = u\zeta$ where ζ is a smooth cutoff function. Then one cannot avoid that the "forbidden quantity" u_t shows up in the calculation of φ_t. In most cases one can easily overcome this complication by using an equivalent definition in terms of Steklov averages, as in chapter 2 of [11]. We have found the convolution

$$u^*(x,t) = \frac{1}{\sigma} \int_0^t e^{(s-t)/\sigma} u(x,s)\, ds, \qquad \sigma > 0, \qquad (2.6)$$

to be very useful, see page 36 in [10]. The notation hides the dependece on σ. The advantage is that no values of $u(x,t)$ outside $Q \times (0,T)$ are needed for the calculation of $u^*(x,t)$ in $Q \times (0,T)$. For continuous or bounded and semicontinuous functions u the averaged function u^* is defined at each point. We have

$$u^* + \sigma \frac{\partial u^*}{\partial t} = u. \qquad (2.7)$$

This implies the expedient fact that

$$(|u|^{m-1}u - |u^*|^{m-1}u^*)\frac{\partial u^*}{\partial t} \ge 0 \qquad (2.8)$$

when $m \ge 1$.

Some properties are listed in the following lemma.

Lemma 2.9. (i) *If $u \in L^p(Q_T)$, then*

$$\|u^*\|_{p,Q_T} \le \|u\|_{p,Q_T}$$

and

$$\frac{\partial u^*}{\partial t} = \frac{u - u^*}{\sigma} \in L^p(Q_T).$$

Moreover, $u^* \to u$ in $L^p(Q_T)$ as $\sigma \to 0$.

(ii) *If, in addition, $\nabla u \in L^p(Q_T)$, then $\nabla(u^*) = (\nabla u)^*$ componentwise,*

$$\|\nabla u^*\|_{p,Q_T} \le \|\nabla u\|_{p,Q_T},$$

and $\nabla u^* \to \nabla u$ in $L^p(Q_T)$ as $\sigma \to 0$.

(iii) *Furthermore, if $u_k \to u$ in $L^p(Q_T)$, then also*

$$u_k^* \to u^* \qquad and \qquad \frac{\partial u_k^*}{\partial t} \to \frac{\partial u^*}{\partial t}$$

in $L^p(Q_T)$.

(iv) *If $\nabla u_k \to \nabla u$ in $L^p(Q_T)$, then $\nabla u_k^* \to \nabla u^*$ in $L^p(Q_T)$.*

(v) *Analogous results hold for weak convergence in $L^p(Q_T)$.*

(vi) *Finally, if $\varphi \in C(\overline{Q_T})$, then*

$$\varphi^*(x,t) + e^{-t/\sigma}\varphi(x,0) \to \varphi(x,t)$$

uniformly in Q_T as $\sigma \to 0$.

Proof. The proof is rather straightforward and we leave it as an exercise. See [10] and [8]. $\qquad\square$

The averaged equation for a weak supersolution u in Ω is the following. If $\overline{Q_T} \subset \Omega$, then

$$\int_0^T \int_Q \left(\nabla(|u|^{m-1}u)^* \cdot \nabla\varphi - u^* \frac{\partial\varphi}{\partial t} \right) dx\,dt$$
$$+ \int_Q u^*(x,T)\varphi(x,T)\,dx \qquad (2.10)$$
$$\geq \int_Q u(x,0)\left(\frac{1}{\sigma} \int_0^T \varphi(x,s)e^{-s/\sigma}\,ds \right) dx$$

for all test functions $\varphi \geq 0$ vanishing on the lateral boundary $\partial Q \times [0,T]$ of Q_T. The reader can easily verify that we obtain (2.3) as $\sigma \to 0$. The averaged equation can also be written as

$$\int_0^T \int_Q \left(\nabla(|u|^{m-1}u)^* \cdot \nabla\varphi + \varphi \frac{\partial u^*}{\partial t} \right) dx\,dt$$
$$\geq \int_Q u(x,0)\left(\frac{1}{\sigma} \int_0^T \varphi(x,s)e^{-s/\sigma}\,ds \right) dx. \qquad (2.11)$$

By approximation this is valid for all non-negative $\varphi \in L^2(0,T;H_0^1(Q))$. For positive weak supersolutions many a priori estimates can be derived from the simpler inequality

$$\int_0^T \int_Q \left(\nabla(u^m)^* \cdot \nabla\varphi + \varphi \frac{\partial u^*}{\partial t} \right) dx\,dt \geq 0 \qquad (2.12)$$

valid for all non-negative $\varphi \in L^2(0,T;H_0^1(Q))$. We point out that (2.11) and (2.12) hold without any assumption about φ_t.

The following lemma contains a Caccioppoli type estimate. For the reader's convenience, we give a proof of this well-known result.

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Lemma 2.13 (Caccioppoli)**.** *Let $|u|^{m-1}u \in L^2(0,T; H^1_0(Q))$ and suppose that $|u| \le M$ in Q_T. If u is a weak supersolution, then*

$$\int_0^T \int_Q \zeta^2 |\nabla(|u|^{m-1}u)|^2 \, dx \, dt$$

$$\le 16M^{2m}T \int_Q |\nabla\zeta|^2 \, dx + 6M^{m+1} \int_Q \zeta^2 \, dx,$$

where ζ depends only on x, $\zeta \in C_0^\infty(Q)$ and $\zeta \ge 0$.

Proof. In the averaged equation (2.11) we use the test function

$$\varphi = (M^m - |u|^{m-1}u)\zeta^2.$$

The crucial integral containing $\dfrac{\partial u^*}{\partial t}$ can be written as

$$\int_0^T \int_Q \varphi \frac{\partial u^*}{\partial t} \, dx \, dt$$

$$= M^m \int_Q \zeta^2(x) u^*(x,T) \, dx - \int_0^T \int_Q \zeta^2 |u|^{m-1}u \frac{\partial u^*}{\partial t} \, dx \, dt$$

$$= M^m \int_Q \zeta^2(x) u^*(x,T) \, dx$$

$$- \int_0^T \int_Q \zeta^2 \big(|u|^{m-1}u - |u^*|^{m-1}u^*\big) \frac{\partial u^*}{\partial t} \, dx \, dt$$

$$- \int_0^T \int_Q \zeta^2 |u^*|^{m-1}u^* \frac{\partial u^*}{\partial t} \, dx \, dt.$$

Now we have come to a decisive point. In the second integral on the right-hand side

$$\big(|u|^{m-1}u - |u^*|^{m-1}u^*\big)\frac{\partial u^*}{\partial t} = \big(|u|^{m-1}u - |u^*|^{m-1}u^*\big)\frac{u - u^*}{\sigma} \ge 0$$

since both factors have the same sign (recall that $m \ge 1$), see (2.7) and (2.8). It follows that

$$\int_0^T \int_Q \varphi \frac{\partial u^*}{\partial t} \, dx \, dt \le \int_Q \zeta^2(x)\Big(M^m u^*(x,T) - \frac{|u^*(x,T)|^{m+1}}{m+1}\Big) \, dx$$

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and hence we obtain an estimate free of the time derivative $\dfrac{\partial u^*}{\partial t}$. Taking this estimate into account and letting $\sigma \to 0$, we can write (2.11) as

$$\int_0^T \int_Q \nabla(|u|^{m-1}u) \cdot \nabla \varphi \, dx \, dt$$

$$+ \int_Q \zeta^2(x) \Big(M^m u(x,T) - \frac{|u(x,T)|^{m+1}}{m+1} \Big) dx$$

$$\geq \int_Q \zeta^2(x) \big(M^m u(x,0) - |u(x,0)|^{m+1} \big) dx$$

after some simplification. The two single integrals are not symmetric! A deviation when $t = 0$ is due to the omission of the term $e^{-t/\sigma} u(x,0)$ in the definition of $u^*(x,t)$, see (vi) in Lemma 2.9. A simple estimation of the two single integrals leads to

$$- \int_0^T \int_Q \nabla(|u|^{m-1}u) \cdot \nabla \varphi \, dx \, dt \leq 3 M^{m+1} \int_Q \zeta^2 \, dx.$$

In the elliptic term we write

$$-\nabla(|u|^{m-1}u) \cdot \nabla \varphi$$
$$= \zeta^2 |\nabla(|u|^{m-1}u)|^2 - 2\zeta \nabla(|u|^{m-1}u) \cdot (M^m - |u|^{m-1}u)\nabla\zeta.$$

The first term on the right-hand side is of the desired type. We estimate the second term on the right-hand side. The elementary inequality $2ab \leq \varepsilon^2 a^2 + \varepsilon^{-2} b^2$ gives

$$2 \int_0^T \int_Q |\zeta \nabla(|u|^{m-1}u) \cdot (M^m - |u|^{m-1}u)\nabla\zeta| \, dx \, dt$$

$$\leq \varepsilon^2 \int_0^T \int_Q \zeta^2 |\nabla(|u|^{m-1}u)|^2 \, dx \, dt$$

$$+ \varepsilon^{-2} \int_0^T \int_Q (M^m - |u|^{m-1}u)^2 |\nabla\zeta|^2 \, dx \, dt$$

$$\leq \varepsilon^2 \int_0^T \int_Q \zeta^2 |\nabla(|u|^{m-1}u)|^2 \, dx \, dt + \varepsilon^{-2}(2M^m)^2 T \int_Q |\nabla\zeta|^2 \, dx.$$

We choose $\varepsilon = 1/\sqrt{2}$ so that the first term on the right-hand side can be absorbed (the so-called Peter-Paul Principle). The result follows.
□

It will be crucial for us to be able to move from from one moment of time to another. The following estimate connects the future to the past. An excess term appears.

Theorem 2.14. *Let v be a weak supersolution in $\Omega \subset \mathbb{R}^{N+1}$ and $Q_{t_1,t_2} \Subset \Omega$. Suppose that $v \geq 0$ and $v^m \in L^2(t_1, t_2; H_0^1(Q))$. Then*

$$\frac{1}{m+1} \int_Q v(x, t_2)^{m+1}\, dx - \frac{1}{m+1} \int_Q v(x, t_1)^{m+1}\, dx$$

$$+ \limsup_{\sigma \to 0} \int_{t_1}^{t_2} \int_Q (v^m - (v^*)^m) \frac{\partial v^*}{\partial t}\, dx\, dt$$

$$+ \int_{t_1}^{t_2} \int_Q |\nabla v^m|^2\, dx\, dt \geq 0.$$

Proof. Choosing $\varphi = v^m$ in (2.12) we have the basic estimate

$$\int_{t_1}^{t_2} \int_Q \nabla(v^m)^* \cdot \nabla(v^m)\, dx\, dt + \int_{t_1}^{t_2} \int_Q v^m \frac{\partial v^*}{\partial t}\, dx\, dt \geq 0.$$

Since

$$\int_{t_1}^{t_2} v^m \frac{\partial v^*}{\partial t}\, dt = \int_{t_1}^{t_2} (v^*)^m \frac{\partial v^*}{\partial t}\, dt + \int_{t_1}^{t_2} \left(v^m - (v^*)^m\right) \frac{\partial v^*}{\partial t}\, dt$$

$$= \frac{1}{m+1}\left(v^*(x, t_2)^{m+1} - v^*(x, t_1)^{m+1}\right) + \int_{t_1}^{t_2} \left(v^m - (v^*)^m\right) \frac{\partial v^*}{\partial t}\, dt,$$

we may safely let $\sigma \to 0$. $\qquad\square$

Remark 2.15. (1) Limes superior can be replaced with limes inferior, since the actual limes exists for all other terms.

(2) The *excess term*

$$\limsup_{\sigma \to 0} \int_{t_1}^{t_2} \int_Q (v^m - (v^*)^m) \frac{\partial v^*}{\partial t}\, dx\, dt \geq 0$$

is not negligible. To see this, consider the essentially one-dimensional example, where $v(x, t) = 0$ if $t \leq 0$ and $v(x, t) = 1$ if $t > 0$. Then

$$v(x, t)^m - v^*(x, t)^m = 1 - \left(\frac{1}{\sigma} \int_0^t e^{(s-t)/\sigma}\, ds\right)^m = 1 - \left(1 - e^{-t/\sigma}\right)^m$$

and

$$\frac{\partial v^*}{\partial t}(x, t) = \frac{v(x, t) - v^*(x, t)}{\sigma} = \frac{1}{\sigma} e^{-t/\sigma}$$

when $t > 0$. Hence

$$\int_0^T (v^m - (v^*)^m) \frac{\partial v^*}{\partial t}\, dt = 1 - e^{-T/\sigma} - \frac{(1 - e^{-T/\sigma})^{m+1}}{m+1}$$

upon integration. Thus the excess term is

$$\lim_{\sigma \to 0} \int_0^T \int_Q (v^m - (v^*)^m) \frac{\partial v^*}{\partial t}\, dx\, dt = \frac{m}{m+1} |Q| > 0,$$

a positive quantity.

The *obstacle problem* in the calculus of variations is a basic tool in our study of viscosity supersolutions. Let $\psi \in C^\infty(\mathbb{R}^{N+1})$ and consider the class $\mathcal{F}_\psi$ of all functions $w \in C(\overline{Q_T})$ such that

$$|w|^{m-1}w \in L^2(0, T; H^1(Q)), \quad w = \psi \text{ on } \Gamma_T, \quad \text{and } w \geq \psi \text{ in } Q_T.$$

The function ψ acts as an obstacle and also prescribes the boundary values.

The following existence theorem will be useful for us later.

Lemma 2.16. *There is a unique $w \in \mathcal{F}_\psi$ such that*

$$\int_0^T \int_Q \left(\nabla(|w|^{m-1}w) \cdot \nabla(\phi - w) + (\phi - w)\frac{\partial \phi}{\partial t} \right) dx \, dt \tag{2.17}$$
$$\geq \frac{1}{2} \int_Q |\phi(x, T) - w(x, T)|^2 \, dx$$

for all smooth functions ϕ in the class $\mathcal{F}_\psi$. In particular, w is a continuous weak supersolution. Moreover, in the open set $\{w > \psi\}$ the function w is a solution.

Proof. The existence can be shown as in the proof of Theorem 3.2 in [1]. Continuity follows from standard regularity theory, but it seems to be difficult to find a convenient reference. $\square$

3. Viscosity supersolutions

In this section we define the class of viscosity supersolutions and prove Theorem 1.3. We need approximating weak supersolutions. They are constructed via an obstacle problem in the calculus of variations. The procedure will be discussed below.

Let us begin with the definition, which is similar to the classical definition of superharmonic functions, due to F. Riesz. We remark once more that the word "viscosity" is only used as a label by us.

Definition 3.1. A function $v : \Omega \to (-\infty, \infty]$ is called a *viscosity supersolution* if

    (1) v is lower semicontinuous in Ω,
    (2) v is finite in a dense subset of Ω,
    (3) v satisfies the following comparison principle in each subdomain $D_{t_1,t_2} \Subset \Omega$: if $h \in C(\overline{D_{t_1,t_2}})$ is a solution in D_{t_1,t_2} and if $h \leq v$ on the parabolic boundary of D_{t_1,t_2}, then $h \leq v$ in D_{t_1,t_2}.

It follows immediately that the pointwise minimum

$$v(x,t) = \min(v_1(x,t), v_2(x,t), \ldots, v_j(x,t))$$

of finitely many viscosity supersolutions is a viscosity supersolution. Another useful construction for a non-negative viscosity supersolution v is to redefine it as 0 till a given instant t_0. In other words

$$v_0(x,t) = \begin{cases} 0, & t \le t_0, \\ v(x,t), & t \ge t_0, \end{cases}$$

is a viscosity supersolution. In section 5 a further construction, the so-called Poisson modification, is discussed.

Notice that a viscosity supersolution is defined at every point in its domain. No differentiability is presupposed in the definition. The only tie to the differential equation is through the comparison principle.

It turns out that a viscosity supersolution satisfies the comparison principle in more general domains than the cylinders D_{t_1,t_2}. For our purposes it is sufficient that the comparison principle holds for a finite union of boxes. The proof is a matter of successive comparisons, starting with the earliest boxes.

There is a relation between weak supersolutions and viscosity supersolutions. Roughly speaking, the weak supersolutions are viscosity supersolutions, provided the issue about lower semicontinuity is properly handled. In particular, a continuous supersolution is a viscosity supersolution. On the other hand, a bounded viscosity supersolution is a weak supersolution.

The Barenblatt solution clearly shows that the class of viscosity supersolutions contains more than weak supersolutions. Nevertheless, it turns out that a viscosity supersolution can be approximated pointwise with an increasing sequence of weak supersolutions, constructed through successive obstacle problems. Let us describe this procedure.

Theorem 3.2. *Suppose that v is a viscosity supersolution in Ω and let $Q_{t_1,t_2} \Subset \Omega$. Then there is a sequence of weak supersolutions $v_k \in C(\overline{Q_{t_1,t_2}})$, $|v_k|^{m-1}v_k \in L^2(t_1, t_2; H^1(Q))$, $k = 1, 2, \ldots$, such that $v_1 \le v_2 \le \cdots \le v$ and $v_k \to v$ pointwise in Q_{t_1,t_2} as $k \to \infty$. If, in addition, v is locally bounded in Ω, then $|v|^{m-1}v \in L^2(t_1, t_2; H^1(Q))$ and v itself is a weak supersolution.*

Proof. The lower semicontinuity implies that there is a sequence of functions $\psi_k \in C^\infty(\Omega)$, $k = 1, 2, \ldots$, such that

$$\psi_1 < \psi_2 < \ldots \qquad \text{and} \qquad \lim_{k \to \infty} \psi_k = v$$

at every point of Ω. It is decisive here that the inequality $\psi_k < \psi_{k+1}$ is strict. Next, using the functions ψ_k as obstacles, we construct supersolutions of (1.1) that approximate v from below. This has to be done locally, say in a given box Q_{t_1,t_2} with $Q_{t_1,t_2} \Subset \Omega$. To simplify the notation we consider Q_T, assuming that $Q_T \Subset \Omega$. Let $v_k \in C(\overline{Q_T})$, $|v|^{m-1}v \in L^2(0, T; H^1(Q))$, $k = 1, 2, \ldots$, denote the solution of the obstacle problem in Q_T with the obstacle ψ_k, see Lemma 3.2. Thus $v_k \in \mathcal{F}_{\psi_k}$.

We claim that

$$v_1 \leq v_2 \leq \ldots \qquad \text{and} \qquad v_k \leq v, \qquad k = 1, 2, \ldots,$$

in Q_T. In particular, $\psi_k \leq v_k \leq v$ then gives the desired convergence. Due to a technical difficulty, we choose an arbitrarily small $\varepsilon > 0$ and prove that $v_k(x, t) \leq v(x, t)$ when $0 < t < T - \varepsilon$ and $x \in Q$. The set

$$K_k = \{(x, t) : x \in \overline{Q}, 0 \leq t \leq T - \varepsilon, v_k(x, t) \geq \psi_{k+1}(x, t)\},$$

$k = 1, 2, \ldots$, is compact. The distance of K_k to the set where $v_k(x, t) = \psi_k(x, t)$ is positive, say $\delta = \delta(k, \varepsilon)$. (We tacitly assume that the set K_k is not empty.) This is due to the continuity of the functions and the strict inequality $\psi_{k+1} > \psi_k$. Recall that in the set where $v_k(x, t) > \psi_k(x, t)$ the function v_k is a solution of the porous medium equation. Now we want to use the comparison principle.

Suppose that $\mathbb{R}^{N+1}$ has been divided into dyadic cubes in the standard way. The set K_k can be covered with a finite number of (closed) dyadic cubes, all of the same size and with a sufficiently small diameter, say that the diameter of each cube is smaller than $\delta/2$. The interior of the union of the cubes has its parabolic boundary in the set where $\psi_k < v_k \leq \psi_{k+1}$. Thus $v_k \leq \psi_{k+1} < v$ on this parabolic boundary. By the comparison principle we conclude that $v_k \leq v$ at least in K_k. Outside K_k the inequality $v_k < \psi_{k+1} < v$ holds trivially. This shows that $v_k \leq v$ in Q_T since $\varepsilon > 0$ was arbitrary. The inequality $v_k \leq v_{k+1}$ can be shown in the same way.

If v is locally bounded, a compactness argument is available. The Caccioppoli estimate (Lemma 2.13) gives

$$\int_0^T \int_Q \zeta^2 |\nabla v_k^m|^2 \, dx \, dt \leq 16 M^{2m} T \int_Q |\nabla \zeta|^2 \, dx + 6 M^{m+1} \int_Q \zeta^2 \, dx,$$

where ζ depends only on x, $\zeta \in C_0^\infty(Q)$, $\zeta \geq 0$ and M is the supremum of v in the support of ζ. By weak compactness, $\nabla(|v|^{m-1}v)$ exists in Sobolev's sense and $\nabla(|v_k|^{m-1}v_k) \to \nabla(|v|^{m-1}v)$ weakly in L^2 as $k \to \infty$. Hence we may proceed to the limit under the integral sign in

$$\int_0^T \int_Q \left(\nabla(|v_k|^{m-1}v_k) \cdot \nabla\varphi - v_k \frac{\partial\varphi}{\partial t} \right) dx \, dt \geq 0,$$

where $\varphi \in C_0^\infty(Q_T)$. This shows that v is a weak supersolution. $\quad\square$

Theorem 1.3 follows immediately from the previous theorem.

We seize the opportunity to mention that the following result can be extracted from the end of the previous proof.

Proposition 3.3. *Consider a sequence*

$$v_1 \le v_2 \le \dots$$

of weak supersolutions in Q_T such that $|v_k| \le M$ and $|v_k|^{m-1} v_k \in L^2(0, T; H^1(Q))$ when $k = 1, 2, \dots$ Then also the limit function

$$v = \lim_{k \to \infty} v_k$$

is a weak supersolution. In particular, $|v|^{m-1} v \in L^2(0, T; H^1(Q'))$ whenever $Q' \subset\subset Q$.

The following Harnack type convergence theorem holds for solutions of the porous medium equation.

Lemma 3.4 (Harnack)**.** *Suppose that h_k, $k = 1, 2, \dots$, is a sequence of weak solutions in Ω and that $0 \le h_1 \le h_2 \le \cdots$ pointwise in Ω. If the limit function*

$$h(x, t) = \lim_{k \to \infty} h_k(x, t)$$

is finite in a dense subset of Ω, then h is a weak solution in Ω.

Proof. The intrinsic Harnack inequalities proved by DiBenedetto in [4] can be passed over from the sequence h_k, $k = 1, 2, \dots$, to the limit function h. It follows that h is locally bounded. Then we may use the compactness argument at the end of the proof of Theorem 3.2 to conclude that the limit of the equations

$$\iint_\Omega \left(\nabla h_k^m \cdot \nabla \varphi - h_k \frac{\partial \varphi}{\partial t} \right) dx\, dt = 0$$

as $k \to 0$ is the required equation for h. Here $\varphi \in C_0^\infty(\Omega)$. This proves the lemma. $\quad\square$

4. Preliminary summability estimates

This section is devoted to some technical estimates, where the functions v^m have to be truncated at the level k (the original functions v at the level $k^{1/m}$). The notation

$$w^m = (v^m)_j = \min(v(x, t)^m, j)$$

will be used for a large index j. We also write

$$(w^m)_k = \min(v^m, k)$$

for $k = 0, 1, \ldots, j$. A test function used by Kilpeläinen and Malý in [7] will play an essential role.

Lemma 4.1. *Let $m > 1$ and let $\Omega \subset \mathbb{R}^{N+1}$ be a domain with $Q_T \Subset \Omega$. Suppose that $v \geq 0$ is a supersolution in Ω, $v^m \in L^2(0, T; H_0^1(Q))$ and $v(x, 0) = 0$ when $x \in Q$. Let j be a positive integer and denote $w^m = (v^m)_j = \min(v^m, j)$. Then*

$$\int_0^T \int_Q |\nabla(w^m)|^2 \, dx \, dt + \frac{1}{m+1} \int_Q w^{m+1} \, dx$$

$$+ \limsup_{\sigma \to 0} \int_0^T \int_Q \left(w^m - (w^*)^m\right) \frac{\partial w^*}{\partial t} \, dx \, dt$$

$$\leq j \left(\int_0^T \int_Q |\nabla(w^m)_1|^2 \, dx \, dt + \limsup_{\sigma \to 0} \int_0^T \int_Q (w^m)_1 \frac{\partial w^*}{\partial t} \, dx \, dt \right).$$

Remark 4.2. Observe that the familiar excess term appears on the left-hand side of the estimate. It is needed to counterbalance the excess term in Theorem 2.14.

Proof. Choose the test function

$$\varphi = \left((w^m)_k - (w^m)_{k-1}\right) - \left((w^m)_{k+1} - (w^m)_k\right),$$

where $k = 1, 2, \ldots, j - 1$. Now $\varphi \geq 0$. Since w is a supersolution, we have

$$\int_0^T \int_Q \nabla(w^m)^* \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_Q \varphi \frac{\partial w^*}{\partial t} \, dx \, dt \geq 0$$

according to (2.12). This implies that

$$\int_0^T \int_Q \nabla(w^m)^* \cdot \nabla\left((w^m)_{k+1} - (w^m)_k\right) \, dx \, dt$$

$$+ \int_0^T \int_Q \left((w^m)_{k+1} - (w^m)_k\right) \frac{\partial w^*}{\partial t} \, dx \, dt$$

$$\leq \int_0^T \int_Q \nabla(w^m)^* \cdot \nabla\left((w^m)_k - (w^m)_{k-1}\right) \, dx \, dt$$

$$+ \int_0^T \int_Q \left((w^m)_k - (w^m)_{k-1}\right) \frac{\partial w^*}{\partial t} \, dx \, dt.$$

for every $k = 1, 2, \ldots, j - 1$. We abbreviate the previous expression as

$$a_{k+1} \leq a_k$$

from which it follows, by recursion, that

$$\sum_{k=1}^{j} a_k \le ja_1. \tag{4.3}$$

The notation hides the dangerous fact that a_k depends on j. Because of cancellation the sum on the left-hand side of (4.3) can be computed as

$$\sum_{k=1}^{j} a_k = \int_0^T \int_Q \nabla(w^m)^* \cdot \nabla(w^m)_j \, dx \, dt + \int_0^T \int_Q (w^m)_j \frac{\partial w^*}{\partial t} \, dx \, dt,$$

where $(w^m)_j = w^m$ and consequently

$$(w^m)_j \frac{\partial w^*}{\partial t} = w^m \frac{\partial w^*}{\partial t} = \left(w^m - (w^*)^m\right) \frac{\partial w^*}{\partial t} + (w^*)^m \frac{\partial w^*}{\partial t}.$$

A simple integration gives

$$\int_0^T \int_Q (w^*)^m \frac{\partial w^*}{\partial t} \, dx \, dt = \frac{1}{m+1} \int_Q (w^*)^{m+1} \, dx$$

since $w^*(x, 0) = 0$. This implies that

$$\sum_{k=1}^{j} a_k = \int_0^T \int_Q \nabla(w^m)^* \cdot \nabla w^m \, dx \, dt$$

$$+ \int_0^T \int_Q \left(w^m - (w^*)^m\right) \frac{\partial w^*}{\partial t} \, dx \, dt + \frac{1}{m+1} \int_Q (w^*)^{m+1} \, dx.$$

The left-hand side of (4.3) is

$$ja_1 = j\left(\int_0^T \int_Q \nabla(w^m)^* \cdot \nabla(w^m)_1 \, dx \, dt + \int_0^T \int_Q (w^m)_1 \frac{\partial w^*}{\partial t} \, dx \, dt \right).$$

Thus we have

$$\int_0^T \int_Q \nabla(w^m)^* \cdot \nabla w^m \, dx \, dt + \int_0^T \int_Q \left(w^m - (w^*)^m\right) \frac{\partial w^*}{\partial t} \, dx \, dt$$

$$+ \frac{1}{m+1} \int_Q (w^*)^{m+1} \, dx$$

$$\le j\left(\int_0^T \int_Q \nabla(w^m)^* \cdot \nabla(w^m)_1 \, dx \, dt + \int_0^T \int_Q (w^m)_1 \frac{\partial w^*}{\partial t} \, dx \, dt \right).$$

The claim follows from this letting the smoothing parameter $\sigma \to 0$.
□

Let us proceed a little further under the same assumptions. Let $0 < t_1 < T$ and $t_1 \leq \tau \leq T$. By Theorem 2.14 we have

$$\frac{1}{m+1} \int_Q w(x,t)^{m+1} \, dx$$
$$\leq \int_0^\tau \int_Q |\nabla(w^m)|^2 \, dx \, dt + \frac{1}{m+1} \int_Q w(x,\tau)^{m+1} \, dx$$
$$+ \limsup_{\sigma \to 0} \int_0^\tau \int_Q (w^m - (w^*)^m) \frac{\partial w^*}{\partial t} \, dx \, dt$$

for every $0 < t < t_1$. Together with Lemma 4.1 this implies that

$$\operatorname*{ess\,sup}_{0<t<t_1} \frac{1}{m+1} \int_Q w(x,t)^{m+1} \, dx$$
$$\leq \int_0^\tau \int_Q |\nabla(w^m)|^2 \, dx \, dt + \frac{1}{m+1} \int_Q w(x,\tau)^{m+1} \, dx$$
$$+ \limsup_{\sigma \to 0} \int_0^\tau \int_Q (w^m - (w^*)^m) \frac{\partial w^*}{\partial t} \, dx \, dt$$
$$\leq j \Big(\int_0^\tau \int_Q |\nabla(w^m)_1|^2 \, dx \, dt + \limsup_{\sigma \to 0} \int_0^\tau \int_Q (w^m)_1 \frac{\partial w^*}{\partial t} \, dx \, dt \Big).$$

Using Lemma 4.1 again we conclude that

$$\operatorname*{ess\,sup}_{0<t<t_1} \frac{1}{m+1} \int_Q w(x,t)^{m+1} \, dx + \int_0^{t_1} \int_Q |\nabla(w^m)|^2 \, dx \, dt$$
$$\leq 2j \Big(\int_0^\tau \int_Q |\nabla(w^m)_1|^2 \, dx \, dt \tag{4.4}$$
$$+ \limsup_{\sigma \to 0} \int_0^\tau \int_Q (w^m)_1 \frac{\partial w^*}{\partial t} \, dx \, dt \Big).$$

Observe that the excess term disappeared.

The estimation of the right-hand side of (4.4) is postponed till section 5. We proceed by assuming, for the moment, that we already have archieved the bound

$$\operatorname*{ess\,sup}_{0<t<t_1} \frac{1}{m+1} \int_Q w(x,t)^{m+1} \, dx + \int_0^{t_1} \int_Q |\nabla(w^m)|^2 \, dx \, dt$$
$$\leq cj \Big(\int_Q w(x,\tau) \, dx + T|Q| \Big). \tag{4.5}$$

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We integrate both sides of this inequality with respect to τ over the interval $[t_1, T]$ and divide by $T - t_1$. This implies

$$\operatorname*{ess\,sup}_{0<t<t_1} \frac{1}{m+1} \int_Q w(x,t)^{m+1}\, dx + \int_0^{t_1} \int_Q |\nabla(w^m)|^2\, dx\, dt$$

$$\leq cj\Big(\frac{1}{T-t_1} \int_0^T \int_Q w(x,t)\, dx\, dt + \frac{T}{T-t_1}|Q|\Big).$$

Recall that $w^m = \min(v^m, j)$. Hence the previous estimate implies that

$$\operatorname*{ess\,sup}_{0<t<t_1} \frac{1}{m+1} \int_Q \min(v^m, j)^{1+1/m}\, dx$$

$$+ \int_0^{t_1} \int_Q |\nabla \min(v^m, j)|^2\, dx\, dt$$

$$\leq cj\Big(\frac{1}{T-t_1} \int_0^T \int_Q \min(v^m, j)^{1/m}\, dx\, dt + \frac{T|Q|}{T-t_1}\Big) \qquad (4.6)$$

$$\leq cj\Big(\frac{T|Q|}{T-t_1} j^{1/m} + \frac{T|Q|}{T-t_1}\Big)$$

$$\leq cj^{1+1/m} \frac{T|Q|}{T-t_1}.$$

The constant c may change from one line to the next.

Let $\kappa = 1 + 1/N + 1/(mN)$ and define the sets

$$E_j = \big\{(x,t) \in Q_{t_1} : j \leq v^m(x,t) < 2j\big\}$$

for $j = 1, 2, \ldots$ Sobolev's inequality and (4.6) imply that

$$j^{2\kappa}|E_j| \leq \iint_{E_j} (\min(v^m, 2j))^{2\kappa}\, dx\, dt$$

$$\leq \int_0^T \int_Q (\min(v^m, 2j))^{2\kappa}\, dx\, dt$$

$$\leq c \int_0^T \int_Q |\nabla \min(v^m, 2j)|^2\, dx\, dt$$

$$\cdot \Big(\operatorname*{ess\,sup}_{0<t<T} \int_Q \min(v^m, 2j)^{1+1/m}\, dx\Big)^{2/N}$$

$$\leq cj^{(1+1/m)(1+2/N)} \Big(\frac{T|Q|}{T-t_1}\Big)^{1+2/N},$$

for $j = 1, 2, \ldots$

It follows that

$$|E_j| \leq cj^{-1+1/m} \Big(\frac{T|Q|}{T-t_1}\Big)^{1+2/N}$$

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for $j = 1, 2, \ldots$ Let $\alpha > 0$. From this we conclude that the sum in

$$\int_0^{t_1} \int_Q v^{m\alpha} \, dx \, dt \leq T|Q| + \sum_{j=1}^{\infty} \iint_{E_{2^{j-1}}} v^{m\alpha} \, dx \, dt$$

can be majorized by

$$\sum_{j=1}^{\infty} \iint_{E_{2^{j-1}}} v^{m\alpha} \, dx \, dt \leq \sum_{j=1}^{\infty} 2^{\alpha j} |E_{2^{j-1}}|$$

$$\leq c \Big(\frac{T|Q|}{T - t_1} \Big)^{1+2/n} \sum_{j=1}^{\infty} 2^{-j(-\alpha+1-1/m)}.$$

The series converges if $\alpha < 1 - 1/m$. Thus we have a finite majorant. Indeed, since $m > 1$ there is a small $\alpha_1 > 0$ such that

$$\int_0^{t_1} \int_Q v^{m\alpha_1} \, dx \, dt < \infty.$$

We may assume that $0 < \alpha_1 < 1/m$.

This was the first step. In order to improve the exponent we iterate this procedure. At the next step we split $1/m$ as

$$\frac{1}{m} = \alpha_1 + \Big(\frac{1}{m} - \alpha_1 \Big).$$

Let $0 < t_2 < t_1$. As in (4.6) we have

$$\operatorname*{ess\,sup}_{0 < t < t_2} \frac{1}{m+1} \int_Q \min(v^m, j)^{1+1/m} \, dx$$

$$+ \int_0^{t_2} \int_Q |\nabla \min(v^m, j)|^2 \, dx \, dt$$

$$\leq cj \Big(\frac{1}{t_1 - t_2} \int_0^{t_1} \int_Q \min(v^m, j)^{1/m} \, dx \, dt + \frac{t_1|Q|}{t_1 - t_2} \Big)$$

$$\leq cj \Big(\frac{1}{t_1 - t_2} \int_0^{t_1} \int_Q \min(v^m, j)^{\alpha_1} \min(v^m, j)^{1/m - \alpha_1} \, dx \, dt$$

$$+ \frac{t_1|Q|}{t_1 - t_2} \Big)$$

$$\leq cj^{1+1/m-\alpha_1} \Big(\frac{1}{t_1 - t_2} \int_0^{t_1} \int_Q v^{m\alpha_1} \, dx \, dt + \frac{t_1|Q|}{t_1 - t_2} \Big).$$

The right-hand side of this estimate is a finite number by the first step of the iteration. Hence we may apply Sobolev's inequality and estimate the size of the distribution set E_j as before. It follows that

$$|E_j| \leq cj^{-1+1/m-\alpha_1(1+2/N)}$$

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for $j = 1, 2, \ldots$ where the constant c depends on various parameters. From this we conclude that

$$\int_0^{t_2} \int_Q v^{m\alpha_2} \, dx \, dt < \infty$$

whenever

$$\alpha_2 < 1 - \frac{1}{m} + \alpha_1 \left(1 + \frac{2}{N} \right).$$

Certainly $\alpha_2 > \alpha_1$. At each step of the iteration we obtain $\alpha_{k+1} > a_k$. After a finite number of steps we reach $1/m$ and here we stop. We choose k such that $\alpha_k < 1/m \le a_{k+1}$. We have now reached at least all exponents $\alpha < 1/m$. At the final step of the iteration we use that $a_k < \alpha < 1/m$ from which we conclude that

$$\int_0^{t_{k+1}} \int_Q v^{m\alpha} \, dx \, dt < \infty$$

whenever

$$\alpha < 1 + \frac{2}{mN}.$$

Indeed, the exponent is the supremum of all

$$1 - \frac{1}{m} + \alpha \left(1 + \frac{2}{N} \right),$$

where $\alpha < 1/m$. To reach a given exponent α only a finite number of steps is needed and hence the influence of the t_k's is under control. We have proved the following result, in which the correct exponent is present.

Theorem 4.7. *Let $m > 1$ and let $\Omega \subset \mathbb{R}^{N+1}$ be a domain with $Q_T \Subset \Omega$. Suppose that $v \ge 0$ is a weak supersolution in Ω, $v^m \in L^2(0, T; H_0^1(Q))$ and $v(x, 0) = 0$ when $x \in Q$. Let j be a positive integer and denote $w^m = (v^m)_j = \min(v^m, j)$. Fix $t_1 < T$. If there is a constant c such that*

$$\operatorname*{ess\,sup}_{0 < t < t_1} \frac{1}{m+1} \int_Q w(x, t)^{m+1} \, dx + \int_0^{t_1} \int_Q |\nabla(w^m)|^2 \, dx \, dt$$
$$\le cj \left(\int_Q w(x, \tau) \, dx + T|Q| \right),$$

then $v^m \in L^q(Q_T)$ for every $0 < q < 1 + 2/(mN)$.

Remark 4.8. According to Theorem 3.2 the function v may be a viscosity supersolution in the theorem above.

5. SUMMABILITY ESTIMATES CONCLUDED

In order to complete the proof of Theorem 1.4 we have to modify the viscosity supersolution v near the parabolic boundary of Q_T so that Theorem 4.7 applies. In a strip near Γ_T we will replace v with a solution in such a way that the modified function is a viscosity supersolution. Therefore we assume that $\overline{Q_T} \subset \Omega$. Let $\overline{Q'} \subset Q$ and select t_1 and t_2 so that $0 < t_1 < t_2 < T$. Then $\overline{Q'_{t_1,t_2}} \subset Q_T$. Furthermore, we can redefine v so that $v(x,t) = 0$, when $t \leq t_1$. The obtained function v is a viscosity supersolution in Q_T. We aim at proving the summability in Q'_{t_1,t_2}.

Roughly speaking, we want to redefine v in $Q_T \setminus Q'_{t_1,T}$ in the following way:

$$V = \begin{cases} v & \text{in} \quad \overline{Q'_{t_1,T}}, \\ h & \text{in} \quad Q_T \setminus \overline{Q'_{t_1,T}}, \end{cases}$$

where h is the solution in $Q_T \setminus \overline{Q'_{t_1,T}}$ with zero boundary values on the parabolic boundary of Q_T and $h = v$ on the parabolic boundary of $Q'_{t_1,T}$. Notice that h and v both are zero when $t \leq t_1$. We say that V is the *Poisson modification* of v. We will show that V is a viscosity supersolution.

Let us first construct h. Lower semicontinuity implies that there is a sequence of functions $\psi_k \in C^\infty(\Omega)$, $k = 1, 2, \ldots$, such that

$$0 \leq \psi_1 \leq \psi_2 \leq \ldots \qquad \text{and} \qquad \lim_{k \to \infty} \psi_k = v$$

at every point of Ω. We assume, as we may, that $\psi_k = 0$ in $Q \times [0, t_1]$. Let h_k denote the unique continuous solution of the porous medium equation in $(Q \setminus \overline{Q'}) \times (0, T)$ with the following boundary values

$$h_k = \begin{cases} \psi_k & \text{in} \quad \partial Q' \times [0, T], \\ 0 & \text{in} \quad \partial Q \times [0, T], \\ 0 & \text{in} \quad (Q \setminus Q') \times \{0\}. \end{cases}$$

We can extend h_k continuously up to the boundary so that $h_k \in C((\overline{Q} \setminus Q') \times [0, T])$. Actually, $h_k(x, t) = 0$ when $t \leq t_1$. We have

$$h_1 \leq h_2 \leq \ldots \qquad \text{and} \qquad h_k \leq v \qquad \text{in} \qquad Q_T \setminus \overline{Q'_{t_1,T}}.$$

By Harnack's convergence theorem (Lemma 3.4) the function

$$h = \lim_{k \to \infty} h_k$$

is a continuous solution in $Q_T \setminus \overline{Q'_{t_1,T}}$ and clearly $h \leq v$. Thus $V \leq h$.

It remains to verify the comparison principle for V. Let $D_{a,b} = D \times (a, b)$ be a subdomain of Q_T and suppose that $H \in C(\overline{D_{a,b}})$ is a solution

and $V \geq H$ on the parabolic boundary of $D_{a,b}$. Since $v \geq V$, the comparison principle valid for v yields $v \geq H$ in $D_{a,b}$. In particular, $H(x,t) \leq 0$ when $t \leq t_1$ (if $a < t_1$). If $D \subset Q'$ we are done. If not, then a comparison has to be performed in (each component of) $(D \setminus Q') \times (a,b)$. We have that $h \geq H$ on the parabolic boundary of this set. The points on $\partial Q' \times (a,b)$ require some care. Let (x_0, t_0) be a point on $\partial Q' \times (a,b)$. From the construction of h we can deduce that, given $\varepsilon > 0$, there is an index k such that

$$H(x_0, t_0) < h_k(x_0, t_0) + \varepsilon.$$

This implies that

$$H(x_0, t_0) \leq \liminf_{(x,t) \to (x_0, t_0)} h(x,t).$$

Thus $h \geq H$ by the comparison principle. This concludes the proof of the inequality $V \geq H$ in $D_{a,b}$.

Therefore the function V is a viscosity supersolution in Q_T. In particular, V is continuous in $Q_T \setminus \overline{Q'_{t_1,T}}$ and has zero boundary values on the parabolic boundary of Q_T. Our next goal is to obtain a bound for the right-hand side of (4.4). Notice that V (in place of v) satisfies the assumptions in the next lemma.

Lemma 5.1. *Suppose that v is a weak supersolution in Ω with the properties $v \geq 0$ and $v^m \in L^2(0, T; H_0^1(Q))$. We assume that there is a small $\delta > 0$ such that $v(x,t)$ is a solution when $\operatorname{dist}(x, \partial Q) < \delta < 1$ and $0 < t < T$. In addition, we assume that $v(x,t) < 1$ when $\operatorname{dist}(x, \partial Q) < \delta$ and $v(x,0) = 0$ when $x \in Q$. Then*

$$\int_0^T \int_Q |\nabla (v^m)_1|^2 \, dx \, dt + \limsup_{\sigma \to 0} \int_0^T \int_Q (v^m)_1 \frac{\partial v^*}{\partial t} \, dx \, dt$$

$$\leq 3 \int_Q v(x,T) \, dx + c\delta^{-2} T |Q|.$$

Proof. We write $Q = (a_1, b_1) \times \cdots \times (a_N, b_N)$. Let ζ_i^m, $i = 1, 2, \ldots, n$, be a piecewise linear cutoff function of one variable such that its support is $[a_i, b_i]$, $\zeta_i = 1$ in $[a_i + \delta, b_i - \delta]$, $0 \leq \zeta_i \leq 1$ and $|\zeta_i'| \leq 1/\delta$. We define

$$\zeta(x)^m = \min(\zeta_1(x_1)^m, \zeta_2(x_2)^m, \ldots, \zeta_N(x_N)^m).$$

The graph of this function is a truncated pyramid with the base Q. We aim at using the test function

$$\varphi(x,t) = \zeta(x)^m - (v(x,t)^m)_1$$

in

$$\int_0^T \int_Q \left(\nabla (v^m)^* \cdot \nabla \varphi + \varphi \frac{\partial v^*}{\partial t} \right) dx \, dt \geq 0.$$

We claim that $\varphi \geq 0$. Notice that $\Delta(\zeta^m) \leq 0$. Thus ζ is a super-solution of the porous medium equation. More precisely, the function ζ^m is superharmonic as a minimum of a finite number of planes. By comparison in the domains $\text{dist}(x, \partial Q) < \delta$ we have $\zeta(x) \geq (v(x,t))_1$. Since $(v_1)^m = (v^m)_1$, we conclude that $\varphi \geq 0$. The substitution of φ implies

$$\int_0^T \int_Q \nabla(v^m)^* \cdot \nabla(v^m)_1 \, dx \, dt + \int_0^T \int_Q (v^m)_1 \frac{\partial v^*}{\partial t} \, dx \, dt$$
$$\leq \int_0^T \int_Q \nabla(v^m)^* \cdot \nabla(\zeta^m) \, dx \, dt + \int_0^T \int_Q \zeta^m \frac{\partial v^*}{\partial t} \, dx \, dt.$$

Since $\zeta^m = 1$ and $\nabla(\zeta^m) = 0$ on the set where $v^m \geq 1$, an elementary inequality yields

$$\lim_{\sigma \to 0} \int_0^T \int_Q \nabla(v^m)^* \cdot \nabla(\zeta^m) \, dx \, dt = \int_0^T \int_Q \nabla(v^m) \cdot \nabla(\zeta^m) \, dx \, dt$$
$$\leq \frac{1}{2} \int_0^T \int_Q |\nabla(v^m)_1|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_Q |\nabla(\zeta^m)|^2 \, dx \, dt.$$

The estimate

$$\int_0^T \int_Q |\nabla(\zeta^m)|^2 \, dx \, dt \leq c\delta^{-2} T |Q|$$

holds. (Actually it is of magnitude $\mathcal{O}(\delta^{-1})$). On the other hand

$$\int_0^T \int_Q \zeta^m \frac{\partial v^*}{\partial t} \, dx \, dt = \int_Q \zeta^m \int_0^T \frac{\partial v^*}{\partial t} \, dx \, dt$$
$$= \int_Q \zeta^m v^*(x, T) \, dx \leq \int_Q v^*(x, T) \, dx$$

The claim follows from these estimates letting $\sigma \to 0$. $\qquad\qquad\square$

To conclude the proof of Theorem 1.4 we have to assure that the hypothesis of Theorem 4.7 is valid. The previous considerations yield that Lemma 5.1 is valid for the function V and so is Theorem 4.7. This proves Theorem 1.4 for v restricted to $Q' \times (t_1, t_2)$, because $v = V$ in this cylinder. Such a local result is enough for us. This concludes the proof of Theorem 1.4.

6. Pointwise behaviour

A distinct feature is that the viscosity supersolutions are defined at every point in their domain. Thus it is possible to study their pointwise behaviour. The value taken at a given point is determinate. Indeed, we cannot just change the value of a viscosity supersolution at one single point without destroying the function: the new function is no longer

a viscosity supersolution. As an illustration of this phenomenon, we mention a simple proposition.

Proposition 6.1. *Suppose that v_1 and v_2 are viscosity supersolutions in Ω. If $v_1 = v_2$ almost everywhere in Ω, then $v_1 = v_2$ at each point in Ω.*

The proposition is a direct consequence of the main theorem in this section. So is the fact that a viscosity supersolution is finite almost everywhere (this is much more than what condition (2) in Definition 3.1 directly assures). It is important to pay attention to whether a property holds almost everywhere (with respect to the $(N+1)$-dimensional Lebesgue measure) or at each point. The concept

$$\operatorname*{ess\,lim\,inf}_{(y,\tau)\to(x,t)} v(y,\tau)$$

is central. The notion of "essential limes inferior" means that the sets of $(N+1)$-dimensional measure zero are neglected in the calculation of the limes inferior. For a lower semicontinuous function v defined in Ω it holds that

$$v(x,t) \le \liminf_{(y,\tau)\to(x,t)} v(y,\tau)$$
$$\le \operatorname*{ess\,lim\,inf}_{(y,\tau)\to(x,t)} v(y,\tau) \le \operatorname*{ess\,lim\,inf}_{(y,\tau)\to(x,t),\tau<t} v(y,\tau),$$

when $(x,t) \in \Omega$. We show that for a viscosity supersolution, in fact, equality holds at each step.

Theorem 6.2. *Suppose that v is a viscosity supersolution Ω. Then*

$$v(x,t) = \operatorname*{ess\,lim\,inf}_{(y,\tau)\to(x,t),\tau<t} v(y,\tau)$$

holds at each point $(x,t) \in \Omega$.

For the determination of $v(x,t)$ it is not necessary to include the values at any future points (y,τ) with $\tau \ge t$. Only the past, $\tau < t$, counts. The proof of the theorem is based on the lemma below.

Lemma 6.3. *Let v be a viscosity supersolution in Ω and assume that $\overline{Q_T} \subset \Omega$. Suppose that there is a constant λ such that*

(1) $v \le \lambda$ *at each point in Q_T and*
(2) $v = \lambda$ *at almost every point in Q_T.*

Then $v = \lambda$ at each point in $Q \times (0,T]$.

Proof. Notice that the conclusion is immediate if v happens to be continuous. The idea is to employ an auxiliary continuous function, viz. the solution h with boundary values v. It will turn out that $h = \lambda$.

To this end, we repeat the construction of the approximants, as in the beginning of section 5. Again let $\psi_k \in C^\infty(\Omega)$ be such that

$$\psi_1 < \psi_2 < \dots \qquad \text{and} \qquad \lim_{k \to \infty} \psi_k = v$$

at every point of Ω. Let v_k be the solution to the obstacle problem in Q_T with ψ_k acting as the obstacle. Thus $v_k \in \mathcal{F}_{\psi_k}(Q_T)$,

$$v_1 \le v_2 \le \dots \qquad \text{and} \qquad \psi_k \le v_k \le v$$

at each point in Q_T. To be on the safe side concerning the result also at the terminal instant $t = T$, we can solve the obstacle problem in a slightly larger domain, say in $Q_{T+\delta}$ with an increment $\delta > 0$.

Select $Q' \subset\subset Q$. Let h_k be the solution in $Q'_{T+\delta}$ with boundary values v_k on the parabolic boundary. At each point in $Q'_{T+\delta}$ we have

$$h_1 \le h_2 \le \dots \qquad \text{and} \qquad h_k \le v_k \le v.$$

By Harnack's convergence theorem (Lemma 3.4) the limit function $h = \lim h_k$ is a solution in $Q'_{T+\delta}$. We have

$$h(x,t) \le v(x,t) \le \lambda$$

at each point in $Q' \times (0, T]$. This is clear by (1) when $t < T$ and at the terminal points $t = T$ the lower semicontinuity of v yields $v(x, T) \le \lambda$. So far, the inequalities are valid at each point in question.

We claim that $h(x,t) = \lambda$ in $Q' \times (0, T]$. Let us first see how the theorem follows from this claim. It is clear that it implies that $v = \lambda$ in Q'_T. To extend this to the terminal time $t = T$, one only needs the identity $h(x, T) = \lambda$ at each $x \in Q'$, which fact is true by the continuity of h. Thus the claim implies that $v = \lambda$ in $Q' \times (0, T]$. Since Q' was arbitrary, the desired result follows in the whole $Q \times (0, T]$.

The proof of the claim is based on the fact that both the function h and the constant λ solve the same boundary value problem. By uniqueness we can then conclude that $h = \lambda$. The uniqueness proof is based on Oleinik's celebrated test function

$$\varphi(x,t) = \int_t^T \left(|v_k(x, \tau)|^{m-1} v_k(x, \tau) - |h_k(x, \tau)|^{m-1} h_k(x, \tau) \right) d\tau,$$

which can be used in the two equations

$$\int_0^T \int_{Q'} \left(-h\varphi_t + \nabla(|h|^{m-1}h) \cdot \nabla\varphi \right) dx \, dt = 0$$

and

$$\int_0^T \int_{Q'} \left(-\lambda\varphi_t + \nabla(|\lambda|^{m-1}\lambda) \cdot \nabla\varphi \right) dx \, dt = 0.$$

Subtracting the equations and inserting φ, we obtain

$$-\int_0^T \int_{Q'} (\lambda - h)(|v_k|^{m-1}v_k - |h_k|^{m-1}h_k)\, dx\, dt$$

$$= \int_0^T \int_{Q'} \int_t^T \nabla\big(|\lambda|^{m-1}\lambda - |h(x,t)|^{m-1}h(x,t)\big)$$

$$\cdot \nabla\big(|v_k(x,\tau)|^{m-1}v_k(x,\tau) - |h_k(x,\tau)|^{m-1}h_k(x,\tau)\big)\, d\tau\, dx\, dt.$$

We want to proceed to the limit as $k \to \infty$. Using the Caccioppoli estimate (Lemma 2.13) for v_k we obtain a uniform bound like

$$\int_0^T \int_{Q'} |\nabla(|v_k|^{m-1}v_k)|^2\, dx\, dt \le C_1.$$

Applying the Caccioppoli estimate to the Poisson modification of v_k (described in the beginning of section 5) we also obtain

$$\int_0^T \int_{Q'} |\nabla(|h_k|^{m-1}h_k)|^2\, dx\, dt \le C_2.$$

(In practical terms, h_k is the restriction to $Q'_{T+\delta}$ of a viscosity supersolution defined in the larger set $Q_{T+\delta}$.) Therefore we can extract subsequences so that

$$\nabla(|v_k|^{m-1}v_k) \to \nabla(|v|^{m-1}v) \quad \text{and} \quad \nabla(|h_k|^{m-1}h_k) \to \nabla(|h|^{m-1}h)$$

weakly in $L^2(Q'_T)$. This is sufficient to justify the passage to the limit under the integral sign.

After some manipulations, involving the integral formula

$$\int_0^T \int_t^T f(x,t)f(x,\tau)\, d\tau\, dt = \frac{1}{2}\Big(\int_0^T f(x,t)\, dt\Big)^2,$$

we can write the resulting limit formula as

$$-\int_0^T \int_{Q'} (\lambda - h)(|v|^{m-1}v - |h|^{m-1}h)\, dx\, dt$$

$$= \frac{1}{2}\int_{Q'} \sum_{i=1}^N \Big(\int_0^T \frac{\partial}{\partial x_i}(|\lambda|^{m-1}\lambda - |h|^{m-1}h)\, dt\Big)^2 dx \ge 0.$$

Using assumption (2) about the validity of an identity almost everywhere, we can replace v by λ in the integral. We arrive at

$$\int_0^T \int_{Q'} (\lambda - h)(|\lambda|^{m-1}\lambda - |h|^{m-1}h)\, dx\, dt \le 0.$$

Because of the elementary inequality

$$(\lambda - h)(|\lambda|^{m-1}\lambda - |h|^{m-1}h) \ge 2^{1-m}|h - \lambda|^m,$$

we conclude that $h = \lambda$ almost everywhere in Q'_T. The continuity of the solution h implies that $h(x,t) = \lambda$ at each point in $Q' \times (0,T]$. The claim is proved. This concludes the proof. $\square$

We rewrite the previous lemma in the following more practical way.

Corollary 6.4. *Suppose that v is a viscosity supersolution in Ω and that $Q_T \subset\subset \Omega$. If for some constant λ, it holds that*

$$\min(v(x,t), \lambda) = \lambda$$

for almost every $(x,t) \in Q_T$, then $v(x,t) \geq \lambda$ at each point $(x,t) \in Q \times (0,T]$.

Proof. The function $\min(v(x,t), \lambda)$ is a viscosity supersolution and it satisfies the assumptions in the previous lemma. Hence $\min(v(x,t), \lambda) = \lambda$ at each point in $Q \times (0,T]$. The result follows. $\square$

Now we are ready for the proof of Theorem 6.2. Let (x_0, t_0) be an arbitrary point in Ω. Denote

$$\gamma = \operatorname*{ess\,lim\,inf}_{(x,t) \to (x_0,t_0), t < t_0} v(x,t).$$

We have already seen that $\gamma \geq v(x_0, t_0)$. Thus it is sufficient to prove that $\gamma \leq v(x_0, t_0)$. The case $v(x_0, t_0) = \infty$ is clear. (So is the impossible case $\gamma = -\infty$.) Let us first assume that $-\infty < \gamma < \infty$. Given $\varepsilon > 0$ we can find $\delta > 0$ and a small parallelepiped Q with midpoint x_0 such that the closure of $Q \times (t_0 - \delta, t_0)$ is comprised in Ω and

$$v(x,t) > \gamma - \varepsilon = \lambda$$

for almost every $(x,t) \in Q \times (t_0 - \delta, t_0)$. This is the definition of the essential limes inferior, which assumes the inequality only almost everywhere.

Now the above corollary applies with $\lambda = \gamma - \varepsilon$. It follows that $v(x,t) \geq \lambda$ at each point $(x,t) \in Q \times (t_0 - \delta, t_0]$. In particular, $v(x_0, t_0) \geq \lambda$. Since $\varepsilon > 0$ was arbitrary, we conclude that $v(x_0, t_0) \geq \gamma$. This was our claim.

Finally, the case $\gamma = +\infty$ is easily reached via the functions

$$v_k = \min(v(x,t), k), \qquad k = 1, 2, \dots,$$

for which the lemma has already been proved. $\square$

7. Differentiability of lower powers

The attentive reader has probably noticed that, beginning with the definition of a solution, the theory is formulated in terms of the the gradient of v^m, while the gradient of v itself is avoided. Indeed, the validity of the rule $\nabla(v^m) = mv^{m-1}\nabla v$ is not clear on the interface, but the open set where $v(x,t) > \varepsilon > 0$ offers no problem in this respect. To this we may add that, concerning the mollification (2.6) it may happen that $\nabla(v^*)$ does not exist in Sobolev's sense. In fact, this phenomenon occurs already for the Barenblatt solution $\mathcal{B}(x,t)$ even if the origin is excluded. Thus the use of convolutions like (2.6) requires caution.

Calculations with $\mathcal{B}(x,t)$ reveal that, in the domain where it is a solution $(t > 0)$, the Sobolev gradient $\nabla(\mathcal{B}^q)$ exists if and only if $q > (m-1)/2$. This gives a restriction for the power of a solution. Including the singularity at the origin we obtain a further restriction for viscosity supersolutions. The gradient $\nabla(\mathcal{B}^q)$ exists if and only if

$$\frac{m-1}{2} < q < m + \frac{1}{N}.$$

We also have $\nabla(\mathcal{B}^q) \in L^2_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ if and only if

$$\frac{m-1}{2} < q < \frac{m}{2}.$$

This shows that the bound for the power in the next theorem is sharp.

Theorem 7.1. *Suppose that v is a nonnegative viscosity supersolution in Ω. Then the Sobolev derivative $\nabla(v^q)$ exists and belongs to $L^2_{\mathrm{loc}}(\Omega)$ if*

$$\frac{m-1}{2} < q < \frac{m}{2}.$$

Moreover, we have the estimate

$$
\begin{aligned}
\frac{2m\alpha}{(m-\alpha)^2} &\int_{t_1}^{t_2} \int_D \zeta^2 \big|\nabla(v^{(m-\alpha)/2})\big|^2 \, dx \, dt \\
&+ \frac{1}{1-\alpha} \int_D \zeta^2(x) v(x,t_1)^{1-\alpha} \, dx \\
\leq \frac{2m}{\alpha} &\int_{t_1}^{t_2} \int_D v^{m-\alpha} |\nabla\zeta|^2 \, dx \, dt \\
&+ \frac{1}{1-\alpha} \int_D \zeta^2(x) v(x,t_2)^{1-\alpha} \, dx
\end{aligned}
\tag{7.2}
$$

wheever $0 < \alpha < 1$ and $D \times (t_1, t_2) \Subset \Omega$. Here $\zeta = \zeta(x)$ depends only on x, $\zeta \in C_0^\infty(D)$ and $\zeta \geq 0$.

Proof. The proof requires several steps. First, notice that it is enough to prove the Caccioppoli estimate (7.2). Second, it is enough to consider bounded functions v. Indeed, we can apply the estimate for the truncated functions $\min(v, j)$ and the passage $j \to \infty$ offers no difficulties.

Hence we assume that v is bounded. The test function $v(x,t)^{-\alpha}\zeta^2(x)$ formally yields the desired inequality. We use the modified test function

$$\varphi(x, t) = \frac{\zeta^2(x)}{(v(x,t)^m + \varepsilon)^{\alpha/m}},$$

where $\varepsilon > 0$ is an auxiliary parameter. Here v^m appears in a convenient way. Observe that $\varphi \in L^2(t_1, t_2; H_0^1(D))$ and we may use it as a test function in (2.12). By (2.7) we have

$$\int_{t_1}^{t_2} \varphi \frac{\partial(v^*)}{\partial t} \, dt = \zeta^2(x) \int_{t_1}^{t_2} (v^m + \varepsilon)^{-\alpha/m} \frac{\partial(v^*)}{\partial t} \, dt$$

$$= \zeta^2(x) \int_{t_1}^{t_2} \left((v^m + \varepsilon)^{-\alpha/m} - ((v^*)^m + \varepsilon)^{-\alpha/m} \right) \frac{v - v^*}{\sigma} \, dt$$

$$+ \zeta^2(x) \int_{t_1}^{t_2} ((v^*)^m + \varepsilon)^{-\alpha/m} \frac{\partial(v^*)}{\partial t} \, dt$$

$$\leq \zeta^2(x) \int_{t_1}^{t_2} ((v^*)^m + \varepsilon)^{-\alpha/m} \frac{\partial(v^*)}{\partial t} \, dt.$$

In the inequality, we used the fact that one of the integrands is non-positive. The integral on the right-hand side can, in principle, be evaluated. We have the bound

$$\int_{t_1}^{t_2} \varphi \frac{\partial(v^*)}{\partial t} \, dt \leq \zeta^2(x) \int_{v^*(x,t_1)}^{v^*(x,t_2)} (z^m + \varepsilon)^{-\alpha/m} \, dz, \qquad (7.3)$$

where it is decisive that $\alpha < 1$. Integrating with respect to x and letting $\sigma \to 0$, we arrive at

$$\limsup_{\sigma \to 0} \int_{t_1}^{t_2} \int_D \varphi \frac{\partial(v^*)}{\partial t} \, dt \, dx \leq \int_D \zeta^2(x) \int_{v(x,t_1)}^{v(x,t_2)} (z^m + \varepsilon)^{-\alpha/m} \, dz \, dx.$$

So far, (2.12) takes the form

$$-\int_{t_1}^{t_2} \int_D \nabla\varphi \cdot \nabla(v^m) \, dx \, dt \leq \int_D \zeta(x)^2 \int_{v(x,t_1)}^{v(x,t_2)} (z^m + \varepsilon)^{-\alpha/m} \, dz,$$

where σ is no longer present.

We observe that the obtained upper bound has the limit

$$\frac{1}{1-\alpha} \int_D \zeta^2 \left(v(x, t_2)^{1-\alpha} - v(x, t_1)^{1-\alpha} \right) dx$$

as $\varepsilon \to 0$.

Next we consider the elliptic term of (2.12), where we already wrote $\nabla\varphi \cdot \nabla(v^m)$ instead of $\nabla\varphi \cdot \nabla(v^m)^*$. We have

$$\nabla\varphi \cdot \nabla(v^m) = -\frac{\alpha}{m}\frac{\zeta^2|\nabla(v^m)|^2}{(v^m+\varepsilon)^{1+\alpha/m}} + \frac{2\zeta\nabla\zeta \cdot \nabla(v^m)}{(v^m+\varepsilon)^{\alpha/m}}. \qquad (7.4)$$

The elementary inequality $2ab \leq \beta a^2 + \beta^{-1}b^2$ yields

$$\frac{2\zeta|\nabla\zeta||\nabla(v^m)|}{(v^m+\varepsilon)^{\alpha/m}} \leq \beta\frac{\zeta^2|\nabla(v^m)|^2}{(v^m+\varepsilon)^{1+\alpha/m}} + \frac{1}{\beta}|\nabla\zeta|^2(v^m+\varepsilon)^{1-\alpha/m}.$$

When β is small, say $\beta = \alpha/(2m)$, the first term on the right-hand side is absorbed by the similar term in (7.4). Writing

$$\frac{\zeta^2|\nabla(v^m)|^2}{(v^m+\varepsilon)^{1+\alpha/m}} = \left(\frac{2m}{m-\alpha}\right)^2\big|\nabla(v^m+\varepsilon)^{(m-\alpha)/(2m)}\big|^2,$$

we finally arrive at

$$\frac{2m\alpha}{(m-\alpha)^2}\int_{t_1}^{t_2}\int_D \zeta^2\big|\nabla(v^m+\varepsilon)^{(m-\alpha)/(2m)})\big|^2\,dx\,dt$$

$$\leq \frac{2m}{\alpha}\int_{t_1}^{t_2}\int_D (v^m+\varepsilon)^{(m-\alpha)/m}|\nabla\zeta|^2\,dx\,dt$$

$$+ \int_D \zeta^2(x)\int_{v(x,t_1)}^{v(x,t_2)}(z^m+\varepsilon)^{-\alpha/m}\,dz\,dx$$

after some arithmetic. Now we can obviously let $\varepsilon \to 0$ and use a standard weak compactness argument to conclude that $\nabla(v^{(m-\alpha)/2})$ exists in Sobolev's sense. This concludes the proof. $\qquad\square$

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