Harnack's principle for quasiminimizers

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Abstract. We study Harnack type properties of quasiminimizers of the p-Dirichlet integral on metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality. We show that an increasing sequence of quasiminimizers converges locally uniformly to a quasiminimizer, provided the limit function is finite at some point, even if the quasiminimizing constant and the boundary values are allowed to vary in a bounded way. If the quasiminimizing constants converge to one, then the limit function is the unique minimizer of the p-Dirichlet integral. In the Euclidean case with the Lebesgue measure we obtain convergence also in the Sobolev norm.

Key words and phrases: Metric space, doubling measure, Poincaré inequality, Newtonian space, Harnack inequality, Harnack convergence theorem.

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1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $1 < p < \infty$. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *Q-quasiminimizer*, $Q \geq 1$, of the p-Dirichlet integral in Ω if for all open $\Omega' \Subset \Omega$

and all $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $u - v \in W^{1,p}_0(\Omega')$ we have

$$\int_{\Omega'} |\nabla u|^p \, dx \leq Q \int_{\Omega'} |\nabla v|^p \, dx.$$

In the Euclidean case the problem of minimizing the p-Dirichlet integral

$$\int_{\Omega} |\nabla u|^p \, dx$$

among all functions with given boundary values is equivalent to solving the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

Hence 1-quasiminimizers (called minimizers) are weak solutions of the p-Laplace equation. Being a weak solution is a local property, however, being a quasiminimizer is not a local property, see Kinnunen–Martio [19]. Hence, the theory for quasiminimizers usually differs from the theory for minimizers.

Quasiminimizers have been studied by Giaquinta–Giusti, see [8] and [9]. See also DiBenedetto–Trudinger [7], Tolksdorf [26] and Ziemer [27]. Several fundamental properties of quasiminimizers including local Hölder continuity, higher integrability of the gradient and boundary continuity have been studied. Some of these results have been extended to metric spaces, see [1], [2], [4], [5], [19],[20].

Quasiminimizers have been used as tools in studying regularity of minimizers of variational integrals. Indeed, the quasiminimizing condition applies to the whole class of variational integrals at the same time. For example, if a variational kernel $F(x, \nabla u)$ satisfies the standard growth conditions

$$\alpha |h|^p \leq F(x, h) \leq \beta |h|^p$$

for some $0 < \alpha \leq \beta < \infty$, then the minimizers of $\int F(x, \nabla u) \, dx$ are quasiminimizers of the p-Dirichlet integral. Apart from this quasiminimizers have a fascinating theory in themselves, see for example [19].

It is known that a sequence of locally bounded p-harmonic functions on a domain in $\mathbf{R}^n$ has a locally uniformly convergent subsequence that converges to a p-harmonic function on that domain, see Heinonen et al. [11]. The result has been extended to metric measure spaces by Shanmugalingam in [25].

In this note we prove the Harnack principle for Q-quasiminimizers with varying Q: an increasing sequence of Q_i-quasiminimizers in a domain converge locally uniformly, provided the limit function is finite at some point in that domain, to a Q-quasiminimizer with

$$Q = \liminf_{i \to \infty} Q_i,$$

see Theorem 4.3. Moreover, we show that a sequence (u_i) of Q_i-quasiminimizers in a domain, such that the sequence (u_i) is locally uniformly bounded below, has a locally uniformly convergent subsequence which converges either to ∞ or a Q-quasiminimizer on that domain, see Corollary 4.5. In the last section we let (f_i) be a uniformly bounded sequence of functions in an appropriate space such that $f_i \to f$ as $i \to \infty$, and we consider a sequence of Q_i-quasiminimizers in a bounded domain Ω with boundary values f_i. We study the stability of Q_i-quasiminimizers when Q_i tends to 1. More precisely, we show that in this case the quasiminimizers converge locally uniformly in Ω to the unique minimizer of the p-Dirichlet integral with boundary values f. In the Euclidean case with the Lebesgue measure we obtain convergence also in the Sobolev norm.

Our results seem to be new even in the Euclidean setting, but we study the question in complete metric spaces equipped with a doubling measure and supporting a weak $(1,p)$-Poincaré inequality. We have chosen this more general approach to emphasize the fact that the obtained properties hold in a very general context. Indeed, our approach covers weighted Euclidean spaces, Riemannian manifolds, Carnot–Carathéodory spaces, including Carnot groups such as Heisenberg groups, and graphs.

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2. Notation and preliminaries

Throughout the paper $1 < p < \infty$, a domain is an open connected set $\Omega \subset X$ and $X = (X, d, \mu)$ is a complete metric space endowed with a metric d and a positive complete Borel regular measure μ such that $0 < \mu(B(z,r)) < \infty$ for all balls $B(z,r) := \{z_0 \in X : d(z,z_0) < r\}$ in X. At the end of the section we further assume that X supports a weak $(1,p)$-Poincaré inequality and that μ is doubling, which are then assumed throughout the rest of the paper.

The measure μ is said to be *doubling*, if there exists a constant $c_\mu \geq 1$, called the *doubling constant* of μ, such that

$$\mu(B(z,2r)) \leq c_\mu \mu(B(z,r)),$$

for all $z \in X$ and $r > 0$. A metric space with a doubling measure is proper (i.e., closed and bounded subsets are compact) if and only if it is complete. In addition, a complete metric space with a doubling measure is separable.

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real valued function u on X if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds. \tag{2.1}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2.1) holds for p-almost every path, then g is a *p-weak upper gradient* of u.

By saying that (2.1) holds for p-almost every path we mean that it fails only for a path family with zero p-modulus. The *p-modulus* of a family of paths Γ in X is the number

$$\inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ such that for all rectifiable paths γ which belong to Γ we have

$$\int_\gamma \rho \, ds \ge 1.$$

If u has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_u \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of u, $g_u \le g$ μ-almost everywhere (a.e.), see Corollary 3.7 in Shanmugalingam [24]. The minimal p-weak upper gradient can be obtained by the formula

$$g_u(z) := \inf_g \limsup_{r \to 0+} \frac{1}{\mu(B(z,r))} \int_{B(z,r)} g \, d\mu,$$

where the infimum is taken over all upper gradients $g \in L^p(X)$ of u, see Lemma 2.3 in J. Björn [5].

In the following lemma the function $g = g_u \chi_{\{u>v\}} + g_v \chi_{\{v \ge u\}}$, for example, need not be a Borel function, but it can be replaced with a Borel function $\widetilde{g}$ such that $g \le \widetilde{g}$ and $g = \widetilde{g}$ μ-a.e. We shall use this transition from a μ-measurable function g to a Borel function $\widetilde{g}$ without a change of notation.

Lemma 2.2. *Let u and v be functions with upper gradients in $L^p(X)$. Then $g_u \chi_{\{u>v\}} + g_v \chi_{\{v \ge u\}}$ is a minimal p-weak upper gradient of* $\max\{u, v\}$*, and $g_v \chi_{\{u>v\}} + g_u \chi_{\{v \ge u\}}$ is a minimal p-weak upper gradient of* $\min\{u, v\}$*.*

For a proof we refer, for example, to Lemma 3.2 in Björn–Björn [2] or Lemma 3.5 in Marola [22].

We define Sobolev spaces on the metric space X following Shanmugalingam [23].

Definition 2.3. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

For the properties of Newtonian spaces we refer to [23].

Definition 2.4. The *capacity* of a set $E \subset X$ is the number

$$C_p(E) = \inf \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

The capacity is countably subadditive. For this and other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen et al. [15] and Kinnunen–Martio [16, 17].

We say that a property regarding points in X holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [23] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$.

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values.

Definition 2.5. Let E be a measurable subset of X. The *Newtonian space with zero boundary values* is the space

$$N_0^{1,p}(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \text{ q.e. on } X \setminus E\}.$$

Note that if $C_p(X \setminus E) = 0$, then $N_0^{1,p}(E) = N^{1,p}(X)$. The space $N_0^{1,p}(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space, see Theorem 4.4 in Shanmugalingam [24].

We say that u belongs to the *local Newtonian space* $N_{\rm loc}^{1,p}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for every open $\Omega' \Subset \Omega$ (or equivalently that $u \in N^{1,p}(E)$ for every measurable $E \Subset \Omega$).

Definition 2.6. We say that X supports a *weak* $(1,p)$*-Poincaré inequality* if there exist constants $c > 0$ and $\lambda \geq 1$ such that for all balls $B(z,r) \subset X$, all measurable functions f on X and for all p-weak upper gradients g of f,

$$\fint_{B(z,r)} |f - f_{B(z,r)}| \, d\mu \leq cr \biggl(\fint_{B(z,\lambda r)} g^p \, d\mu \biggr)^{1/p}, \tag{2.2}$$

where $f_{B(z,r)} := \fint_{B(z,r)} f \, d\mu := \int_{B(z,r)} f \, d\mu / \mu(B(z,r))$.

By the Hölder inequality it is easy to see that if X supports a weak $(1,p)$-Poincaré inequality, then it supports a weak $(1,q)$-Poincaré inequality for every $q > p$. If X is complete and μ doubling then it is shown in Keith–Zhong [14] that a weak $(1,p)$-Poincaré inequality implies a weak $(1,q)$-Poincaré inequality for some $q < p$.

We shall need a Sobolev type inequality for Newtonian functions with zero boundary values. Assume that E is bounded and $C_p(X \setminus E) > 0$. Then there exists a constant $c = c(p, E) > 0$ such that for all $u \in N_0^{1,p}(E)$,

$$\int_E |u|^p \, d\mu \le c \int_E g_u^p \, d\mu. \tag{2.3}$$

For this result we refer to Kinnunen–Shanmugalingam [20].

Throughout the rest of the paper let us assume that X is complete, supports a weak $(1,p)$-Poincaré inequality, and μ is doubling. It then follows that Lipschitz functions are dense in $N^{1,p}(X)$ and that the functions in $N^{1,p}(X)$ are quasicontinuous, see Björn et al. [3]. This means that in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space as defined on p. 96 of Heinonen et al. [11].

We shall use the following compactness result. For a proof see Kilpeläinen et al. [15]. Let Ω be an open subset of X.

Lemma 2.7. (i) *Let (u_i) be a uniformly bounded sequence in $L^p(X)$ such that $u_i \to u$ μ-a.e. in X. Then $u \in L^p(X)$ and $u_i \to u$ weakly in $L^p(X)$.*

(ii) *Let (u_i) be a uniformly bounded sequence in $N^{1,p}(\Omega)$ such that $u_i \to u$ μ-a.e. in Ω. Then $u \in N^{1,p}(\Omega)$, $u_i \to u$ and $g_{u_i} \to g$ weakly in $L^p(\Omega)$, where g is a p-weak upper gradient of u. Moreover, if $u_i \in N_0^{1,p}(\Omega)$, then $u \in N_0^{1,p}(\Omega)$.*

We close this section by recalling that $E \Subset \Omega$ if $\overline{E}$ is a compact subset of Ω, and that $\mathrm{Lip}_c(\Omega) = \{f \in \mathrm{Lip}(X) : \operatorname{supp} f \Subset \Omega\}$.

Unless otherwise stated, the letter c denotes various positive constants whose values may vary.

3. Quasiminimizers

We start by following Section 3 of Kinnunen–Martio [19]. Let $\Omega \subset X$ be an open set.

Definition 3.1. Let $Q \ge 1$ and $1 < p < \infty$. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *Q-quasiminimizer* in Ω if for all open $\Omega' \Subset \Omega$ and all $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ with $u - v \in N_0^{1,p}(\Omega')$ we have

$$\int_{\overline{\Omega'}} g_u^p \, d\mu \le Q \int_{\overline{\Omega'}} g_v^p \, d\mu. \tag{3.1}$$

Here g_u and g_v are the minimal p-weak upper gradients of u and v in Ω, respectively.

Let $f \in N^{1,p}(\Omega)$. We say that u is a *Q-quasiminimizer with boundary values f* in Ω, if u is a Q-quasiminimizer in Ω such that $u - f \in N_0^{1,p}(\Omega)$.

In (3.1) Ω' is sometimes used instead of $\overline{\Omega'}$, however, these definitions are equivalent, see Björn [1]. In metric measure spaces $\overline{\Omega'}$ is, at times, easier to use.

Proposition 3.2. *Let $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Then the following claims are equivalent:*

(i) *The function u is a Q-quasiminimizer in Ω;*

(ii) *For all $\varphi \in N^{1,p}_0(\Omega)$ we have*

$$\int_{\varphi\neq 0} g_u^p \, d\mu \le Q \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu.$$

Characterizations of quasiminimizers including this can be found in [1].

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *Q-quasisuperminimizer* in Ω if (3.1) holds for all open $\Omega' \Subset \Omega$ and all $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $v \ge u$ μ-a.e. in Ω' and $u - v \in N^{1,p}_0(\Omega')$. A function u is a Q-quasiminimizer if and only if both u and $-u$ are Q-quasisuperminimizers.

If $Q = 1$, then 1-quasiminimizers and 1-quasisuperminimizers are called *minimizers* and *superminimizers*, respectively. Observe, that for minimizers and superminimizers it is enough to test (3.1) with $\Omega' = \Omega$. If Ω is a bounded open set in X so that $C_p(X \setminus \Omega) > 0$ and $f \in N^{1,p}(\Omega)$, then there is a unique minimizer $u \in N^{1,p}(\Omega)$ such that $u - f \in N^{1,p}_0(\Omega)$. In other words, the Dirichlet problem has a unique solution with the given boundary values, see Shanmugalingam [24], and J. Björn [5], Cheeger [6] and Kinnunen–Martio [18].

The following DeGiorgi type estimate is well known, see for example Kinnunen–Shanmugalingam [20] or Latvala [21, Lemma 3.3].

Lemma 3.3. *Let u be a Q-quasiminimizer in Ω and let $0 < r < R$ with $B(z,R) \Subset \Omega$. Then*

$$\int_{B(z,r)} g_u^p \, d\mu \le \frac{c}{(R-r)^p} \int_{B(z,R)} |u-k|^p \, d\mu$$

for all $k \in \mathbf{R}$ and $z \in \Omega$. The constant c depends only on p and Q.

It is known that quasiminimizers can be modified on a set of capacity zero so that they become locally Hölder continuous, and satisfy the strong maximum principle and that they satisfy the following Harnack's inequality, see Kinnunen–Shanmugalingam [20].

Theorem 3.4. *Suppose that u is a nonnegative Q-quasiminimizer in Ω. Then there exists a constant $c \ge 1$ so that*

$$\sup_{B(z,r)} u \le c \inf_{B(z,r)} u$$

for every ball $B(z,r)$ for which $B(z, 20\lambda r) \subset \Omega$. Here the constant c is independent of the ball $B(z,r)$ and function u.

Here λ is the dilation constant in the weak Poincaré inequality. The dilation constant from the weak Poincaré inequality is essential in the condition on the balls in the Harnack inequality, see Section 10 in Björn–Marola [4].

4. Harnack's principle

In this section we consider a sequence of Q-quasiminimizers with varying Q. We prove that an increasing sequence (u_i) of Q_i-quasiminimizers defined in a domain of X converges locally uniformly to a Q-quasiminimizer with $Q = \liminf_{i\to\infty} Q_i$ provided the limit function is finite at some point. This is sometimes referred to as Harnack's principle or Harnack's convergence theorem.

Let Ω be an open set in X. We shall need the following lemma in the proofs of Theorem 4.3 and Theorem 5.1.

Lemma 4.1. *Let (u_i) be a sequence of Q_i-quasiminimizers in Ω such that $u_i \to u$ μ-a.e. in Ω, (u_i) is locally uniformly bounded in $L^p(\Omega)$ and the sequence (Q_i) is uniformly bounded. Let $K \subset \Omega$ be a compact set and for $t > 0$ define*

$$K(t) = \{x \in \Omega \ : \ \operatorname{dist}(x, K) < t\}.$$

Then $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and for almost every $t \in (0, t_0)$ we have

$$\limsup_{i\to\infty} \int_{K(t)} g_{u_i}^p \, d\mu \le Q \int_{K(t)} g_u^p \, d\mu,$$

where $Q = \limsup_{i\to\infty} Q_i$ and $t_0 = \operatorname{dist}(K, X \setminus \Omega)$.

Proof. First note that $K(t) \Subset \Omega$ for $0 < t < \operatorname{dist}(K, X \setminus \Omega) = t_0$ and $K(t)$ is open. By Lemma 3.3, for every ball $B(z, r) \Subset \Omega$ we have

$$\int_{B(z,r/2)} g_{u_i}^p \, d\mu \le \frac{c}{r^p} \int_{B(z,r)} |u_i|^p \, d\mu < \infty.$$

Set $\delta = t_0/3$. Using the fact that X is a doubling space we can find a finite cover of $K(t)$ by balls $B(z_j, \delta)$ with $z_j \in K(t)$. We obtain

$$\int_{B(z_j,\delta)} g_{u_i}^p \, d\mu \le \frac{c}{\delta^p} \int_{B(z_j,2\delta)} |u_i|^p \, d\mu < \infty.$$

Since the cover is finite, we see that the sequence (g_{u_i}) is uniformly bounded in $L^p(K(t))$ and consequently (u_i) is uniformly bounded in $N^{1,p}(K(t))$. Lemma 2.7 implies that $u \in N^{1,p}(K(t))$.

Let $0 < t' < t < t_0$ and choose a Lipschitz cutoff function η such that $0 \le \eta \le 1$, $\eta = 0$ in $\Omega \setminus K(t)$ and $\eta = 1$ in $K(t')$. Let

$$w_i = u_i + \eta(u - u_i), \quad i = 1, 2, \dots$$

Then $u_i - w_i \in N^{1,p}_0(K(t))$ and as in Lemma 2.4 of Kinnunen–Martio [18] we get

$$g_{w_i} \le (1 - \eta) g_{u_i} + \eta g_u + g_\eta |u - u_i|, \quad i = 1, 2, \dots,$$

μ-a.e. in $K(t)$. Thus

$$
\begin{aligned}
&\left(\int_{K(t)} g_{w_i}^p \, d\mu\right)^{1/p} \\
&\qquad \le \left(\int_{K(t)} ((1-\eta)g_{u_i} + \eta g_u)^p \, d\mu\right)^{1/p} + \left(\int_{K(t)} g_\eta^p |u-u_i|^p \, d\mu\right)^{1/p} \\
&\qquad =: \alpha_i + \beta_i.
\end{aligned}
$$

We use the elementary inequality

$$(\alpha_i + \beta_i)^p \le \alpha_i^p + p\beta_i(\alpha_i+\beta_i)^{p-1},$$

from which it follows that

$$\int_{K(t)} g_{w_i}^p \, d\mu \le \int_{K(t)} (1-\eta) g_{u_i}^p \, d\mu + \int_{K(t)} \eta g_u^p \, d\mu + p\beta_i(\alpha_i+\beta_i)^{p-1},$$

where we also used the convexity of the function $s \mapsto s^p$.

The quasiminimizing property of u_i gives

$$
\begin{aligned}
\int_{K(t')} g_{u_i}^p \, d\mu &\le \int_{K(t)} g_{u_i}^p \, d\mu \le Q_i \int_{K(t)} g_{w_i}^p \, d\mu \\
&\le Q_i \int_{K(t)} (1-\eta) g_{u_i}^p \, d\mu + Q_i \int_{K(t)} \eta g_u^p \, d\mu + Q_i p\beta_i(\alpha_i+\beta_i)^{p-1}.
\end{aligned}
$$

Adding the term

$$Q_i \int_{K(t')} g_{u_i}^p \, d\mu$$

to the both sides and taking into account that $\eta = 1$ in $K(t')$ we obtain

$$
\begin{aligned}
(1+Q_i) &\int_{K(t')} g_{u_i}^p \, d\mu \\
&\le Q_i \int_{K(t)} g_{u_i}^p \, d\mu + Q_i \int_{K(t)} g_u^p \, d\mu + Q_i p\beta_i(\alpha_i+\beta_i)^{p-1}.
\end{aligned}
\tag{4.1}
$$

Set $\Psi : (0, t_0) \to \mathbf{R}$,

$$\Psi(t) = \limsup_{i\to\infty} \int_{K(t)} g_{u_i}^p \, d\mu.$$

Now Ψ is a finite valued and increasing function of t. Thus the points of discontinuities of Ψ form a countable set. Let t, $0 < t < t_0$, be a point of continuity of Ψ. Inequality (4.1) implies that

$$
\begin{aligned}
&\int_{K(t')} g_{u_i}^p \, d\mu \\
&\qquad \le C_i \int_{K(t)} g_{u_i}^p \, d\mu + C_i \int_{K(t)} g_u^p \, d\mu + C_i p\beta_i(\alpha_i+\beta_i)^{p-1},
\end{aligned}
\tag{4.2}
$$

where $C_i = Q_i/(Q_i + 1)$. Set $C = \limsup_{i\to\infty} C_i$. Now

$$\beta_i^p = \int_{K(t)} g_\eta^p |u - u_i|^p \, d\mu \to 0$$

as $i \to \infty$ by the Lebesgue dominated convergence theorem; in fact, since (u_i) is locally uniformly bounded in $L^p(\Omega)$, the quasiminimizing property of u_i implies that the sequence (u_i) is bounded in $K(t)$, see e.g. Kinnunen–Shanmugalingam [20, Theorem 4.3]. Note also that the sequence α_i is uniformly bounded.

Letting $i \to \infty$, we obtain from (4.2) the estimate

$$\Psi(t') \le C\Psi(t) + C \int_{K(t)} g_u^p \, d\mu. \tag{4.3}$$

On the other hand, the function $s \mapsto s/(s+1)$ is increasing and hence $C = Q/(Q+1)$, where $Q = \limsup_{i\to\infty} Q_i$. Since t is a point of continuity of Ψ, we conclude from (4.3) that

$$\Psi(t) \le Q \int_{K(t)} g_u^p \, d\mu$$

as required. $\square$

Remark 4.2. The following weaker estimate

$$\liminf_{i\to\infty} \int_{K(t)} g_{u_i}^p \, d\mu \le Q \int_{K(t)} g_u^p \, d\mu,$$

where $Q = \liminf_{i\to\infty} Q_i$, follows immediately from the previous lemma. We shall use this in the proof of Theorem 4.3.

Now we are ready to prove Harnack's principle for Q-quasiminimizers with varying Q.

Theorem 4.3. *Let (u_i) be an increasing sequence of Q_i-quasiminimizers in a domain Ω of X and let the sequence (Q_i) be uniformly bounded. Then either $u_i \to \infty$ locally uniformly or $u_i \to u$ locally uniformly in Ω, where u is a Q-quasiminimizers in Ω with $Q = \liminf_{i\to\infty} Q_i$.*

The proof resembles the proof of Theorem 6.1 in Kinnunen–Martio [19]. However, that proof contains a gap which will be settled here. The authors would like to thank professor Fumi-Yuki Maeda for pointing out the error in [19].

Proof. Let $\Omega' \Subset \Omega$ be a domain. There are two possibilities: Either $u(z) = \infty$ for some $z \in \Omega'$ or $u(z) < \infty$ for all $z \in \Omega'$. Now if $u_i(z) \to \infty$ as $i \to \infty$ for some $z \in \Omega'$, it follows from the Harnack inequality (Theorem 3.4) that $u_i \to \infty$ uniformly on $\overline{\Omega'}$ and we conclude that $u_i \to \infty$ locally uniformly on Ω.

Let $z \in \Omega'$ be such that $u(z) < \infty$. The Harnack inequality implies that the sequence (u_i) is uniformly bounded in $\overline{\Omega'}$. Furthermore, as in the proof of Lemma 4.1 we conclude that the sequence (g_{u_i}) is locally uniformly bounded in $L^p(\overline{\Omega'})$. It follows from Lemma 2.7 that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and (g_{u_i}) converges weakly to g in $L^p(\overline{\Omega'})$, where g is a p-weak upper gradient of u.

As in Lemma 4.1, let $K \subset \Omega$ be a compact set and for $t > 0$ write

$$K(t) = \{x \in \Omega \ : \ \operatorname{dist}(x, K) < t\}.$$

Then $K(t) \Subset \Omega$ for $0 < t < \operatorname{dist}(K, X \setminus \Omega) = t_0$ and $K(t)$ is open.

Let $v \in N^{1,p}(\Omega')$ such that $u - v \in N^{1,p}_0(\Omega')$. We need to show that

$$\int_{\overline{\Omega'}} g_u^p \, d\mu \le Q \int_{\overline{\Omega'}} g_v^p \, d\mu. \tag{4.4}$$

To this end let $\varepsilon > 0$ and choose open sets Ω'' and Ω_0 such that

$$\Omega' \Subset \Omega'' \Subset \Omega_0 \Subset \Omega$$

and

$$\int_{\Omega_0 \setminus \overline{\Omega'}} g_u^p \, d\mu < \varepsilon. \tag{4.5}$$

Next choose a Lipschitz cutoff function η with the properties that $\eta = 1$ in an open set containing $\overline{\Omega'}$, $0 \le \eta \le 1$ and $\eta = 0$ on $\Omega \setminus \Omega''$. We choose a subsequence (i_j) such that $Q_{i_j} \to Q$ as $j \to \infty$. We denote, for simplicity, this subsequence by (Q_i). Set

$$w_i = u_i + \eta(v - u_i), \quad i = 1, 2, \ldots$$

Then $u_i - w_i \in N^{1,p}_0(\Omega'')$. We have, as in Lemma 2.4 of Kinnunen–Martio [18],

$$g_{w_i} \le (1 - \eta) g_{u_i} + \eta g_v + g_\eta |v - u_i|, \quad i = 1, 2, \ldots,$$

μ-a.e. in Ω''. As in the proof of Lemma 4.1 we get

$$\int_{\Omega''} g_{w_i}^p \, d\mu \le \int_{\Omega''} (1 - \eta) g_{u_i}^p \, d\mu + \int_{\Omega''} \eta g_v^p \, d\mu + p\beta_i(\alpha_i + \beta_i)^{p-1}, \tag{4.6}$$

where

$$\alpha_i^p := \int_{\Omega''} ((1 - \eta) g_{u_i} + \eta g_v)^p \, d\mu \quad \text{and} \quad \beta_i^p := \int_{\Omega''} g_\eta^p |v - u_i|^p \, d\mu.$$

We estimate the terms on the right hand side separately. Since $g_\eta = 0$ μ-a.e. in $\overline{\Omega'}$ and $v = u$ in $\Omega'' \setminus \Omega'$, we have

$$\begin{aligned} \beta_i^p &= \int_{\Omega''} g_\eta^p |v - u_i|^p \, d\mu \\ &= \int_{\Omega'' \setminus \Omega'} g_\eta^p |v - u_i|^p \, d\mu + \int_{\Omega'} g_\eta^p |v - u_i|^p \, d\mu \\ &\le \int_{\Omega'' \setminus \Omega'} g_\eta^p |u - u_i|^p \, d\mu \le \int_{\Omega''} g_\eta^p |u - u_i|^p \, d\mu. \end{aligned} \tag{4.7}$$

The Lebesgue monotone convergence theorem implies that β_i^p tends to 0 as $i \to \infty$.

Next, since $\eta = 1$ in an open set containing $\overline{\Omega'}$, there is a compact set $K \subset \overline{\Omega''}$ such that $K \cap \overline{\Omega'} = \emptyset$ and

$$\int_{\Omega''} (1-\eta) g_{u_i}^p \, d\mu \le \int_K g_{u_i}^p \, d\mu.$$

We may choose $K = \overline{\Omega''} \setminus \Omega'(t)$ for sufficiently small $t > 0$; observe that K is independent of i. Next choose $t > 0$ such that

$$\liminf_{i\to\infty} \int_{K(t)} g_{u_i}^p \, d\mu \le Q \int_{K(t)} g_u^p \, d\mu,$$

where $Q = \liminf_{i\to\infty} Q_i$, and $K(t) \subset \Omega_0 \setminus \overline{\Omega'}$. This is possible by Lemma 4.1 (see also Remark 4.2). We have

$$\begin{aligned} \liminf_{i\to\infty} \int_{\Omega''} (1-\eta) g_{u_i}^p \, d\mu &\le \liminf_{i\to\infty} \int_K g_{u_i}^p \, d\mu \\ &\le \liminf_{i\to\infty} \int_{K(t)} g_{u_i}^p \, d\mu \le Q \int_{K(t)} g_u^p \, d\mu \le Q\varepsilon, \end{aligned} \tag{4.8}$$

where the last inequality follows from (4.5). Since the sequence α_i, $i = 1, 2, ...$, is bounded as $i \to \infty$, it follows from (4.6), (4.7) and (4.8) that

$$\liminf_{i\to\infty} \int_{\Omega''} g_{w_i}^p \, d\mu \le Q\varepsilon + \int_{\Omega''} g_v^p \, d\mu.$$

Now g_{u_i} converges weakly to g in $L^p(\Omega'')$ and $g_u \le g$ μ-a.e. Thus the quasiminimizing property of u_i together with the lower semicontinuity of the L^p-norm give

$$\begin{aligned} \int_{\overline{\Omega'}} g_u^p \, d\mu &\le \int_{\Omega''} g^p \, d\mu \\ &\le \liminf_{i\to\infty} \int_{\Omega''} g_{u_i}^p \, d\mu \le \liminf_{i\to\infty} Q_i \int_{\Omega''} g_{w_i}^p \, d\mu \\ &= Q \liminf_{i\to\infty} \int_{\Omega''} g_{w_i}^p \, d\mu \le Q^2\varepsilon + Q \int_{\Omega''} g_v^p \, d\mu \\ &\le Q^2\varepsilon + Q \int_{\overline{\Omega'}} g_v^p \, d\mu + Q \int_{\Omega''\setminus\overline{\Omega'}} g_v^p \, d\mu \\ &\le Q(Q+1)\varepsilon + Q \int_{\overline{\Omega'}} g_v^p \, d\mu, \end{aligned}$$

where we used the facts that $Q_i \to Q$ as $i \to \infty$ and $g_u = g_v$ μ-a.e. in $\Omega'' \setminus \overline{\Omega'}$ together with inequality (4.5). Letting $\varepsilon \to 0$ we obtain (4.4), hence, u is a Q-quasiminimizer in Ω with $Q = \liminf_{i\to\infty} Q_i$.

Now u and u_i are continuous for every i. The local Hölder continuity estimate for quasiminimizers, see Kinnunen–Shanmugalingam [20] (see also the proof of Theorem 5.1 in Section 5), shows that the family (u_i) is equicontinuous. Since the sequence (u_i) is increasing and $u_i \to u$ μ-a.e. it follows that $u_i \to u$ locally uniformly in Ω. Note that this applies to the whole sequence (u_i) and not just to the subsequence emplyed in the proof. The theorem follows. □

Remark 4.4. An easy modification of the previous proof shows the following: If (u_i) is an increasing sequence of Q_i-quasisuperminimizers in Ω and $u = \lim_{i\to\infty} u_i$ is either locally bounded above in Ω or belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$, then u is a Q-quasisuperminimizer with $Q = \liminf_{i\to\infty} Q_i$. See also Theorem 6.1 in Kinnunen–Martio [19].

Corollary 4.5. *Let (u_i) be a sequence of Q_i-quasiminimizers in a domain Ω of X such that the sequence (u_i) is locally uniformly bounded below and the sequence (Q_i) is uniformly bounded. If $Q = \liminf_{i\to\infty} Q_i$, then there is a subsequence (u_{i_j}) of (u_i) such that either $u_{i_j} \to \infty$ locally uniformly or u_{i_j} converges locally uniformly to a Q-quasiminimizer in Ω.*

Proof. Choose a subsequence (Q_{i_k}) of (Q_i) such that $Q = \lim_{k\to\infty} Q_{i_k}$. Suppose that for some $x \in \Omega$ there is a subsequence (v_j) of (u_{i_k}) such that $(v_j(x))$ is bounded. The Harnack inequality implies that the sequence (v_j) is locally uniformly bounded, and the same reasoning as in the proof of Theorem 5.1 in Section 5, shows that (v_j) is equicontinuous. Hence the Arzela–Ascoli theorem implies a subsequence $(\widetilde{v}_j)$ that converges locally uniformly to a continuous function u in Ω.

Let $\Omega' \Subset \Omega$ be open. Then $\widetilde{v}_j \to u$ uniformly in Ω'. By passing to a subsequence we may assume that

$$\sup_{\Omega'} |\widetilde{v}_j - u| < j^{-2}, \quad j = 2, 3, \dots .$$

Let

$$w_j = \widetilde{v}_j - \frac{1}{j}, \quad j = 2, 3, ...$$

Then $w_j \le w_{j+1}$ in Ω' and $w_j \to u$ uniformly in Ω'. By Theorem 4.3 u is a Q-quasiminimizer in Ω' and $(\widetilde{v}_j)$ is the required subsequence.

If for some $x \in \Omega$ there is a subsequence (v_j) of (u_{i_k}) such that $v_j(x) \to \infty$ as j tends to infinity, the Harnack inequality implies that $v_j \to \infty$ locally uniformly in Ω and the proof is complete.

□

5. Stability of quasiminimizers

In this section we shall assume that Ω is a bounded domain in X with $C_p(X\setminus\Omega) > 0$. Since there is no uniqueness for the Dirichlet problem when $C_p(X \setminus \Omega) = 0$, stability problems with fixed boundary values are not important in this case.

We are interested in the stability properties of Q_i-quasiminimizers when the sequence (Q_i) tends to 1.

Theorem 5.1. *Let (f_i) be a uniformly bounded sequence of functions in $N^{1,p}(\Omega)$ such that $f_i \to f$ μ-a.e. Furthermore, let (u_i) be a sequence of Q_i-quasiminimizers in Ω with boundary values f_i in Ω and let the sequence (Q_i) be uniformly bounded. If $Q_i \to 1$, then $u_i \to u$ locally uniformly in Ω and u is a minimizer in Ω with boundary values f.*

Proof. It follows from Lemma 2.7(ii) that $f \in N^{1,p}(X)$. Since $u_i - f_i \in N_0^{1,p}(\Omega)$, quasiminimizing property and Proposition 3.2 implies that

$$\int_\Omega g_{u_i}^p \, d\mu \le Q_i \int_\Omega g_{f_i}^p \, d\mu,$$

and thus the sequence (g_{u_i}) is uniformly bounded in $L^p(\Omega)$ because the sequences (Q_i) and $(\|g_{f_i}\|_{L^p(\Omega)})$ are uniformly bounded. Passing to a subsequence we may assume that (g_{u_i}) converges weakly in $L^p(\Omega)$ to a function $g \in L^p(\Omega)$. Moreover, the sequence (u_i) is uniformly bounded in $L^p(\Omega)$. Indeed, by inequality (2.3) we get

$$\begin{aligned}
\Big(\int_\Omega |u_i|^p \, d\mu\Big)^{1/p} &\le \Big(\int_\Omega |u_i - f_i|^p \, d\mu\Big)^{1/p} + \Big(\int_\Omega |f_i|^p \, d\mu\Big)^{1/p} \\
&\le c\Big(\int_\Omega g_{u_i - f_i}^p \, d\mu\Big)^{1/p} + \Big(\int_\Omega |f_i|^p \, d\mu\Big)^{1/p} \\
&\le c\Big(\int_\Omega g_{u_i}^p \, d\mu\Big)^{1/p} + c\Big(\int_\Omega g_{f_i}^p \, d\mu\Big)^{1/p} + \Big(\int_\Omega |f_i|^p \, d\mu\Big)^{1/p}
\end{aligned}$$

for every $i = 1, 2, \ldots$. All the sequences on the right hand side of the above inequality are bounded, hence, the sequence (u_i) is uniformly bounded in $N^{1,p}(\Omega)$.

By Kinnunen–Shanmugalingam [20, Theorems 4.3 and 5.2] there is a constant $0 < c < \infty$ and $\alpha > 0$, where α depends on p, c_μ and the constants in the weak Poincaré inequality but not on i, such that

$$|u_i(x) - u_i(y)| \le c\|u_i\|_{L^p(\Omega')} d(x, y)^\alpha,$$

where $x, y \in \Omega' \Subset \Omega$. Hence the sequence (u_i) is locally uniformly bounded and equicontinuous in Ω. The Arzela–Ascoli theorem gives a subsequence, which will be denoted by (u_i), converging locally uniformly to a continuous function u in Ω.

From Lemma 2.7 it now follows that $u \in N^{1,p}(\Omega)$ and that g is a p-weak upper gradient of u. Moreover, since $u_i - f_i \in N_0^{1,p}(\Omega)$, $(u_i - f_i)$ is a uniformly bounded sequence in $N^{1,p}(\Omega)$ and $u_i - f_i \to u - f$ μ-a.e. in Ω, it follows from the same lemma that $u - f \in N_0^{1,p}(\Omega)$.

Let $\Omega' \Subset \Omega$ be open. Let $K \subset \Omega$ be a compact set and for $t > 0$ write

$$K(t) = \{x \in \Omega \ : \ \operatorname{dist}(x, K) < t\}.$$

Then $K(t) \Subset \Omega$ for $0 < t < \operatorname{dist}(K, X \setminus \Omega) = t_0$. Lemma 4.1 implies that

$$\limsup_{i\to\infty} \int_{K(t)} g_{u_i}^p \, d\mu \le \int_{K(t)} g_u^p \, d\mu$$

for almost every $t \in (0, t_0)$.

Let $v \in N^{1,p}(\Omega')$ such that $u - v \in N_0^{1,p}(\Omega')$. Let $\varepsilon > 0$ and choose open sets Ω'' and Ω_0 such that

$$\Omega' \Subset \Omega'' \Subset \Omega_0 \Subset \Omega$$

and

$$\int_{\Omega_0 \setminus \overline{\Omega'}} g_u^p \, d\mu < \varepsilon.$$

Next choose a Lipschitz cutoff function η with the properties $\eta = 1$ in a neighbourhood of $\overline{\Omega'}$, $0 \le \eta \le 1$ and $\eta = 0$ on $\Omega \setminus \Omega''$. Set

$$w_i = u_i + \eta(v - u_i), \quad i = 1, 2, \ldots$$

Then $w_i - u_i \in N_0^{1,p}(\Omega'')$. Exactly as in the proof of Theorem 4.3, we obtain

$$\int_{\Omega''} g_{w_i}^p \, d\mu \le \int_{\Omega''} (1 - \eta) g_{u_i}^p \, d\mu + \int_{\Omega''} \eta g_v^p \, d\mu + p\beta_i(\alpha_i + \beta_i)^{p-1}.$$

Estimating the terms on the right hand side separately, see the proof of Theorem 4.3, we finally arrive at

$$\limsup_{i\to\infty} \int_{\Omega''} g_{w_i}^p \, d\mu \le \varepsilon + \int_{\Omega''} g_v^p \, d\mu.$$

Since u_i is a quasiminimizer and $Q_i \to 1$ as i tends to ∞, it follows that

$$\begin{aligned}
\int_{\overline{\Omega'}} g_u^p \, d\mu &\le \int_{\overline{\Omega'}} g^p \, d\mu \le \liminf_{i\to\infty} \int_{\overline{\Omega'}} g_{u_i}^p \, d\mu \le \limsup_{i\to\infty} \int_{\Omega''} g_{u_i}^p \, d\mu \\
&\le \limsup_{i\to\infty} Q_i \int_{\Omega''} g_{w_i}^p \, d\mu \le \varepsilon + \lim_{i\to\infty} Q_i \int_{\Omega''} g_v^p \, d\mu \\
&\le \varepsilon + \int_{\overline{\Omega'}} g_v^p \, d\mu,
\end{aligned}$$

where we used the fact that $g_u = g_v$ μ-a.e. in $\Omega'' \setminus \overline{\Omega'}$ and that $Q_i \to 1$. Letting $\varepsilon \to 0+$ and since $\Omega' \Subset \Omega$ was an arbitrary open set, we see that u is a minimizer in Ω.

The previous proof shows the result can be obtained for every subsequence (u_{i_j}) of (u_i). This and the uniqueness of minimizers with the given boundary data f imply the assertion for the sequence (u_i). □

Corollary 5.2. *Let $f \in N^{1,p}(\Omega)$ and let (u_i) be a sequence of Q_i-quasiminimizers in Ω with boundary values f in Ω and let the sequence (Q_i) be uniformly bounded. If $Q_i \to 1$, then $u_i \to u$ locally uniformly in Ω and u is a minimizer in Ω with boundary values f.*

Remark 5.3. (1) Equation (5.7) in Holopainen et al. [13] shows that if $u, u_i \in N^{1,q}_{\mathrm{loc}}(X)$ for some $q > p$ then $g_{|u-u_i|^{q/p}} \to 0$ in $L^p(X)$.

Moreover, if $q \geq p$, (u_i) and (g_{u_i}) are uniformly bounded in $\Omega \Subset X$, and u is bounded in X, then there is a constant $t \in (0,1)$ such that for every $x \in X$ and $r < \frac{1}{3} \operatorname{diam} X$

$$\sup_{B(x,r)} |u - u_i|^q \leq c \left(\fint_{B(x,2r)} |u - u_i|^q \, d\mu \right)^t,$$

where c is independent of $x \in X$. See Lemma 5.2 in [13].

(2) It would be interesting to know whether for $X = \mathbf{R}^n$ and μ equals the Lebesgue measure, the p-Dirichlet regularity of the open set Ω implies that the convergence in Theorem 5.1 is uniform in Ω for a single continuous boundary function $f_i = f$ for $i = 1, 2, \ldots$.

The proof of Proposition 5.4 below implies that $\|u_i\|_{N^{1,p}(\Omega)} \to \|u\|_{N^{1,p}(\Omega)}$. If $N^{1,p}(\Omega)$ is uniformly convex it follows $u_i \to u$ in $N^{1,p}(\Omega)$. Examples of metric spaces X for which the space $N^{1,p}(X)$ is uniformly convex include unweighted and weighted Euclidean spaces, Carnot–Carathéodory spaces and spaces with Cheeger derivative structure, see Cheeger [6]. However, in the generality of this paper we do not know that the space $N^{1,p}(X)$ (or $N^{1,p}(\Omega)$) is uniformly convex.

From now on, suppose $X = \mathbf{R}^n$ is equipped with Euclidean distance, the measure μ is the Lebesgue measure and $\Omega \subset \mathbf{R}^n$ is a bounded domain so that $C_p(\mathbf{R}^n \setminus \Omega) > 0$.

We recall that if (f_i) is a sequence of functions in L^p (L^p-spaces, with $1 < p < \infty$, are uniformly convex) and f a function in L^p such that $f_i \to f$ weakly and $\|f_i\|_{L^p} \to \|f\|_{L^p}$, then by the Radon–Riesz theorem, see e.g. Hewitt–Stromberg [12], we have $\|f_i - f\|_{L^p} \to 0$. Using these observations we are able to prove a strong convergence of the gradients in the Euclidean case.

Proposition 5.4. *Let $(u_i), (f_i), (Q_i), u$ and f be as in Theorem 5.1 in which case $u_i \to u$ locally uniformly and u is a minimizer with boundary values f in Ω. Then $u_i \to u$ in $W^{1,p}(\Omega)$.*

Proof. Clearly $u_i \to u$ in $L^p(\Omega)$, see the proof of Theorem 5.1. Observe that the Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, has the property that if $w_i, w \in W^{1,p}(\Omega)$, $w_i \to w$ in $L^p(\Omega)$ and $(|\nabla w_i|)$ is a bounded sequence in $L^p(\Omega)$, then the sequence (∇w_i) converges weakly to ∇w in $L^p(\Omega)$. Thus we obtain that $\nabla u_i \to \nabla u$ weakly in $L^p(\Omega)$. Since $u_i - f, u - f, u - u_i \in W_0^{1,p}(\Omega)$, the quasiminimizing property of u_i and u and Proposition 3.2 imply that

$$\begin{aligned} 0 &\le \|\nabla u_i\|^p_{L^p(\Omega)} - \|\nabla u\|^p_{L^p(\Omega)} \\ &\le Q_i \int_\Omega |\nabla u|^p \, dx - \int_\Omega |\nabla u|^p \, dx \le (Q_i - 1)\|\nabla f\|^p_{L^p(\Omega)}. \end{aligned}$$

It follows that $\|\nabla u_i\|_{L^p(\Omega)} \to \|\nabla u\|_{L^p(\Omega)}$ when $i \to \infty$. Since $\nabla u_i \to \nabla u$ weakly in $L^p(\Omega)$, the Radon–Riesz theorem implies the claim. $\square$

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