SCALING INVARIANT SOBOLEV-LORENTZ CAPACITY ON $\mathbf{R}^n$

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Abstract. We develop a capacity theory based on the definition of Sobolev functions on $\mathbf{R}^n$ with respect to the Lorentz norm. Basic properties of capacity, including monotonicity, finite subadditivity and convergence results are included. We also provide sharp estimates for the capacity of balls. Sobolev-Lorentz capacity and Hausdorff measures are related.

1. Introduction

We recall that for $1 \le p < \infty$ and $0 \le \lambda \le n$, the *Morrey space* $\mathcal{L}^{p,\lambda}(\mathbf{R}^n)$ is defined to be the linear space of measurable functions $u \in L^1_{loc}(\mathbf{R}^n)$ such that

$$||u||_{\mathcal{L}^{p,\lambda}(\mathbf{R}^n)} = \sup_{x\in\mathbf{R}^n}\sup_{r>0}\left(r^{-\lambda}\int_{B(x,r)}|u(y)|^p dy\right)^{1/p} < \infty.$$

In other words, the fractional maximal function

$$M_{n-\lambda}u(x) = \sup_{r>0}\left(r^{n-\lambda}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y)|^p dy\right)^{1/p}$$

is bounded in $\mathbf{R}^n$. In particular, $\mathcal{L}^{n,0}(\mathbf{R}^n) = L^n(\mathbf{R}^n)$. We refer to [Gia83, p. 65] for more information about Morrey spaces and their use in the theory of partial differential equations. One notices that the weak Lebesgue space $L^{n,\infty}(\mathbf{R}^n)$ is contained in $\mathcal{L}^{p,n-p}(\mathbf{R}^n)$ for every $p \in [1, n)$. Similarly we can define the Morrey space $\mathcal{L}^{p,\lambda}(\mathbf{R}^n;\mathbf{R}^m)$ for vector-valued measurable functions. Capacities related to Morrey spaces were studied by Adams and Xiao in [AX04].

We already noticed that the Lorentz spaces embed continuously into the Morrey spaces; that is, $L^{n,q}(\mathbf{R}^n) \hookrightarrow L^{n,\infty}(\mathbf{R}^n) \hookrightarrow \mathcal{L}^{p,n-p}(\mathbf{R}^n)$ whenever $1 \le p < n < q \le \infty$. Sobolev-Lorentz spaces have recently been studied by Kauhanen, Koskela, and Malý in [KKM99] and by Malý, Swanson, and Ziemer in [MSZ05].

Our results concerning the Sobolev-Lorentz capacity generalize some of the results concerning s-capacity on $\mathbf{R}^n$ for $s \in (1, n]$. See [HKM93, Chapter 2] for the s-capacity on $\mathbf{R}^n$ and [KM96], [KM00] for capacity on general metric spaces.

Using [HKM93, 2.13], we provide sharp estimates for the Sobolev-Lorentz n,q-capacity of pairs $(\overline{B}(0,r), B(0,1))$ for $n < q \le \infty$ and small r. The Sobolev-Lorentz capacity and Hausdorff measures are also related; we obtain results that are Sobolev-Lorentz analogues of those obtained by Reshetnyak in [Res69], Martio in [Mar79], Maz'ja in [Maz85] and others.

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2. Preliminaries

Our notation in this paper is standard and generally as in [HKM93]. Here Ω will denote a nonempty open subset of $\mathbf{R}^n$, while $dx = dm_n(x)$ will denote the Lebesgue n-measure in $\mathbf{R}^n$, where $n \geq 2$ is integer. For two sets $A, B \subset \mathbf{R}^n$, we define $\mathrm{dist}(A, B)$, the distance between A and B, by

$$\mathrm{dist}(A, B) = \inf_{a \in A, b \in B} |a - b|.$$

For $n \geq 2$ integer $\Omega_n = |B(0,1)|$ denotes the measure of the n-dimensional unit ball, that is $\Omega_n = |B(0,1)|$. Thus, $\omega_{n-1} = n\Omega_n$, where ω_{n-1} denotes the spherical measure of the $n-1$-dimensional sphere.

For a measurable $u : \Omega \to \mathbf{R}^n$, supp u is the smallest closed set such that u vanishes outside supp u. We also define

$$\begin{aligned} C_0(\Omega) &= \{\varphi \in C(\Omega) : \mathrm{supp}\ \varphi \subset\subset \Omega\} \\ Lip(\Omega) &= \{\varphi : \Omega \to \mathbf{R} : \varphi \text{ is Lipschitz}\}. \end{aligned}$$

For a function $\varphi \in Lip(\Omega) \cap C_0(\Omega)$ we write

$$\nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \dots, \partial_n\varphi)$$

for the gradient of φ. This notation makes sense, since from Rademacher's theorem ([Fed69, Theorem 3.1.6]) every Lipschitz function on $\mathbf{R}^n$ is a.e. differentiable.

Throughout this section we will assume that $m \geq 1$ is a positive integer. Let $f : \Omega \to \mathbf{R}^m$ be a measurable function. We define $\lambda_{[f]}$, the *distribution function* of f as follows (see [BS88, Definition II.1.1] and [SW75, p. 57]):

$$\lambda_{[f]}(t) = |\{x \in \Omega : |f(x)| > t\}|, \qquad t \geq 0.$$

We define f^*, the *nonincreasing rearrangement* of f by

$$f^*(t) = \inf\{v : \lambda_{[f]}(v) \leq t\}, \quad t \geq 0.$$

(See [BS88, Definition II.1.5] and [SW75, p. 189].) We notice that f and f^* have the same distribution function. Moreover, for every positive α we have $(|f|^\alpha)^* = (|f|^*)^\alpha$ and if $|g| \leq |f|$ a.e. on Ω, then $g^* \leq f^*$. (See [BS88, Proposition II.1.7].) We also define f^{**}, the *maximal function* of f^* by

$$f^{**}(t) = m_{f^*}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad t > 0.$$

(See [BS88, Definition II.3.1] and [SW75, p. 203].)

Throughout this paper, we will denote by p' the Hölder conjugate of $p \in [1, \infty]$, that is

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ \frac{p}{p-1} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

The *Lorentz space* $L^{p,q}(\Omega; \mathbf{R}^m)$, $1 < p < \infty$, $1 \leq q \leq \infty$, is defined as follows:

$$L^{p,q}(\Omega; \mathbf{R}^m) = \{f : \Omega \to \mathbf{R}^m : f \text{ is measurable and } ||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} < \infty\},$$

where

$$||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} = ||\,|f|\,||_{p,q} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \, \frac{dt}{t}\right)^{\frac{1}{q}} & 1 \leq q < \infty \\ \sup_{t>0} t\lambda_{[f]}(t)^{\frac{1}{p}} = \sup_{s>0} s^{\frac{1}{p}} f^*(s) & q = \infty. \end{cases}$$

(See [BS88, Definition IV.4.1] and [SW75, p. 191].) If $1 \le q \le p$, then $||\cdot||_{L^{p,q}(\Omega;\mathbf{R}^m)}$ already represents a norm, but for $p < q \le \infty$ it represents a quasinorm, equivalent to the norm $||\cdot||_{L^{(p,q)}(\Omega;\mathbf{R}^m)}$, where

$$||f||_{L^{(p,q)}(\Omega;\mathbf{R}^m)} = ||\,|f|\,||_{(p,q)} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^q \, \frac{dt}{t}\right)^{\frac{1}{q}} & 1 \le q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & q = \infty. \end{cases}$$

(See [BS88, Definition IV.4.4].) Namely, from [BS88, Lemma IV.4.5] we have that

$$||\,|f|\,||_{L^{p,q}(\Omega)} \le ||\,|f|\,||_{L^{(p,q)}(\Omega)} \le \frac{p}{p-1} ||\,|f|\,||_{L^{p,q}(\Omega)}$$

for every $1 \le q \le \infty$ and every measurable function $f : \Omega \to \mathbf{R}^m$.

It is known that $(L^{p,q}(\Omega;\mathbf{R}^m), ||\cdot||_{L^{p,q}(\Omega;\mathbf{R}^m)})$ is a Banach space for $1 \le q \le p$, while $(L^{p,q}(\Omega;\mathbf{R}^m), ||\cdot||_{L^{(p,q)}(\Omega;\mathbf{R}^m)})$ is a Banach space for $1 < p < \infty$, $1 \le q \le \infty$. These spaces are reflexive if $1 < q < \infty$. (See [BS88, Theorem IV.4.7, Corollaries I.4.3 and IV.4.8], the definition of $L^{p,q}(\Omega;\mathbf{R}^m)$ and the discussion after Definition 2.1.)

Definition 2.1. (See [BS88, Definition I.3.1].) Let $1 < p < \infty$ and $1 \le q \le \infty$. Let $X = L^{p,q}(\Omega;\mathbf{R}^m)$. A function f in X is said to have *absolutely continuous norm* in X if and only if $||f\chi_{E_k}||_X \to 0$ for every sequence E_k satisfying $E_k \to \emptyset$ a.e.

Let X_a be the subspace of X consisting of functions of absolutely continuous norm and let X_b be the closure in X of the set of simple functions. It is known that $X_a = X_b$. (See [BS88, Theorem I.3.13].) Moreover, we have $X_a = X_b = X$ whenever $1 \le q < \infty$. (See [BS88, Theorem IV.4.7 and Corollary IV.4.8] and the definition of $L^{p,q}(\Omega;\mathbf{R}^m)$.)

We prove now that $X_a \ne X$ for $X = L^{p,\infty}(\Omega;\mathbf{R}^m)$. Without loss of generality we can assume that $m = 1$ and that $\Omega = B(0,2) \setminus \{0\}$. We define $u : \Omega \to \mathbf{R}$,

$$u(x) = \begin{cases} |x|^{-\frac{n}{p}} & \text{if } 0 < |x| < 1 \\ 0 & \text{if } 1 \le |x| \le 2. \end{cases} \tag{1}$$

It is easy to see that $u \in L^{p,\infty}(\Omega)$ and moreover,

$$||u\chi_{B(0,\alpha)}||_{L^{p,\infty}(\Omega)} = ||u||_{L^{p,\infty}(\Omega)} = \Omega_n^{1/p}$$

for every $\alpha > 0$. This shows that u does not have absolutely continuous weak L^p-norm and therefore $L^{p,\infty}(\Omega)$ does not have absolutely continuous norm. Since $L^{p,\infty}(\Omega)$ can be identified with $(L^{p',1}(\Omega))^*$ (see [BS88, Corollary IV.4.8]), it follows from [BS88, Corollaries I.4.3, I.4.4, IV.4.8 and Theorem IV.4.7] that neither $L^{p,1}(\Omega)$, nor $L^{p,\infty}(\Omega)$ are reflexive whenever $1 < p < \infty$.

Remark 2.2. It is also known (see [BS88, Proposition IV.4.2]) that for every $p \in (1,\infty)$ and $1 \le r < s \le \infty$ there exists a constant $C(p,r,s)$ such that

$$||\,|f|\,||_{L^{p,s}(\Omega)} \le C(p,r,s) ||\,|f|\,||_{L^{p,r}(\Omega)} \tag{2}$$

for all measurable functions $f \in L^{p,r}(\Omega;\mathbf{R}^m)$ and all integers $m \ge 1$. In particular, we have the embedding $L^{p,r}(\Omega;\mathbf{R}^m) \hookrightarrow L^{p,s}(\Omega;\mathbf{R}^m)$.

We have the following generalized Hölder inequality for Lorentz spaces.

Theorem 2.3. *Suppose $\Omega \subset \mathbf{R}^n$ has finite measure. Let $1 < p_1, p_2, p_3 < \infty$, $1 \le q_1, q_2, q_3 \le \infty$ be such that*

$$\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}$$

and either

$$\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$$

whenever $1 \le q_1, q_2, q_3 < \infty$ or $1 \le q_1 = q_2 \le q_3 = \infty$ or $1 \le q_1 = q_3 \le q_2 = \infty$. Then

$$||f||_{L^{p_1,q_1}(\Omega;\mathbf{R}^m)} \le ||f||_{L^{p_2,q_2}(\Omega;\mathbf{R}^m)} \, ||\chi_\Omega||_{L^{p_3,q_3}(\Omega)}.$$

Proof. From the definition of the Lorentz norms and quasinorms for vector-valued functions, it follows that it is enough to assume that $m = 1$. Let $f \in L^{p_2,q_2}(\Omega)$. Since Ω has finite measure, we have $f^*(t) = 0$ for every $t \ge |\Omega|$. We have to analyze few distinct cases.

(i) $1 \le q_1, q_2, q_3 < \infty$. We have

$$\begin{aligned}
||f||_{L^{p_1,q_1}(\Omega)} &= \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_1} - \frac{1}{q_1}})^{q_1} \, dt \right)^{\frac{1}{q_1}} \\
&= \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_2} - \frac{1}{q_2}} \, t^{\frac{1}{p_3} - \frac{1}{q_3}})^{q_1} \, dt \right)^{\frac{1}{q_1}} \\
&\le \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_2} - \frac{1}{q_2}})^{q_2} \, dt \right)^{\frac{1}{q_2}} \left(\int_0^{|\Omega|} (t^{\frac{1}{p_3} - \frac{1}{q_3}})^{q_3} \right)^{\frac{1}{q_3}} \\
&= ||f||_{L^{p_2,q_2}(\Omega)} \, ||\chi_\Omega||_{L^{p_3,q_3}(\Omega)}.
\end{aligned}$$

(ii) $q_1 = q_2 = q_3 = \infty$. Then

$$\begin{aligned}
||f||_{L^{p_1,\infty}(\Omega)} &= \sup_{0 \le t \le |\Omega|} t^{\frac{1}{p_1}} f^*(t) \le |\Omega|^{\frac{1}{p_1} - \frac{1}{p_2}} \sup_{0 \le t \le |\Omega|} t^{\frac{1}{p_2}} f^*(t) \\
&= |\Omega|^{\frac{1}{p_3}} \, ||f||_{L^{p_2,\infty}(\Omega)} = ||f||_{L^{p_2,\infty}(\Omega)} \, ||\chi_\Omega||_{L^{p_3,\infty}(\Omega)}.
\end{aligned}$$

(iii) $1 \le q_1 = q_2 < q_3 = \infty$. Then

$$\begin{aligned}
||f||_{L^{p_1,q_1}(\Omega)} &= \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_1} - \frac{1}{q_1}})^{q_1} \, dt \right)^{\frac{1}{q_1}} \\
&= \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_2} - \frac{1}{q_1}})^{q_1} \, t^{\frac{q_1}{p_3}} \, dt \right)^{\frac{1}{q_1}} \\
&\le |\Omega|^{\frac{1}{p_3}} \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_2} - \frac{1}{q_1}})^{q_1} \, dt \right)^{\frac{1}{q_1}} \\
&= ||f||_{L^{p_2,q_1}(\Omega)} \, ||\chi_\Omega||_{L^{p_3,\infty}(\Omega)} = ||f||_{L^{p_2,q_2}(\Omega)} \, ||\chi_\Omega||_{L^{p_3,\infty}(\Omega)}.
\end{aligned}$$

(iv) $1 \leq q_1 = q_3 < q_2 = \infty$. Then

$$\begin{aligned}
||f||_{L^{p_1,q_1}(\Omega)} &= \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_1}-\frac{1}{q_1}})^{q_1}\, dt \right)^{\frac{1}{q_1}} \\
&= \left(\int_0^{|\Omega|} (f^*(t)\, t^{\frac{1}{p_2}})^{q_1}\, (t^{\frac{1}{p_3}-\frac{1}{q_1}})^{q_1}\, dt \right)^{\frac{1}{q_1}} \\
&\leq \sup_{0 \leq t \leq |\Omega|} f^*(t)\, t^{\frac{1}{p_2}} \left(\int_0^{|\Omega|} (t^{\frac{1}{p_3}-\frac{1}{q_1}})^{q_1}\, dt \right)^{\frac{1}{q_1}} \\
&= ||f||_{L^{p_2,\infty}(\Omega)}\, ||\chi_\Omega||_{L^{p_3,q_1}(\Omega)} = ||f||_{L^{p_2,\infty}(\Omega)}\, ||\chi_\Omega||_{L^{p_3,q_3}(\Omega)}.
\end{aligned}$$

This finishes the proof. □

As an application of Theorem 2.3 we have the following result.

Corollary 2.4. *Let $1 < p < q \leq \infty$ and $\varepsilon \in (0, p-1)$ be fixed. Suppose $\Omega \subset \mathbf{R}^n$ has finite measure. Then*

$$||f||_{L^{p-\varepsilon}(\Omega;\mathbf{R}^m)} \leq C(p,q,\varepsilon)\, |\Omega|^{\frac{\varepsilon}{p(p-\varepsilon)}}\, ||f||_{L^{p,q}(\Omega;\mathbf{R}^m)} \tag{3}$$

for every integer $m \geq 1$, where

$$C(p,q,\varepsilon) = \begin{cases} \left(\frac{p(q-p+\varepsilon)}{q}\right)^{\frac{1}{p-\varepsilon}-\frac{1}{q}} \varepsilon^{\frac{1}{q}-\frac{1}{p-\varepsilon}}, & p < q < \infty \\ p^{\frac{1}{p-\varepsilon}}\, \varepsilon^{-\frac{1}{p-\varepsilon}}, & q = \infty. \end{cases}$$

Proof. From the definition of the Lorentz norms and quasinorms for vector-valued functions, it follows that it is enough to assume that $m = 1$. A simple application of Theorem 2.3 gives us the desired conclusion. □

We have two interesting results concerning Lorentz spaces.

Theorem 2.5. *Suppose $1 < p < q \leq \infty$. Let $\Omega \subset \mathbf{R}^n$ and let $f_1, f_2 \in L^{p,q}(\Omega)$. We let $f_3 = \max(|f_1|, |f_2|)$. Then $f_3 \in L^{p,q}(\Omega)$ and*

$$||f_3||^p_{L^{p,q}(\Omega)} \leq ||f_1||^p_{L^{p,q}(\Omega)} + ||f_2||^p_{L^{p,q}(\Omega)}.$$

Proof. Without loss of generality we can assume that both f_1 and f_2 are nonnegative. We have to consider two cases, depending on whether $p < q < \infty$ or $q = \infty$.

Suppose $p < q < \infty$. We have ([KKM99, Proposition 2.1])

$$||f_i||^p_{L^{p,q}(\Omega)} = \left(p \int_0^\infty s^{q-1} \lambda_{[f_i]}(s)^{\frac{q}{p}}\, ds \right)^{\frac{p}{q}},$$

where $\lambda_{[f_i]}$ is the distribution function of f_i for $i = 1, 2, 3$. From the definition of f_3 we obviously have $\lambda_{[f_3]}(s) \leq \lambda_{[f_1]}(s) + \lambda_{[f_2]}(s)$ for every $s \geq 0$, which implies that

$$\begin{aligned}
||f_3||^p_{L^{p,q}(\Omega)} &\leq \left(p \int_0^\infty s^{q-1} (\lambda_{[f_1]}(s) + \lambda_{[f_2]}(s))^{\frac{q}{p}}\, ds \right)^{\frac{p}{q}} \\
&\leq \left(p \int_0^\infty s^{q-1} \lambda_{[f_1]}(s)^{\frac{q}{p}}\, ds \right)^{\frac{p}{q}} + \left(p \int_0^\infty s^{q-1} \lambda_{[f_2]}(s)^{\frac{q}{p}}\, ds \right)^{\frac{p}{q}} \\
&= ||f_1||^p_{L^{p,q}(\Omega)} + ||f_2||^p_{L^{p,q}(\Omega)}.
\end{aligned}$$

Suppose now $q = \infty$. From the definition of f_3 we obviously have as before $\lambda_{[f_3]}(s) \le \lambda_{[f_1]}(s) + \lambda_{[f_2]}(s)$ for every $s \ge 0$. Therefore

$$s^p\, \lambda_{[f_3]}(s) \le s^p\, \lambda_{[f_1]}(s) + s^p\, \lambda_{[f_3]}(s)$$

for every $s \ge 0$ which implies

$$s^p\, \lambda_{[f_3]}(s) \le ||f_1||^p_{L^{p,\infty}(\Omega)} + ||f_2||^p_{L^{p,\infty}(\Omega)} \tag{4}$$

for every $s \ge 0$. By taking the supremum over all $s \ge 0$ in (4), we get the desired conclusion. □

Theorem 2.6. *Suppose $1 < p < q \le \infty$ and $\varepsilon \in (0,1)$. Let $\Omega \subset \mathbf{R}^n$ and let $f_1, f_2 \in L^{p,q}(\Omega)$. We denote $f_3 = f_1 + f_2$. Then $f_3 \in L^{p,q}(\Omega)$ and*

$$||f_3||^p_{L^{p,q}(\Omega)} \le (1-\varepsilon)^{-p}||f_1||^p_{L^{p,q}(\Omega)} + \varepsilon^{-p}||f_2||^p_{L^{p,q}(\Omega)}.$$

Proof. Without loss of generality we can assume that both f_1 and f_2 are nonnegative. We have to consider two cases, depending on whether $p < q < \infty$ or $q = \infty$.

Suppose $p < q < \infty$. We have ([KKM99, Proposition 2.1])

$$||f_i||^p_{L^{p,q}(\Omega)} = \left(p\int_0^\infty s^{q-1}\lambda_{[f_i]}(s)^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}},$$

where $\lambda_{[f_i]}$ is the distribution function of f_i for $i = 1,2,3$. From the definition of f_3 we obviously have $\lambda_{[f_3]}(s) \le \lambda_{[f_1]}((1-\varepsilon)s) + \lambda_{[f_2]}(\varepsilon s)$ for every $s \ge 0$, which implies that

$$\begin{aligned}
||f_3||^p_{L^{p,q}(\Omega)} &\le \left(p\int_0^\infty s^{q-1}(\lambda_{[f_1]}((1-\varepsilon)s) + \lambda_{[f_2]}(\varepsilon s))^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}} \\
&\le \left(p\int_0^\infty s^{q-1}\lambda_{[f_1]}((1-\varepsilon)s)^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}} + \left(p\int_0^\infty s^{q-1}\lambda_{[f_2]}(\varepsilon s)^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}} \\
&= (1-\varepsilon)^{-p}||f_1||^p_{L^{p,q}(\Omega)} + \varepsilon^{-p}||f_2||^p_{L^{p,q}(\Omega)}.
\end{aligned}$$

Suppose now $q = \infty$. From the definition of f_3 we obviously have as before $\lambda_{[f_3]}(s) \le \lambda_{[f_1]}((1-\varepsilon)s) + \lambda_{[f_2]}(\varepsilon s)$ for every $s \ge 0$. Therefore

$$s^p\, \lambda_{[f_3]}(s) \le s^p\, \lambda_{[f_1]}((1-\varepsilon)s) + s^p\, \lambda_{[f_3]}(\varepsilon s)$$

for every $s \ge 0$ which implies

$$s^p\, \lambda_{[f_3]}(s) \le (1-\varepsilon)^{-p}||f_1||^p_{L^{p,\infty}(\Omega)} + \varepsilon^{-p}||f_2||^p_{L^{p,\infty}(\Omega)} \tag{5}$$

for every $s \ge 0$. By taking the supremum over all $s \ge 0$ in (5), we get the desired conclusion. □

Theorem 2.6 has an interesting corollary.

Corollary 2.7. *Let $\Omega \subset \mathbf{R}^n$ be open. Suppose $1 < p < \infty$ and $1 \le q \le \infty$. Let f_k be a sequence of functions in $L^{p,q}(\Omega;\mathbf{R}^m)$ converging to f with respect to the p,q-quasinorm and pointwise a.e. in Ω. Then*

$$\lim_{k\to\infty} ||f_k||_{L^{p,q}(\Omega;\mathbf{R}^m)} = ||f||_{L^{p,q}(\Omega;\mathbf{R}^m)}.$$

Proof. We can assume without loss of generality that $m = 1$. Since for $1 \le q \le p$ $||\cdot||_{L^{p,q}(\Omega)}$ is already a norm, the claim is trivial in this case. Hence we can assume without loss of generality that $p < q \le \infty$. The proof for the case $q = \infty$ was presented to me by Jan Malý.

Since $f^* \le \liminf f_k^*$ (see [BS88, Proposition II.1.7]), it follows easily that

$$\liminf_{k\to\infty} ||f_k||_{L^{p,q}(\Omega)} \ge ||f||_{L^{p,q}(\Omega)}.$$

We would be done if we show that

$$\limsup_{k\to\infty} ||f_k||_{L^{p,q}(\Omega)} \le ||f||_{L^{p,q}(\Omega)}. \tag{6}$$

In order to do that we fix $\varepsilon \in (0, 1)$. From Theorem 2.6 we have

$$||f_k||^p_{L^{p,q}(\Omega)} \le (1-\varepsilon)^{-p}||f||^p_{L^{p,q}(\Omega)} + \varepsilon^{-p}||f_k - f||^p_{L^{p,q}(\Omega)}$$

for every $k = 1, 2, \ldots$ Taking $\limsup$ on both sides and using the fact that f_k converges to f with respect to the $L^{p,q}$-quasinorm, we get

$$\limsup_{k\to\infty} ||f_k||^p_{L^{p,q}(\Omega)} \le (1-\varepsilon)^{-p}||f||^p_{L^{p,q}(\Omega)}. \tag{7}$$

Letting $\varepsilon \to 0$ in (7) yields (6). This finishes the proof.

□

We use the notation

$$u^+ = \max(u, 0) \text{ and } u^- = \min(u, 0).$$

If $u \in C_0(\Omega) \cap Lip(\Omega)$, then obviously $u^+ \in C_0(\Omega) \cap Lip(\Omega)$ and from [HKM93, Lemmas 1.11 and 1.19] we have

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0 \\ 0 & \text{if } u \le 0. \end{cases} \tag{8}$$

3. Sobolev-Lorentz n, q relative capacity

Suppose $1 < q \le \infty$. Let $\Omega \subset \mathbf{R}^n$ be an open set. Let $K \subset \Omega$ be compact. The Sobolev-Lorentz n, q-capacity of the pair (K, Ω) is denoted

$$\mathrm{cap}_{n,q}(K, \Omega) = \inf \{||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} : u \in W(K, \Omega)\},$$

where

$$W(K, \Omega) = \{u \in C_0^\infty(\Omega) : u \ge 1 \text{ in a neighborhood of } K\}.$$

We call $W(K, \Omega)$ the *set of admissible functions for the condenser* (K, Ω).

Lemma 3.1. *If $K \subset \Omega$ is compact, then we can get the same capacity if we restrict ourselves to a bigger set, namely*

$$W_0(K, \Omega) = \{u \in C_0(\Omega) \cap Lip(\Omega) : u \ge 1 \text{ on } K\}.$$

Proof. Let $u \in W_0(K, \Omega)$. We can assume without loss of generality that $u \ge 1$ in a neighborhood $U \subset\subset \Omega$ of K and that Ω is bounded. Let $\eta \in C_0^\infty(B(0, 1))$ be a mollifier. For every integer $j \ge 1$ let $\eta_j(x) = j^n\eta(jx)$ and let $u_j = \eta_j * u$ be the convolution defined by

$$u_j(x) = (\eta_j * u)(x) = \int_{\mathbf{R}^n} \eta_j(x - y)u(y)dy.$$

For the basic properties of a mollifier see [Zie89, Theorems 1.6.1 and 2.1.3]. Let $\widetilde{U}$ be a neighborhood of K such that $\widetilde{U} \subset\subset U$ and let j_0 be a positive integer such that

$$1/j_0 < \min\{\operatorname{dist}(\operatorname{supp} u, \partial\Omega), \operatorname{dist}(\widetilde{U}, \partial U)\}.$$

It is easy to see that $u_j, j \geq j_0$ is a sequence in $W(K, \Omega)$ and since $u \in C_0(\Omega) \cap Lip(\Omega)$, we have from [HKM93, Lemma 1.11] that

$$\lim_{j\to\infty} (||u_j - u||_{L^{n+1}(\Omega)} + ||\nabla u_j - \nabla u||_{L^{n+1}(\Omega;\mathbf{R}^n)}) = 0.$$

This together with (2) and Theorem 2.3 yields

$$\lim_{j\to\infty} (||u_j - u||_{L^{n,q}(\Omega)} + ||\nabla u_j - \nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)}) = 0. \tag{9}$$

An appeal to Corollary 2.7 applied for $p = n$ establishes the assertion, since $W(K, \Omega) \subset W_0(K, \Omega)$. □

Since truncation decreases the n, q-quasinorm whenever $1 < q \leq \infty$, it follows from Lemma 3.1 that we can choose only functions $u \in W_0(K, \Omega)$ that satisfy $0 \leq u \leq 1$ when computing the n, q relative capacity.

3.1. **Basic properties of the n, q relative capacity.** Usually, a capacity is a monotone and subadditive set function. The following theorem will show, among other things, that this is true in the case of the n, q relative capacity. We follow [HKM93].

Theorem 3.2. *Suppose $1 < q \leq \infty$. Let $\Omega \subset \mathbf{R}^n$ be open. The set function $K \mapsto \operatorname{cap}_{n,q}(K, \Omega)$, $K \subset \Omega$, K compact, enjoys the following properties:*

(i) *If $K_1 \subset K_2$, then $\operatorname{cap}_{n,q}(K_1, \Omega) \leq \operatorname{cap}_{n,q}(K_2, \Omega)$.*

(ii) *If $\Omega_1 \subset \Omega_2$ are open and K is a compact subset of Ω_1, then*

$$\operatorname{cap}_{n,q}(K, \Omega_2) \leq \operatorname{cap}_{n,q}(K, \Omega_1).$$

(iii) *If K_i is a decreasing sequence of compact subsets of Ω with $K = \bigcap_{i=1}^{\infty} K_i$, then*

$$\operatorname{cap}_{n,q}(K, \Omega) = \lim_{i\to\infty} \operatorname{cap}_{n,q}(K_i, \Omega).$$

(iv) *Suppose $n \leq q \leq \infty$. If $K = \bigcup_{i=1}^{k} K_i \subset \Omega$ then*

$$\operatorname{cap}_{n,q}(K, \Omega) \leq \sum_{i=1}^{k} \operatorname{cap}_{n,q}(K_i, \Omega),$$

where $k \geq 1$ is a positive integer.

(v) *If $K = \bigcup_{i=1}^{k} K_i \subset \Omega$ then*

$$\operatorname{cap}_{n,q}^{1/n}(K, \Omega) \leq \sum_{i=1}^{k} \operatorname{cap}_{n,q}^{1/n}(K_i, \Omega),$$

where $k \geq 1$ is a positive integer.

Proof. Properties (i) and (ii) are immediate consequences of the definition.

(iii) Let $b =: \lim_{i\to\infty} \operatorname{cap}_{n,q}(K_i, \Omega)$. We fix a small $\varepsilon > 0$ and we pick a function $u \in W(K, \Omega)$ such that

$$||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} < \operatorname{cap}_{n,q}(K, \Omega) + \varepsilon.$$

When i is large, the sets K_i lie in the compact set $\{u \geq 1-\varepsilon\}$. Therefore

$$\lim_{i\to\infty} \operatorname{cap}_{n,q}(K_i,\Omega) \leq \operatorname{cap}_{n,q}(\{u \geq 1-\varepsilon\},\Omega) \leq \frac{1}{(1-\varepsilon)^{2n}} ||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)}.$$

Letting $\varepsilon \to 0$ yields $b \leq \operatorname{cap}_{n,q}(K,\Omega)$, whence (iii) follows because obviously $b \geq \operatorname{cap}_{n,q}(K,\Omega)$.

It is enough to prove (iv) and (v) for $k=2$ because then the general finite case follows by induction.

(iv) When $q=n$ we are in the case of the n-capacity and then the claim holds. (See for example [HKM93, Theorem 2.2 (iii)].) So we can assume without loss of generality that $n < q \leq \infty$.

Let $u_i \in W_0(K_i,\Omega)$, $i=1,2$, such that

$$||\nabla u_i||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} < \operatorname{cap}_{n,q}(K_i,\Omega) + \varepsilon.$$

We define $u = \max(u_1,u_2)$. Since $u = (u_1-u_2)^+ + u_2$, it follows from the discussion after Corollary 2.7 and (8) that $u \in W_0(K_1 \cup K_2,\Omega)$ with $|\nabla u| \leq \max(|\nabla u_1|,|\nabla u_2|)$. This and Theorem 2.5 imply

$$\begin{aligned}\operatorname{cap}_{n,q}(K_1\cup K_2,\Omega) &\leq ||\nabla u||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} \leq ||\nabla u_1||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} + ||\nabla u_2||^n_{L^{n,q}(\Omega;\mathbf{R}^n)} \\ &\leq \operatorname{cap}_{n,q}(K_1,\Omega) + \operatorname{cap}_{n,q}(K_2,\Omega) + 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we complete the proof in the case of two sets, and hence the general finite case.

(v) We notice that (iv) implies (v) when $n \leq q \leq \infty$. So we can assume without loss of generality that $1 < q < n$.

Let $u_i \in W_0(K_i,\Omega)$, $i=1,2$, such that

$$0 \leq u_1 \leq 1 \text{ and } ||\nabla u_i||_{L^{n,q}(\Omega;\mathbf{R}^n)} < \operatorname{cap}^{1/n}_{n,q}(K_i,\Omega) + \varepsilon.$$

Then $u = u_1 + u_2 \in W_0(K_1 \cup K_2,\Omega)$ and since $||\cdot||_{L^{n,q}(\Omega;\mathbf{R}^n)}$ is a norm when $1 < q < n$, we have

$$\begin{aligned}\operatorname{cap}^{1/n}_{n,q}(K_1\cup K_2,\Omega) &\leq ||\nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)} \leq ||\nabla u_1||_{L^{n,q}(\Omega;\mathbf{R}^n)} + ||\nabla u_2||_{L^{n,q}(\Omega;\mathbf{R}^n)} \\ &\leq \operatorname{cap}^{1/n}_{n,q}(K_1,\Omega) + \operatorname{cap}^{1/n}_{n,q}(K_2,\Omega) + 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we complete the proof in the case of two sets, and hence the general finite case. The theorem is proved. □

Remark 3.3. The definition of the n,q-capacity easily implies

$$\operatorname{cap}_{n,q}(K,\Omega) = \operatorname{cap}_{n,q}(\partial K,\Omega)$$

whenever K is a compact set in Ω.

3.2. **Estimates for the n,q relative capacity.** Suppose $1 < q \leq \infty$. Obviously, $\operatorname{cap}_{n,q}(E,\Omega) = \operatorname{cap}_{n,q}(E+x,\Omega+x)$ for every $x \in \mathbf{R}^n$. Indeed, the n,q-quasinorm is invariant under translations.

Lemma 3.4. *Suppose $1 < q \leq \infty$. Let Ω be bounded and $K \subset \Omega$ be compact. Then*

$$\operatorname{cap}_{n,q}(K,\Omega) = \operatorname{cap}_{n,q}(\alpha K,\alpha\Omega), \tag{10}$$

where $\alpha > 0$ and $\alpha A = \{\alpha a : a \in A\}$.

Proof. We have to analyze two cases, depending on whether $1 < q < \infty$ or $q = \infty$.

We assume first that $1 < q < \infty$. Let $u \in C_0^\infty(\Omega)$. We define $u_{(\alpha)} : \alpha\Omega \to \mathbf{R}$ by $u_{(\alpha)}(x) = u(\frac{x}{\alpha})$. Then $u \in W(E, \Omega)$ if and only if $u_{(\alpha)} \in W(\alpha E, \alpha\Omega)$. We notice that $\nabla u_{(\alpha)}(x) = \frac{1}{\alpha} \nabla u(\frac{x}{\alpha})$. We have

$$\begin{aligned} |\{x \in \alpha\Omega : |\nabla u_{(\alpha)}(x)| \geq t\}| &= |\{x \in \alpha\Omega : \frac{1}{\alpha}|\nabla u(\frac{x}{\alpha})| \geq t\}| \\ = \quad |\{x \in \alpha\Omega : |\nabla u(\frac{x}{\alpha})| \geq \alpha t\}| &= \alpha^n |\{\frac{x}{\alpha} \in \Omega : |\nabla u(\frac{x}{\alpha})| \geq \alpha t\}|. \end{aligned}$$

So $\lambda_{[|\nabla u_{(\alpha)}|]}(t) = \alpha^n \lambda_{[|\nabla u|]}(\alpha t)$ for every $t \geq 0$. Therefore

$$\begin{aligned} |\nabla u_{(\alpha)}|^*(t) &= \inf\{v \geq 0 : \lambda_{[|\nabla u_{(\alpha)}|]}(v) \leq t\} = \inf\{v \geq 0 : \alpha^n \lambda_{[|\nabla u|]}(\alpha v) \leq t\} \\ &= \frac{1}{\alpha} \inf\{\alpha v \geq 0 : \lambda_{[|\nabla u|]}(\alpha v) \leq \frac{t}{\alpha^n}\} = \frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n}). \end{aligned}$$

Hence we just proved that $|\nabla u_{(\alpha)}|^*(t) = \frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})$ for every $t \geq 0$. Therefore

$$||\nabla u_{(\alpha)}||^q_{L^{n,q}(\alpha\Omega;\mathbf{R}^n)} = \int_0^\infty t^{\frac{q}{n}} \left(|\nabla u_{(\alpha)}|^*(t)\right)^q \frac{dt}{t} = \int_0^\infty t^{\frac{q}{n}} \left(\frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})\right)^q \frac{dt}{t}.$$

By making the substitution $\frac{t}{\alpha^n} = s$, we have

$$\int_0^\infty t^{\frac{q}{n}} \left(\frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})\right)^q \frac{dt}{t} = \int_0^\infty (s\alpha^n)^{\frac{q}{n}} \left(\frac{1}{\alpha}|\nabla u|^*(s)\right)^q \frac{ds}{s} = ||\nabla u||^q_{L^{n,q}(\Omega;\mathbf{R}^n)}.$$

Thus we get $||\nabla u_{(\alpha)}||_{L^{n,q}(\alpha\Omega;\mathbf{R}^n)} = ||\nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)}$. This proves the claim when $1 < q < \infty$.

Now assume that $q = \infty$. We let $u \in C_0^\infty(\Omega)$ and we define $u_{(\alpha)}$ as before. Then as before, we have $u \in W(K, \Omega)$ if and only if $u_{(\alpha)} \in W(\alpha K, \alpha\Omega)$ and $|\nabla u_{(\alpha)}|^*(t) = \frac{1}{\alpha}|\nabla u|^*(\frac{t}{\alpha^n})$ for every $t \geq 0$. This implies

$$\begin{aligned} (11) \qquad ||\nabla u_{(\alpha)}||^n_{L^{n,\infty}(\alpha\Omega)} &= \sup_{t \geq 0} t\,(|\nabla u_{(\alpha)}|^*(t))^n = \sup_{t \geq 0} \frac{t}{\alpha^n}\,(|\nabla u|^*(\frac{t}{\alpha^n}))^n \\ &= \sup_{s \geq 0} s\,(|\nabla u|^*(s))^n = ||\nabla u||^n_{L^{n,\infty}(\Omega)}. \end{aligned}$$

This finishes the proof.

□

Corollary 2.4 yields the following Hölder inequality for capacities:

Theorem 3.5. *Let $\Omega \subset \mathbf{R}^n$ be bounded, let $n < q \leq \infty$, and let $\varepsilon \in (0, n-1)$ be fixed. Then for every $K \subset \Omega$ compact we have*

$$(12) \qquad \mathrm{cap}_{n-\varepsilon}^{1/(n-\varepsilon)}(K, \Omega) \leq C(n, q, \varepsilon)\, |\Omega|^{\frac{\varepsilon}{n(n-\varepsilon)}}\, \mathrm{cap}_{n,q}^{1/n}(K, \Omega).$$

Proof. Let K be compact in Ω. Let $u \in W(K, \Omega)$. Then from Corollary 2.4 applied for $p = n$ and the definition of the $|| \cdot ||_{L^{n-\varepsilon}(\Omega;\mathbf{R}^n)}$-norm and $|| \cdot ||_{L^{(n,q)}(\Omega;\mathbf{R}^n)}$-quasinorm we have

$$||\nabla u||_{L^{n-\varepsilon}(\Omega;\mathbf{R}^n)} \leq C(n, q, \varepsilon)\, |\Omega|^{\frac{\varepsilon}{n(n-\varepsilon)}}\, ||\nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)}.$$

Taking the infimum on both sides over such functions u, we get the claim for $K \subset \Omega$ compact. This finishes the proof. □

Theorem 3.6. *Let $n < q \le \infty$ be fixed. There exists a constant $C(n,q) > 0$ such that*

$$C(n,q)^{-1}\left(\ln\frac{1}{r}\right)^{-\frac{n}{q'}} \le \operatorname{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \le C(n,q)\left(\ln\frac{1}{r}\right)^{-\frac{n}{q'}}$$

for every $0 < r < e^{-\frac{1}{n-1}}$, where q' is the Hölder conjugate of q.

Proof. We get some lower estimates for $\operatorname{cap}_{n,q}(\overline{B}(0,r), B(0,1))$, where $r > 0$ is small. We have to consider two cases, depending on whether $n < q < \infty$ or $q = \infty$.

First we consider the case $n < q < \infty$. From (12) applied for $p = n$ and $n < q < \infty$, there exists a constant

$$C(n,\varepsilon,q) = \Omega_n^{\frac{\varepsilon}{n(n-\varepsilon)}}\,\varepsilon^{-\frac{1}{n-\varepsilon}+\frac{1}{q}}\left(\frac{n(q-n+\varepsilon)}{q}\right)^{\frac{1}{n-\varepsilon}-\frac{1}{q}}$$

such that

$$\operatorname{cap}_{n-\varepsilon}^{1/(n-\varepsilon)}(\overline{B}(0,r), B(0,1)) \le C(n,\varepsilon,q)\operatorname{cap}_{n,q}^{1/n}(\overline{B}(0,r), B(0,1))$$

for every $\varepsilon \in (0, n-1)$ and every $r \in (0,1)$. From [HKM93, 2.13] we have

$$\operatorname{cap}_{n-\varepsilon}(\overline{B}(0,r), B(0,1)) = \omega_{n-1}\left(\frac{\varepsilon}{n-\varepsilon-1}\right)^{n-\varepsilon-1}(r^{-\frac{\varepsilon}{n-\varepsilon-1}}-1)^{1-n+\varepsilon}.$$

Therefore,

$$\operatorname{cap}_{n,q}^{1/n}(\overline{B}(0,r), B(0,1)) \ge C_1(n,\varepsilon,q)\,\varepsilon^{1-\frac{1}{q}}\,r^{\frac{\varepsilon}{n-\varepsilon}} \tag{13}$$

for every $0 < \varepsilon < n-1$, where

$$C_1(n,\varepsilon,q) = \omega_{n-1}^{\frac{1}{n-\varepsilon}}\frac{\Omega_n^{-\frac{\varepsilon}{n(n-\varepsilon)}}}{(n-\varepsilon-1)^{\frac{n-\varepsilon-1}{n-\varepsilon}}}\left(\frac{n(q-n+\varepsilon)}{q}\right)^{\frac{1}{q}-\frac{1}{n-\varepsilon}}.$$

We define

$$C_1(n,q) = \inf_{0<\varepsilon<n-1} C_1(n,\varepsilon,q).$$

We notice that $C_1(n,q) > 0$. This together with (13) implies

$$\operatorname{cap}_{n,q}^{1/n}(\overline{B}(0,r), B(0,1)) \ge C_1(n,q)\,\varepsilon^{1-\frac{1}{q}}\,r^{\frac{\varepsilon}{n-\varepsilon}}. \tag{14}$$

For $r \in (0, e^{-\frac{1}{n-1}})$, we let $\varepsilon = \frac{1}{\ln\frac{1}{r}}$. Then $0 < \varepsilon < n-1$ and from (14) it follows that

$$\operatorname{cap}_{n,q}(\overline{B}(0,r), B(0,r)) \ge \frac{C_1(n,q)^n}{e^n}\left(\ln\frac{1}{r}\right)^{\frac{n}{q}-n} \tag{15}$$

for every $r \in (0, e^{-\frac{1}{n-1}})$. This yields the desired lower bound for the relative capacity whenever $n < q < \infty$ and $r \in (0, e^{-\frac{1}{n-1}})$.

Now we assume $q = \infty$. From (12) we have

$$\operatorname{cap}_{n-\varepsilon}^{1/(n-\varepsilon)}(\overline{B}(0,r), B(0,1)) \le \Omega_n^{\frac{\varepsilon}{n(n-\varepsilon)}}\,\varepsilon^{-\frac{1}{n-\varepsilon}}\,n^{\frac{1}{n-\varepsilon}}\operatorname{cap}_{n,\infty}^{1/n}(\overline{B}(0,r), B(0,1))$$

for every $\varepsilon \in (0, n-1)$. This together with [HKM93, 2.13] gives

$$\operatorname{cap}_{n,\infty}^{1/n}(\overline{B}(0,r), B(0,1)) \ge C_1(n,\varepsilon)\,\varepsilon\, r^{\frac{\varepsilon}{n-\varepsilon}} \tag{16}$$

for every $0 < \varepsilon < n-1$, where

$$C_1(n,\varepsilon) = \omega_{n-1}^{\frac{1}{n-\varepsilon}}\,\Omega_n^{-\frac{\varepsilon}{n(n-\varepsilon)}}\,(n-\varepsilon-1)^{-\frac{n-\varepsilon-1}{n-\varepsilon}}\,n^{-\frac{1}{n-\varepsilon}}.$$

We define

$$C_1(n) = \inf_{0<\varepsilon<n-1} C_1(n,\varepsilon).$$

We notice that $C_1(n) > 0$. This together with (16) implies

$$\text{(17)} \qquad \mathrm{cap}_{n,\infty}^{1/n}(\overline{B}(0,r), B(0,1)) \geq C_1(n)\, \varepsilon\, r^{\frac{\varepsilon}{n-\varepsilon}}.$$

For $r \in (0, e^{-\frac{1}{n-1}})$ we let $\varepsilon = \frac{1}{\ln \frac{1}{r}}$. Then $0 < \varepsilon < n-1$ and from (17) it follows that

$$\text{(18)} \qquad \mathrm{cap}_{n,\infty}(\overline{B}(0,r), B(0,1)) \geq \frac{C_1(n)^n}{e^n} \left(\ln \frac{1}{r} \right)^{-n}$$

for every $r \in (0, e^{-\frac{1}{n-1}})$. We let $C_1(n,q) = C_1(n)$ when $q = \infty$. This yields the desired lower bound for the relative capacity when $q = \infty$ and $r \in (0, e^{-\frac{1}{n-1}})$.

We shall get an upper estimate for $\mathrm{cap}_{n,q}(\overline{B}(0,r), B(0,1))$ whenever $r \in (0, e^{-\frac{1}{n-1}})$ and $1 < q \leq \infty$. We use the function $u : B(0,1) \to \mathbf{R}$ defined by

$$u(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq r \\ \frac{\ln |x|}{\ln r} & \text{if } r < |x| < 1. \end{cases}$$

Then

$$|\nabla u(x)| = \begin{cases} 0 & \text{if } 0 \leq |x| < r \\ \frac{1}{\ln \frac{1}{r}} \frac{1}{|x|} & \text{if } r < |x| < 1. \end{cases}$$

We notice that $u \notin W_0(\overline{B}(0,r), B(0,1))$. However,

$$\text{(19)} \qquad \mathrm{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \leq ||\nabla u||^n_{L^{n,q}(B(0,1);\mathbf{R}^n)}$$

because

$$||\nabla u||_{L^{n,q}(B(0,1);\mathbf{R}^n)} = \lim_{\delta \to 0} ||\nabla u_\delta||_{L^{n,q}(B(0,1);\mathbf{R}^n)},$$

where u_δ, $0 < \delta < \frac{1-r}{r}$ is a sequence in $W_0(\overline{B}(0,r), B(0,1))$ defined by

$$u_\delta(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq r \\ \frac{\ln (1+\delta)|x|}{\ln r(1+\delta)} & \text{if } r < |x| < \frac{1}{1+\delta} \\ 0 & \text{if } \frac{1}{1+\delta} \leq |x| \leq 1. \end{cases}$$

We want to get an upper estimate for $||\nabla u||_{L^{n,q}(B(0,1);\mathbf{R}^n)}$ whenever $1 < q \leq \infty$. We define $v : B(0,1) \to \mathbf{R}$ by $v(x) = -\ln r\, |\nabla u(x)|$. We compute $\lambda_{[v]}$. We recall that $\Omega_n = |B(0,1)|$. We have

$$\lambda_{[v]}(t) = |\{x \in B(0,1) \setminus B(0,r) : \frac{1}{|x|} > t\}| = |\{x \in B(0,1) \setminus B(0,r) : |x| < \frac{1}{t}\}|.$$

Hence

$$\lambda_{[v]}(t) = \begin{cases} 0 & \text{if } t > \frac{1}{r} \\ \Omega_n \left(\frac{1}{t^n} - r^n \right) & \text{if } 1 \leq t \leq \frac{1}{r} \\ \Omega_n \left(1 - r^n\right) & \text{if } 0 \leq t \leq 1. \end{cases}$$

We notice that

$$v^*(t) = \begin{cases} \left(\frac{1}{t/\Omega_n + r^n} \right)^{\frac{1}{n}} & \text{if } 0 \leq t < \Omega_n \left(1 - r^n\right) \\ 0 & \text{if } t \geq \Omega_n \left(1 - r^n\right). \end{cases}$$

We compute $||v||_{L^{n,q}(B(0,1))}$. We have to consider two cases, depending on whether $1 < q < \infty$ or $q = \infty$.

We assume first that $1 < q < \infty$. Let

$$J =: ||v||^q_{L^{n,q}(B(0,1))} = \int_0^{\Omega_n(1-r^n)} t^{\frac{q}{n}} (v^*(t))^q \frac{dt}{t}.$$

By making the substitution $t = s\,\Omega_n r^n$, we get

$$\begin{aligned} J &= \int_0^{\Omega_n(1-r^n)} t^{\frac{q}{n}} \left(\frac{1}{t/\Omega_n + r^n}\right)^{\frac{q}{n}} \frac{dt}{t} = \Omega_n^{\frac{q}{n}} \int_0^{\frac{1-r^n}{r^n}} s^{\frac{q}{n}} \left(\frac{1}{s+1}\right)^{\frac{q}{n}} \frac{ds}{s} \\ &= \Omega_n^{\frac{q}{n}} \left(\int_0^1 s^{\frac{q}{n}-1} \left(\frac{1}{s+1}\right)^{\frac{q}{n}} ds + \int_1^{\frac{1-r^n}{r^n}} \left(\frac{s}{s+1}\right)^{\frac{q}{n}} \frac{ds}{s} \right) \\ &\leq \Omega_n^{\frac{q}{n}} \left(\frac{n}{q} + \ln \frac{1-r^n}{r^n} \right) \leq \Omega_n^{\frac{q}{n}} \left(\frac{n}{q} + n \ln \frac{1}{r} \right) \leq C_2(n,q) \ln \frac{1}{r} \end{aligned}$$

if $0 < r < e^{-\frac{1}{n-1}}$. Therefore, from (19) and the fact that $v = -\ln r\,|\nabla u|$ we get

$$\text{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \leq C_2(n,q)^{\frac{n}{q}} \left(\ln \frac{1}{r}\right)^{\frac{n}{q}-n} \tag{20}$$

for every $r \in (0, e^{-\frac{1}{n-1}})$ whenever $1 < q < \infty$.

From (15) and (20) it follows that there exists a constant

$$C(n,q) =: \max \left(C_2(n,q)^{\frac{n}{q}}, \frac{e^n}{C_1(n,q)^n} \right)$$

such that

$$C(n,q)^{-1} \left(\ln \frac{1}{r}\right)^{\frac{n}{q}-n} \leq \text{cap}_{n,q}(\overline{B}(0,r), B(0,1)) \leq C(n,q) \left(\ln \frac{1}{r}\right)^{\frac{n}{q}-n}$$

for every $0 < r < e^{-\frac{1}{n-1}}$ whenever $n < q < \infty$.

Now assume $q = \infty$. We have

$$||v||^n_{L^{n,\infty}(B(0,1))} = \sup_{t \geq 0} t\,(v^*(t))^n = \sup_{0 \leq t \leq \Omega_n\,(1-r^n)} \frac{t}{t/\Omega_n + r^n} = \Omega_n\,(1-r^n).$$

Therefore

$$||\nabla u||^n_{L^{n,\infty}(B(0,1);\mathbf{R}^n)} = \left(\ln \frac{1}{r}\right)^{-n} ||v||^n_{L^{n,\infty}(B(0,1))} = \Omega_n\,(1-r^n) \left(\ln \frac{1}{r}\right)^{-n}$$

and from (19) we get

$$\text{cap}_{n,\infty}(\overline{B}(0,r), B(0,1)) \leq \Omega_n\,(1-r^n) \left(\ln \frac{1}{r}\right)^{-n} \tag{21}$$

for every $r \in (0,1)$.

From (18) and (21) it follows that there exists a constant

$$C(n,q) =: \max \left(\Omega_n, \frac{e^n}{C_1(n,q)^n} \right)$$

such that

$$C(n,q)^{-1}\left(\ln\frac{1}{r}\right)^{-\frac{n}{q'}} \le \mathrm{cap}_{n,q}(\overline{B}(0,r),B(0,1)) \le C(n,q)\left(\ln\frac{1}{r}\right)^{-\frac{n}{q'}}$$

for every $0<r<e^{-\frac{1}{n-1}}$ when $q=\infty$. This finishes the proof of the theorem.

□

Remark 3.7. We actually showed that the upper estimate (20) holds in fact for every $q\in(1,\infty)$ as long as $r\in(0,e^{-\frac{1}{n-1}})$. When $q=n$ we are in the case of the n-capacity and then (20) is known. (See for example [HKM93, 2.13].) Consequently, for every $1<q\le\infty$ there exists a constant $C(n,q)>0$ such that

$$\mathrm{cap}_{n,q}(\overline{B}(0,r),B(0,1)) \le C(n,q)\left(\ln\frac{1}{r}\right)^{-\frac{n}{q'}}$$

for every $r\in(0,e^{-\frac{1}{n-1}})$. We do not know whether a similar lower bound exists when $1<q<n$.

4. Hausdorff measure and the Sobolev-Lorentz n,q-capacity

In this section we examine the relationship between Hausdorff measures and the Sobolev-Lorentz n,q-capacity.

Definition 4.1. Let $1<q<\infty$. Let K be a compact set in $\mathbf{R}^n$. We say that K is of *n,q-capacity zero* if

$$\mathrm{cap}_{n,q}(K,\Omega)=0$$

whenever Ω is an open neighborhood of K. In this case we write $\mathrm{cap}_{n,q}(K)=0$.

Before proceeding, we recall the following version of the Poincaré inequality.

Theorem 4.2. Poincaré inequality for Sobolev-Lorentz spaces. *Let $\Omega\subset\mathbf{R}^n$ be bounded. Let $1\le q\le\infty$ be fixed. Then there exists a constant $C(n,q)$ such that*

$$||u||_{L^{n,q}(\Omega)} \le C(n,q)\,|\Omega|^{\frac{1}{n}}\,||\nabla u||_{L^{n,q}(\Omega;\mathbf{R}^n)} \tag{22}$$

for every $u\in C_0^\infty(\Omega)$.

Proof. For every $u\in C_0^\infty(\Omega)$ we have (see [GT83, Lemma 7.14]):

$$|u(x)| \le \frac{1}{\omega_{n-1}}(I_1|\nabla u|)(x) \tag{23}$$

for every $x\in\mathbf{R}^n$. We recall that for every measurable function f in $\mathbf{R}^n$, $I_1 f$ is its Riesz potential of order 1. (See [BS88, Definition IV.4.17] and [Hei01, p. 20].) An application of Hardy-Littlewood-Sobolev theorem of fractional integration ([BS88, Theorem IV.4.18]) together with Theorem 2.3, [BS88, Proposition II.1.7] and (23) yields the desired conclusion. □

Theorem 4.3. *Suppose $1<q<\infty$. Let E be a compact set in $\mathbf{R}^n$. If there exists a constant $M>0$ such that*

$$\mathrm{cap}_{n,q}(E,\Omega)\le M<\infty$$

for all open sets Ω containing E, then $\mathrm{cap}_{n,q}(E)=0$.

Proof. When $q = n$ we are in the case of the n-capacity and then the claim holds. (See for example [HKM93, Lemma 2.34]). So we can assume without loss of generality that $q \neq n$. We let Ω be a fixed open neighborhood of E. We can assume without loss of generality that Ω is bounded. We choose a descending sequence of open sets

$$\Omega = \Omega_1 \supset\supset \Omega_2 \supset\supset \cdots \supset\supset \cap_i \Omega_i = E$$

and we choose $\varphi_i \in W(E, \Omega_i)$, $0 \leq \varphi_i \leq 1$ with $\varphi_i = 1$ on E and

$$||\nabla \varphi_i||^n_{L^{n,q}(\Omega_i;\mathbf{R}^n)} < M + 1.$$

From the Poincaré inequality for Sobolev-Lorentz spaces (22) we have that $(\varphi_i, \nabla\varphi_i)$ is bounded in the space $L^{n,q}(\Omega) \times L^{n,q}(\Omega; \mathbf{R}^n)$. We notice that φ_i converges pointwise to a function ψ which is 1 on E and 0 on $\mathbf{R}^n \setminus E$. Hence, from Mazur's lemma ([Yos80, p. 120]), [BS88, Lemma IV.4.5], and the reflexivity of $L^{n,q}(\Omega) \times L^{n,q}(\Omega; \mathbf{R}^n)$ it follows that there exists a subsequence denoted again by φ_i such that $(\varphi_i, \nabla\varphi_i)$ converges weakly to $(\psi, 0)$ in $L^{n,q}(\Omega) \times L^{n,q}(\Omega; \mathbf{R}^n)$ and a sequence $\widetilde{\varphi}_i$ of convex combinations of φ_i,

$$\widetilde{\varphi}_i = \sum_{j=i}^{j_i} \lambda_{i,j}\varphi_j, \quad \lambda_{i,j} \geq 0, \quad \sum_{j=i}^{j_i} \lambda_{i,j} = 1,$$

such that $(\widetilde{\varphi}_i, \nabla\widetilde{\varphi}_i)$ converges to $(\psi, 0)$ in $L^{n,q}(\Omega) \times L^{n,q}(\Omega; \mathbf{R}^n)$. The closedness of $W(E, \Omega_i)$ under finite convex combinations implies that $\widetilde{\varphi}_i \in W(E, \Omega_i)$ for every integer $i \geq 1$. Therefore

$$0 \leq \mathrm{cap}_{n,q}(E, \Omega) \leq \limsup_{i\to\infty} ||\nabla\widetilde{\varphi}_i||^n_{L^{n,q}(\Omega_i;\mathbf{R}^n)} = 0.$$

□

Theorem 4.4. *Suppose that $1 < q \leq \infty$ and that E is a compact set in $\mathbf{R}^n$. For $1 < q \leq \infty$ we let $h_{n,q} : [0, \infty) \to \mathbf{R}$ be defined by*

$$h_{n,q}(t) = \begin{cases} 0 & \text{if } t = 0 \\ \left(\ln \frac{1}{t}\right)^{-\frac{n}{q'}} & \text{if } 0 < t < \frac{1}{2} \\ 2\,(\ln 2)^{-\frac{n}{q'}}\, t & \text{if } t \geq \frac{1}{2}. \end{cases}$$

(i) *If $1 < q < n$, then $\Lambda_{h^{1/n}_{n,q}}(E) < \infty$ implies $\mathrm{cap}_{n,q}(E) = 0$.*
(ii) *If $n \leq q < \infty$, then $\Lambda_{h_{n,q}}(E) < \infty$ implies $\mathrm{cap}_{n,q}(E) = 0$.*
(iii) *If $q = \infty$, then $\Lambda_{h_{n,q}}(E) = 0$ implies $\mathrm{cap}_{n,\infty}(E, \Omega) = 0$ whenever Ω is an open neighborhood of E.*

Proof. We have to analyze three cases, depending on whether $1 < q < n$ or $n \leq q < \infty$ or $q = \infty$. It is enough to prove that $\mathrm{cap}_{n,q}(E, \Omega) = 0$ whenever Ω is a bounded open neighborhood of E. So let Ω be a bounded open set containing E. We denote by δ the distance from E to the complement of Ω. Without loss of generality we can assume that $0 < \delta < e^{-\frac{1}{2(n-1)}}$. Fix $0 < \varepsilon < 1$ such that $\varepsilon < \frac{1}{4}\delta^2$; then $r < \varepsilon$ implies $\ln(\frac{\delta}{2r}) \geq \frac{1}{2}\ln(\frac{1}{r})$. We cover E by open balls $B(x_i, r_i)$ such that $r_i < \frac{1}{2}\varepsilon$. Since we may assume that the balls $B(x_i, r_i)$ intersect E, we have $B(x_i, \frac{\delta}{2}) \subset \Omega$. In fact, since E is compact, E is covered by finitely many of the balls $B(x_i, r_i)$.

We assume first that $1 < q < n$. Using Theorem 3.2 (ii) and (v) we obtain

$$\begin{aligned}
\mathrm{cap}_{n,q}^{1/n}(E,\Omega) &\leq \sum_i \mathrm{cap}_{n,q}^{1/n}(\overline{B}(x_i,r_i),\Omega) \\
&\leq \sum_i \mathrm{cap}_{n,q}^{1/n}(\overline{B}(x_i,r_i),B(x_i,\frac{\delta}{2})) \\
&\leq C(n,q)\sum_i \left(\ln\frac{1}{r_i}\right)^{\frac{1}{q}-1},
\end{aligned}$$

where in the last step we also used Remark 3.7 together with our choice of ε. Taking the infimum over all such coverings and letting $\varepsilon \to 0$, we conclude

$$\mathrm{cap}_{n,q}^{1/n}(E,\Omega) \leq C(n,q)\Lambda_{h_{n,q}^{1/n}}(E) < \infty.$$

Since Ω was an arbitrary bounded open set containing E, the desired conclusion follows from Theorems 3.2 (ii) and 4.3 when $1 < q < n$.

We assume now that $n \leq q < \infty$. When $q = n$ we are in the case of the n-capacity and then the claim holds. (See for example [HKM93, Theorem 2.27].) So we can assume without loss of generality that $n < q < \infty$. Using the finite subadditivity and the monotonicity property of the n,q-capacity we obtain

$$\begin{aligned}
\mathrm{cap}_{n,q}(E,\Omega) &\leq \sum_i \mathrm{cap}_{n,q}(B(x_i,r_i),\Omega) \leq \sum_i \mathrm{cap}_{n,q}(B(x_i,r_i),B(x_i,\frac{\delta}{2})) \\
&= \sum_i \mathrm{cap}_{n,q}(B(0,r_i),B(0,\frac{\delta}{2})) \leq C(n,q)\sum_i \left(\ln\frac{1}{r_i}\right)^{\frac{n}{q}-n},
\end{aligned}$$

where in the last step we also used Remark 3.7 for the n,q-capacity of spherical condensers together with our choice of ε. Taking the infimum over all such coverings, we conclude

$$\mathrm{cap}_{n,q}(E,\Omega) \leq C(n,q)\Lambda_{h_{n,q}}(E) < \infty.$$

Since Ω was an arbitrary bounded open set containing E, the desired conclusion follows from Theorems 3.2 (ii) and 4.3 when $n < q < \infty$.

We assume now that $q = \infty$. Using the finite subadditivity and the monotonicity property of the n,∞-capacity we obtain

$$\begin{aligned}
\mathrm{cap}_{n,\infty}(E,\Omega) &\leq \sum_i \mathrm{cap}_{n,\infty}(B(x_i,r_i),\Omega) \leq \sum_i \mathrm{cap}_{n,\infty}(B(x_i,r_i),B(x_i,\frac{\delta}{2})) \\
&= \sum_i \mathrm{cap}_{n,\infty}(B(0,r_i),B(0,\frac{\delta}{2})) \leq C(n)\sum_i \left(\ln\frac{1}{r_i}\right)^{-n},
\end{aligned}$$

where in the last step we also used formula (21) for the n,∞-capacity of spherical condensers together with our choice of ε. Taking the infimum over all such coverings, we conclude

$$\mathrm{cap}_{n,\infty}(E,\Omega) \leq C(n)\Lambda_{h_{n,\infty}}(E) = 0.$$

□

Remark 4.5. It is known that if $\text{cap}_n(E) = 0$, then $\Lambda_h(E) = 0$ whenever E is a compact set in $\mathbf{R}^n$ and h is an increasing function on $[0, \infty)$ such that $h(0) = 0$, and

$$\int_0^1 h(r)^{1/(n-1)} \frac{dr}{r} < \infty.$$

(See [AH96, p. 20 and Theorem 5.1.13] and [HKM93, Corollary 2.40].) This corresponds to the case $q = n$. It is not known if we have similar results for $q \neq n$. A possible result would be the following:

Conjecture 4.6. *Let E be a compact set in $\mathbf{R}^n$ and let $1 < q \leq \infty$ be such that $q \neq n$. Then, if there exists a bounded open neighborhood Ω of E such that $\text{cap}_{n,q}(E, \Omega) = 0$, we have $\Lambda_h(E) = 0$ whenever h is an increasing function on $[0, \infty)$ such that $h(0) = 0$, and*

$$\int_0^1 h(r)^{\frac{q'}{n}} \frac{dr}{r} < \infty.$$

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