Parameter estimation for stochastic equations with additive fractional Brownian sheet

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November 21, 2005

Abstract

We study the maximum likelihood estimator for stochastic equations with additive fractional Brownian sheet. We use the Girsanov transform for the two-parameter fractional Brownian motion, as well as the Malliavin calculus and Gaussian regularity theory.

Mathematics Subject Classification (2000): 60G15, G0H07, 60G35, 62M40

Key words and phrases: maximum likelihood estimator, fractional Brownian sheet, Malliavin calculus, Girsanov transform

1 Introduction

The recent development in the stochastic calculus with respect to the fractional Brownian motion has led to the study of parameter estimation problems for stochastic equations driven by this process. Several authors has studied these aspects; see e.g. [6, 7, 13, 8, 19]. An obvious extension is to study the two-parameter case. Elements of the stochastic calculus with respect to the fractional Brownian sheet has recently been considered by [17, 18] and stochastic equations with fractional Brownian sheet has emerged in [3] and [15].

The aim of this work is to construct the maximum likelihood estimator (MLE) for the parameter θ in the equation

$$X_{t,s} = \theta \int_0^t \int_0^s b(X_{v,u})\,\mathrm{d}u\mathrm{d}v + W^{\alpha,\beta}_{t,s}, \qquad t,s \in [0,T],$$

where $W^{\alpha,\beta}$ is a fractional Brownian sheet with Hurst parameters $\alpha,\beta \in (0,1)$ and b is a Lipschitz function. Our construction of the estimator is based on the Girsanov transform and uses the connection between the fractional Brownian sheet and the

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standard one, Malliavin calculus, and Gaussian regularity theory. A related work on a two-parameter model with standard Brownian sheet is the paper [2].

The paper is organized as follows. Section 2 contains some preliminaries on the fractional Brownian sheet. In section 3, using the techniques of the Malliavin calculus, we prove that the solution is sub-Gaussian. Section 4 contains the proof of the existence of the MLE for the parameter θ and we separate this proof following the values of α and β. Finally, in Section 5, we present a different expression of the MLE and relate our work with the approach of [7].

2 Fractional Brownian sheet as a Volterra sheet

We recall how the fractional Brownian sheet $W^{\alpha,\beta}$ can be represented by a standard Brownian sheet $W = W^{\frac{1}{2},\frac{1}{2}}$ that is constructed from it. For details and references see [14, 15].

We start with the one-parameter case. Define a Volterra kernel (i.e. a kernel which vanishes if the second variable is greater than the first one)

$$K_\alpha(t,s) = c_\alpha \left(\left(\frac{t}{s}\right)^{\alpha-\frac{1}{2}} (t-s)^{\alpha-\frac{1}{2}} - (\alpha - \frac{1}{2}) s^{\frac{1}{2}-\alpha} \int_s^t u^{\alpha-\frac{3}{2}} (u-s)^{\alpha-\frac{1}{2}} \, \mathrm{d}u \right),$$

where the normalising constant is

$$c_\alpha = \sqrt{\frac{(2\alpha+\frac{1}{2})\Gamma(\frac{1}{2}-\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2-2\alpha)}}$$

and Γ is the Euler's gamma function. It was shown in [10] that the fractional Brownian motion W^α with Hurst index $\alpha \in (0,1)$ can be represented by using a standard Brownian motion $W = W^{\frac{1}{2}}$ as

$$W_t^\alpha = \int_0^t K_\alpha(t,s) \, \mathrm{d}W_s.$$

This Wiener integral can be understood both in pathwise and L^2-sense. The Brownian motion W is actually constructed from the fractional one by the (pathwise or L^2) integral

$$W_t = \int_0^t K_\alpha^{-1}(t,s) \, \mathrm{d}W_s^\alpha.$$

Here the Volterra kernel K_α^{-1} is

$$K_\alpha^{-1}(t,s) = c'_\alpha \left(\left(\frac{t}{s}\right)^{\alpha-\frac{1}{2}} (t-s)^{\frac{1}{2}-\alpha} - (\alpha - \frac{1}{2}) s^{\frac{1}{2}-\alpha} \int_s^t u^{\alpha-\frac{3}{2}} (u-s)^{\frac{1}{2}-\alpha} \, \mathrm{d}u \right),$$

$$c'_\alpha = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(2-2\alpha)}{\mathrm{B}(\frac{1}{2}-\alpha)\sqrt{(2\alpha+\frac{1}{2})\Gamma(\frac{1}{2}-\alpha)}},$$

and B is the beta function.

Now we turn to the two-parameter case. Let

$$\begin{aligned} K_{\alpha,\beta}(t,s;v,u) &= K_\alpha(t,v)K_\beta(s,u), \\ K_{\alpha,\beta}^{-1}(t,s;v,u) &= K_\alpha^{-1}(t,v)K_\beta^{-1}(s,u). \end{aligned}$$

The kernels $K_{\alpha,\beta}$ and $K_{\alpha,\beta}^{-1}$ are of Volterra type: they vanish if $v \geq t$ or $u \geq s$. Then from the one-parameter case it follows that we have the following (pathwise and L^2-sense) transformations connecting the fractional Brownian sheet $W^{\alpha,\beta}$ and the standard one $W = W^{\frac{1}{2},\frac{1}{2}}$:

$$W_{t,s}^{\alpha,\beta} = \int_0^t \int_0^s K_{\alpha,\beta}(t,s;v,u)\,\mathrm{d}W_{u,v}, \tag{2.1}$$

$$W_{t,s} = \int_0^t \int_0^s K_{\alpha,\beta}^{-1}(t,s;v,u)\,\mathrm{d}W_{u,v}^{\alpha,\beta}. \tag{2.2}$$

We shall use this connection in the following way: The fractional Brownian sheet $W^{\alpha,\beta}$ is assumed to be given, the standard Brownian sheet W is constructed from the fractional one $W^{\alpha,\beta}$ by the formula (2.2), and then $W^{\alpha,\beta}$ is represented in terms of W by the formula (2.1).

In what follows we shall denote by $K_{\alpha,\beta}$ also the operator on $L^2([0,T]^2)$ induced by the kernel $K_{\alpha,\beta}$:

$$K_{\alpha,\beta}[f](t,s) = \int_0^t \int_0^s K_{\alpha,\beta}(t,s;v,u) f(v,u)\,\mathrm{d}u\mathrm{d}v,$$

and similarly for $K_{\alpha,\beta}^{-1}$. Note that as an operator $K_{\alpha,\beta}^{-1}$ is indeed the inverse of the operator $K_{\alpha,\beta}$.

Finally, we note the following connection with deterministic fractional calculus. The *fractional Riemann–Liouville integral* of order $\gamma > 0$ is

$$I^\gamma[f](t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-v)^{\gamma-1} f(v)\,\mathrm{d}v.$$

The *fractional Weyl derivative* of order $\gamma \in (0,1)$ is

$$I^{-\gamma}[f](t) = \frac{1}{\Gamma(1-\gamma)} \left[\frac{f(t)}{t^\gamma} + \gamma \int_0^t \frac{f(t)-f(v)}{(t-v)^{\gamma+1}}\,\mathrm{d}v \right].$$

The mixed fractional integral–differential operator $I^{\gamma,\eta}$ acts on two-argument functions $f = f(t,s)$ argumentwise: $I^{\gamma,\eta}[f](t,s) = I^\gamma[f_\eta(\cdot,s)](t)$, where $f_\eta(t,s) = I^\eta[f(t,\cdot)](s)$. Now, we have the representation (see [3])

$$K_{\alpha,\beta}^{-1}[f](t,s) = c''_{\alpha,\beta} t^{\alpha-\frac{1}{2}} s^{\beta-\frac{1}{2}} I^{\frac{1}{2}-\alpha,\frac{1}{2}-\beta} \left[t^{\frac{1}{2}-\alpha} s^{\frac{1}{2}-\beta} \frac{\partial^2 f}{\partial t \partial s} \right](t,s), \tag{2.3}$$

where $c''_{\alpha,\beta}$ is a certain normalising constant.

3 On the solution

Now we shall focus our attention to the stochastic differential equation

$$X_{t,s} = \theta \int_0^t \int_0^s b(X_{v,u})\,\mathrm{d}u\mathrm{d}v + W^{\alpha,\beta}_{t,s}, \qquad t,s \in [0,T]. \tag{3.1}$$

The equation (3.1) has been considered in [3] for parameters $(\alpha,\beta) \in (0,\frac{1}{2})^2$ in the more general context $b(x) = b(t,s;x)$ with $t,s \in [0,T]$. It has been proved that, if b satisfy the linear growth condition

$$|b(t,s;x)| \leq C(1+|x|)$$

then (3.1) admits a unique weak solution, and if b is nondecreasing in the second variable and bounded, then (3.1) has a unique strong solution.

We are here interesting in the case when the drift coefficient b is Lipschitz (more exactly, we will assume that b is differentiable with bounded derivative). It is clear that in this case the method of standard Picard iterations can be applied to obtain the existence and the uniqueness of the solution for all Hurst parameters α,β belonging to $(0,1)$. As far as we know there are not existence and uniqueness results in the non-Lipschitz case if α or β are bigger than $\frac{1}{2}$.

Since our main objective is the construction of a maximum likehood estimator from the observation of the trajectory of the process X that satisfies (3.1), we will need some estimates on the supremum of this processes, and even more generally, on the variations of this process. For this purpose, we follow the ideas developped in [19] to prove that the solution of (3.1) is a *sub-Gaussian process* with respect to a certain canonical metric. The proof is based on the Poincaré inequality and uses some elements of the Malliavin calculus.

We make the following assumption on the drift b throughout the rest of the paper:

(C1) *b is differentiable with bounded derivative.*

Denote by D the Malliavin derivative with respect to the Brownian sheet W. We refer to [11] details of Malliavin calculus and just recall two basic facts:

(i) If F is a Wiener integral of the form

$$F = \int_0^T \int_0^T f(t,s)\,\mathrm{d}W_{s,t}$$

with $f \in L^2([0,T]^2)$ then

$$D_{t,s}F = f(t,s).$$

(ii) If F is a random variable differentiable in the Malliavin sense and b is a function satisfying the condition (C1) then $b(F)$ is Malliavin differentiable and we have the chain rule

$$D_{t,s}b(F) = b'(F)D_{t,s}F.$$

We will need two auxiliary lemmas.

3.2 Lemma. *There exists a constant C_θ depending on T, α, β, $\|b'\|_\infty$ and the parameter θ such that for every $t,s\in[0,T]$ we have the bound*

$$\|D_{\cdot,\cdot}X_{t,s}\|^2_{L^2(\Omega\times[0,T]^2)}\le C_\theta. \tag{3.3}$$

Proof. Taking Malliavin derivatives $D_{v,u}$ on the both sides of the equation (3.1) we obtain

$$D_{v,u}X_{t,s}=\theta\int_v^t\int_u^s b'(X_{v',u'})D_{v,u}X_{v',u'}\,\mathrm{d}u'\mathrm{d}v'+K_{\alpha,\beta}(t,s;v,u).$$

Denote

$$M_{t,s}=\int_0^T\int_0^T|D_{v,u}X_{t,s}|^2\,\mathrm{d}u\mathrm{d}v=\int_0^t\int_0^s|D_{v,u}X_{t,s}|^2\,\mathrm{d}u\mathrm{d}v.$$

Then, by Fubini theorem and the estimate $(x+y)^2\le 2x^2+2y^2$, we obtain

$$\begin{aligned}
M_{t,s} &= \theta^2\int_0^t\int_0^s\left|\int_v^t\int_u^s b'(X_{v',u'})D_{v,u}X_{v',u'}\,\mathrm{d}u'\mathrm{d}v'+K_{\alpha,\beta}(t,s;v,u)\right|^2\mathrm{d}u\mathrm{d}v\\
&\le 2\theta^2\int_0^t\int_0^s\left|\int_v^t\int_u^s b'(X_{v',u'})D_{v,u}X_{v',u'}\,\mathrm{d}u'\mathrm{d}v'\right|^2\mathrm{d}u\mathrm{d}v\\
&\quad +2\int_0^t\int_0^s K_{\alpha,\beta}(t,s;v,u)^2\,\mathrm{d}u\mathrm{d}v\\
&\le 2\theta^2\|b'\|_\infty^2\int_0^t\int_0^s\left\{\int_v^t\int_u^s|D_{v,u}X_{v',u'}|^2\,\mathrm{d}u'\mathrm{d}v'\right\}\mathrm{d}u\mathrm{d}v+2t^{2\alpha}s^{2\beta}\\
&= 2\theta^2\|b'\|_\infty^2\int_0^t\int_0^s\left\{\int_0^{v'}\int_0^{u'}|D_{v,u}X_{v',u'}|^2\,\mathrm{d}u\mathrm{d}v\right\}\mathrm{d}u'\mathrm{d}v'+2t^{2\alpha}s^{2\beta}\\
&= 2\theta^2\|b'\|_\infty^2\int_0^t\int_0^s M_{v',u'}\,\mathrm{d}u'\mathrm{d}v'+2t^{2\alpha}s^{2\beta}.
\end{aligned}$$

So, the claim follows by a two-parameter version of the Gronwall lemma. □

3.4 Lemma. *Let X be the unique solution of (3.1). Then there exists a constant C_θ depending on T, α, β, $\|b'\|_\infty$ and the parameter θ such that for every $s\le s'$, $t\le t'$ it holds that*

$$\|D_{\cdot,\cdot}\left(X_{t',s'}-X_{t,s}\right)\|^2_{L^2(\Omega\times[0,T]^2)}\le C_\theta(|t-t'|^{2\alpha}+|s-s'|^{2\beta}). \tag{3.5}$$

Proof. For every $s\le s'$ and $t\le t'$, we have

$$X_{t',s'}-X_{t,s}=X_{t',s'}-X_{t,s'}+X_{t,s'}-X_{t,s}.$$

So, it is enough to show that

$$\|D_{\cdot,\cdot}\left(X_{t',s'}-X_{t,s'}\right)\|^2_{L^2(\Omega\times[0,T]^2)}\le C_\theta(|t-t'|^{2\alpha}). \tag{3.6}$$

Now

$$X_{t',s'} - X_{t,s'} = \theta \int_t^{t'} \int_0^{s'} b(X_{v,u})\, \mathrm{d}u \mathrm{d}v + W^{\alpha,\beta}_{t',s'} - W^{\alpha,\beta}_{t,s'}$$

and thus

$$\begin{aligned} D_{a,b}\left[X_{t',s'} - X_{t,s'}\right] &= \theta \int_t^{t'} \int_0^{s'} b'(X_{v,u}) D_{a,b} X_{v,u}\, \mathrm{d}u \mathrm{d}v \\ &\quad + K_\beta(s',b)\Big(K_\alpha(t',a) - K_\alpha(t,a)\Big). \end{aligned}$$

By using the fact that

$$\int_0^T \left(K_\alpha(t',a) - K_\alpha(t,a)\right)^2 \, \mathrm{d}a = |t'-t|^{2\alpha}$$

we obtain

$$\begin{aligned} &\mathbf{E}\left[\int_0^T \int_0^T \left|D_{a,b}\left[X_{t',s'} - X_{t,s'}\right]\right|^2 \mathrm{d}b\mathrm{d}a\right] \\ &\leq 2\theta^2 \mathbf{E}\left[\left|\int_t^{t'} \int_0^{s'} b'(X_{v,u}) D_{a,b} X_{v,u}\, \mathrm{d}u\mathrm{d}v\right|^2 \mathrm{d}b\mathrm{d}a\right] + 2(s')^{2\beta}|t'-t|^{2\alpha} \\ &\leq 2\theta^2 \|b'\|_\infty^2 \mathbf{E}\left[\int_0^T \int_0^T |D_{a,b}X_{v,u}|^2 \, \mathrm{d}b\mathrm{d}a\right](t-t')^2 + 2(s')^{2\beta}|t'-t|^{2\alpha}. \end{aligned}$$

The claim follows now from Lemma 3.2 and the fact that $\alpha, \beta < 1$. □

Recall that a sheet X is *sub-Gaussian* with respect to metric δ if for all $\lambda \in \mathbb{R}$

$$\mathbf{E}\left[\exp\left\{\lambda\left(X_{t,s} - X_{t',s'}\right)\right\}\right] \leq \exp\left\{\frac{\lambda^2}{2}\delta(t,s;t',s')^2\right\}.$$

3.7 Proposition. *Suppose that b satisfies condition (C1). Then the solution X of (3.1) is a sub-Gaussian process with respect to the metric δ given by*

$$\delta(t,s;t',s')^2 = C_\theta\left(|t-t'|^{2\alpha} + |s-s'|^{2\beta}\right),$$

where the constant C_θ comes from Lemma 3.4.

Proof. Recall the Poincaré inequality (see [20], page 76): if F is a functional of the Brownian sheet W, then

$$\mathbf{E}\left[\exp\{F\}\right] \leq \mathbf{E}\left[\exp\left\{\frac{\pi^2}{8}\|DF\|^2_{L^2([0,T]^2)}\right\}\right].$$

The claim follows from this and Lemma 3.4. □

Proposition 3.7 says that, in the case of the Lipschitz coefficient b, the variations of the process X are dominated, in distribution, by those of the Gaussian process with canonical metric (3.7); this prcosess is actually the so-called isotropic fractional Brownian sheet. As a consequence, the sub-Gaussian regularity theory (see [4] or [9]) can be applied to obtain supremum estimates on the process X. As an immediate consequence, we get, using the results in [9], Chapter 12 and the methods in [19]).

$$\mathbf{E}\left[\sup_{v\leq t,u\leq s}|X_{v,u}|\right] \leq C_\theta\sqrt{t^{2\alpha}+s^{2\beta}}. \tag{3.8}$$

and, for any positive $a_\theta = a(\alpha,\beta,T,\|b'\|_\infty,\theta)$ small enough,

$$\mathbf{E}\left[\exp\left\{a_\theta \sup_{t,s\in[0,T]}|X_{t,s}|^2\right\}\right] < \infty. \tag{3.9}$$

These estimates will be explicitly used in the next section.

4 Maximum likelihood estimator

First we recall how the Girsanov theorem for the shifted fractional Brownian sheet

$$\tilde{W}^{\alpha,\beta}_{t,s} = W^{\alpha,\beta}_{t,s} + \int_0^t\int_0^s a_{u,v}\,\mathrm{d}u\mathrm{d}v \tag{4.1}$$

can be recovered from the Girsanov theorem for the standard shifted Brownian sheet by the correspondence (2.1)–(2.2).

Since the shift term in (4.1) is differentiable we can operate pathwise with the kernel $K^{-1}_{\alpha,\beta}$ on the both sides of the equation (4.1). So, we can set

$$\tilde{W}_{t,s} = \int_0^t\int_0^s K^{-1}_{\alpha,\beta}(t,s;v,u)\,\mathrm{d}\tilde{W}^{\alpha,\beta}_{u,v} \tag{4.2}$$

and we have the following inverse relation for the transfer (4.2):

$$\tilde{W}^{\alpha,\beta}_{t,s} = \int_0^t\int_0^s K_{\alpha,\beta}(t,s;v,u)\,\mathrm{d}\tilde{W}_{u,v}. \tag{4.3}$$

Now we want to find a shift b such that

$$\tilde{W}_{t,s} = W_{t,s} + \int_0^t\int_0^s b_{v,u}\,\mathrm{d}u\mathrm{d}v, \tag{4.4}$$

where W is a standard Brownian sheet constructed from the fractional one $W^{\alpha,\beta}$ by

$$W_{t,s} = \int_0^t\int_0^s K^{-1}_{\alpha,\beta}(t,s;v,u)\,\mathrm{d}W^{\alpha,\beta}_{u,v}. \tag{4.5}$$

Comparing equations (4.1)–(4.5) we see that the connection is

$$\int_0^t\int_0^s a_{v,u}\,\mathrm{d}u\mathrm{d}v = \int_0^t K_{\alpha,\beta}(t,s;v,u)b_{v,u}\,\mathrm{d}u\mathrm{d}v$$

So, we conclude that

$$b_{t,s} = \int_0^t \int_0^s K^{-1}_{\alpha,\beta}(t,s;v,u) \left(\int_0^v \int_0^u a_{v',u'}\,\mathrm{d}u'\mathrm{d}v' \right) \mathrm{d}u\mathrm{d}v \tag{4.6}$$

or, in operator notation,

$$b = K^{-1}_{\alpha,\beta}\left[\int_0^{\cdot} \int_0^{\cdot} a_{t,s}\,\mathrm{d}s\mathrm{d}t \right].$$

Now, comparing (4.1) to (4.4) with the connection (4.6) we obtain the following Girsanov theorem from the classical Girsanov theorem for the shifted standard Brownian sheet.

4.7 Theorem. *Let $W^{\alpha,\beta}$ be a fractional Brownian sheet and let a be a process adapted to the filtration generated by $W^{\alpha,\beta}$. Let W be a standard Brownian sheet constructed from $W^{\alpha,\beta}$ by (2.2) and let b b constructed from a by (4.6).*

Assume that $b \in L^2(\Omega \times [0,T]^2)$, and $\mathbf{E}[V_{T,T}] = 1$ where

$$V_{T,T} = \exp\left\{ -\int_0^T \int_0^T b_{t,s}\,\mathrm{d}W_{s,t} - \frac{1}{2}\int_0^T \int_0^T b^2_{t,s}\,\mathrm{d}s\mathrm{d}t \right\}.$$

Then under the new probability $\tilde{\mathbf{P}}$ with $\frac{\mathrm{d}\tilde{\mathbf{P}}}{\mathrm{d}\mathbf{P}} = V_{T,T}$ the process $\tilde{W}$ given by (4.2) is a Brownian sheet and the process $\tilde{W}^{\alpha,\beta}$ given by (4.1) is a fractional Brownian sheet.

The rest of this paper is devoted to construct a maximum likelihood estimator for the parameter θ in (3.1) by using the Girsanov theorem (Theorem 4.7). The existence and the expression of this MLE are given by the following result.

4.8 Proposition. *Assume that one of the following holds:*

(i) *At least one of the parameters α and β belongs to $(0, \frac{1}{2})$ and b satisfies (C1)*

(ii) *The parameters α and β both belong to $(\frac{1}{2}, 1)$ and b is linear.*

Denote

$$Q_{t,s} = K^{-1}_{\alpha,\beta}\left[\int_0^{\cdot} \int_0^{\cdot} b(X_{v,u})\,\mathrm{d}u\mathrm{d}v \right](t,s). \tag{4.9}$$

Then given observation over $[0,t]^2$ the MLE for θ in (3.1) is

$$\theta_t = -\frac{\int_0^t \int_0^t Q_{v,u}\,\mathrm{d}W_{u,v}}{\int_0^t \int_0^t Q^2_{v,u}\,\mathrm{d}u\mathrm{d}v}. \tag{4.10}$$

Before going into the proof of Proposition 4.8 let us note that we can also write

$$\theta_t = \frac{\int_0^t \int_0^t Q_{v,u}\,\mathrm{d}\tilde{W}_{u,v}}{\int_0^t \int_0^t Q^2_{v,u}\,\mathrm{d}u\mathrm{d}v}. \tag{4.11}$$

This shows that the estimator can be deduced by the observed process X since

$$\tilde{W}_{t,s} = \int_0^t \int_0^s K^{-1}_{\alpha,\beta}(t,s;v,u)\mathrm{d}X_{u,v}.$$

Proof. Let us denote by $\mathbf{P}_\theta$ the law of the process $X_{t,s}$ that is the unique solution of (3.1). Then the MLE is obtained by taking the $\sup_\theta F_\theta$, where

$$F_\theta = \frac{\mathrm{d}\mathbf{P}_\theta}{\mathrm{d}\mathbf{P}_0}.$$

The conclusion (4.9) then follows by the Girsanov theorem (Theorem 4.7) if we show that $V_{t,t}$ is well-defined and $\mathbf{E}[V_{t,t}] = 1$. The proof is separated into three cases. In what follows $\mathfrak{c}_{\alpha,\beta}$ is a generic constant depending on α and β that may change even within a line.

The case $\alpha,\beta \in (0,\frac{1}{2})$: In this case we use the expression (2.3) with $I^{\alpha-\frac{1}{2},\beta-\frac{1}{2}}$ as a double fractional integral. We get

$$Q_{t,s} = \mathfrak{c}_{\alpha,\beta}\, t^{\alpha-\frac{1}{2}} s^{\beta-\frac{1}{2}} \int_0^t \int_0^s (t-v)^{-\frac{1}{2}-\alpha} v^{\frac{1}{2}-\alpha} (s-u)^{-\frac{1}{2}-\beta} u^{\frac{1}{2}-\beta} b(X_{u,v})\, \mathrm{d}u\mathrm{d}v.$$

Since $|b(x)| \le |b(x)-b(0)| + |b(0)| \le K(|x|+K')$, we see that

$$|Q_{t,s}| \le \mathfrak{c}_{\alpha,\beta} \left(K + \sup_{u\le t, v\le s} |X_{u,v}| \right). \tag{4.12}$$

Clearly (4.12) and the estimates (3.8), (3.9) show that $Q \in L^2([0,T]^2)$ and so $V_{t,t}$ is well-defined. To prove that $\mathbf{E}[V_{t,t}] = 1$, it suffices to invoke Theorem 1.1, page 152 in [5] and to note that, by (4.12), there exists $a > 0$ such that

$$\sup_{v\le t, u\le s} \mathbf{E}\left[\exp\left\{aQ_{v,u}^2\right\}\right] < \infty.$$

The case $\alpha,\beta \in (\frac{1}{2},1)$: In this case we use the operator $I^{\alpha-\frac{1}{2},\beta-\frac{1}{2}}$ in the expression (2.3) is a double fractional derivative. We get

$$\begin{aligned}
Q_{t,s} &= \mathfrak{c}_{\alpha,\beta}\, t^{\frac{1}{2}-\alpha} s^{\frac{1}{2}-\beta} b(X_{t,s}) + \mathfrak{c}_{\alpha,\beta}\, t^{\alpha-\frac{1}{2}} s^{\frac{1}{2}-\beta} \int_0^t \frac{t^{\frac{1}{2}-\alpha} b(X_{t,s}) - v^{\frac{1}{2}-\alpha} b(X_{v,s})}{(t-v)^{\alpha+\frac{1}{2}}}\, \mathrm{d}v \\
&\quad + \mathfrak{c}_{\alpha,\beta}\, t^{\frac{1}{2}-\alpha} s^{\beta-\frac{1}{2}} \int_0^s \frac{s^{\frac{1}{2}-\beta} b(X_{t,s}) - u^{\frac{1}{2}-\beta} b(X_{t,u})}{(s-u)^{\beta+\frac{1}{2}}}\, \mathrm{d}u \\
&\quad + \mathfrak{c}_{\alpha,\beta} \int_0^t \int_0^s \Big\{ t^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} b(X_{t,s}) - v^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} b(X_{v,s}) - t^{\frac{1}{2}-\alpha} u^{\frac{1}{2}-\beta} b(X_{t,u}) \\
&\qquad + v^{\frac{1}{2}-\alpha} u^{\frac{1}{2}-\beta} b(X_{v,u}) \Big\} \left((t-v)^{-\alpha-\frac{1}{2}} (s-u)^{-\beta-\frac{1}{2}} \right) \mathrm{d}u\mathrm{d}v \\
&:= A_1(t,s) + A_2(t,s) + A_3(t,s) + A_4(t,s).
\end{aligned}$$

The term $A_1(t,s)$ can be easily treated, since by (C1),

$$|A_1(t,s)| \le \mathfrak{c}_{\alpha,\beta}\, t^{\frac{1}{2}-\alpha} s^{\frac{1}{2}-\beta} \left(K + \|X\|_\infty \right)$$

which is finite by (4.12). The term $A_2(t,s)$ and $A_3(t,s)$ are rather similar to those appearing in the one-parameter case (see [12] and [19]). For the sake of completeness, we illustrate how to treat them. For $A_2(t,s)$ write

$$\begin{aligned} A_2(t,s) &= \mathfrak{c}_{\alpha,\beta}\, t^{\alpha-\frac{1}{2}} s^{\frac{1}{2}-\beta} b(X_{t,s}) \int_0^t \frac{t^{\frac{1}{2}-\alpha} - v^{\frac{1}{2}-\alpha}}{(t-v)^{\alpha+\frac{1}{2}}}\, \mathrm{d}v \\ &\quad + \mathfrak{c}_{\alpha,\beta}\, t^{\alpha-\frac{1}{2}} s^{\frac{1}{2}-\beta} \int_0^t \frac{b(X_{t,s}) - b(X_{v,s})}{(t-v)^{\alpha+\frac{1}{2}}}\, \mathrm{d}v \\ &=: A_{21}(t,s) + A_{22}(t,s). \end{aligned}$$

Since

$$\int_0^t \frac{t^{\frac{1}{2}-\alpha} - u^{\frac{1}{2}-\alpha}}{(t-v)^{\alpha+\frac{1}{2}}}\, \mathrm{d}v = \mathfrak{c}_{\alpha,\beta}\, t^{1-2\alpha},$$

the summand $A_{21}(t,s)$ is clearly almost surely finite using condition (C1). The second summand A_{22} can be bounded as follows: for ε small enough,

$$|A_{2,2}(t,s)| \le \mathfrak{c}_{\alpha,\beta}\, t^{\alpha-\frac{1}{2}} s^{\frac{1}{2}-\beta} \sup_{v\le t} \left[\frac{|X_{t,s} - X_{v,s}|}{(t-v)^{\alpha-\varepsilon}} \right] \int_0^t u^{\frac{1}{2}-\alpha} (t-v)^{-\frac{1}{2}-\varepsilon}\, \mathrm{d}v$$

By the Fernique theorem the supremum above has exponential moments. So, it follows that the Novikov criterium is satisfied. Let us study now the term $A_4(t,s)$. It is not difficult to see that the expression

$$\begin{aligned} I &= t^{\frac{1}{2}-\alpha} s^{\frac{1}{2}-\beta} b(X_{t,s}) - v^{\frac{1}{2}-\alpha} s^{\frac{1}{2}-\beta} b(X_{v,s}) \\ &\quad - t^{\frac{1}{2}-\alpha} u^{\frac{1}{2}-\beta} b(X_{t,u}) + u^{\frac{1}{2}-\alpha} u^{\frac{1}{2}-\beta} b(X_{v,u}) \end{aligned}$$

can be written as

$$\begin{aligned} I &= \left(t^{\frac{1}{2}-\alpha} - u^{\frac{1}{2}-\alpha}\right) b(X_{t,s}) \left(s^{\frac{1}{2}-\beta} - v^{\frac{1}{2}-\beta}\right) \\ &\quad + \left(t^{\frac{1}{2}-\alpha} - u^{\frac{1}{2}-\alpha}\right) v^{\frac{1}{2}-\beta} \Big(b(X_{t,s}) - b(X_{t,v})\Big) \\ &\quad + u^{\frac{1}{2}-\alpha} \left(s^{\frac{1}{2}-\beta} - v^{\frac{1}{2}-\beta}\right) \Big(b(X_{t,s}) - b(X_{u,s})\Big) \\ &\quad + u^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} \Big(b(X_{t,s}) - b(X_{t,v}) - b(X_{u,s}) + b(X_{u,v})\Big). \end{aligned}$$

This gives a decomposition of the term $A_4(t,s)$ in four summands; The first three summands can be handled by using similar arguments to those already used throughout this proof. For the last term actually we need to assume the linearity of the function b (and obviously the solution of (3.1) is then Gaussian). If b is linear, then

$$\begin{aligned} & b(X_{t,s}) - b(X_{t,u}) - b(X_{v,s}) + b(X_{v,u}) \\ &= X_{t,s} - X_{t,u} - X_{v,s} + X_{v,u} \\ &= \int_v^t \int_u^s X_{b,a}\, \mathrm{d}b \mathrm{d}a + W^{\alpha,\beta}((z,z']) \end{aligned}$$

where $W^{\alpha,\beta}((z,z'])$ denotes the planar increments of $W^{\alpha,\beta}$ between $z=(v,u)$ and $z'=(t,s)$. We finish again by an application of the Fernique theorem and observing that the process $W^{\alpha,\beta}$ (and thus X) is Hölder continuous of order (α,β) (see Proposition 5 in [1]).

The case $\alpha\in(0,\frac{1}{2})$, $\beta\in(\frac{1}{2},1)$*:* In this case, it is not difficult to see that

$$
\begin{aligned}
Q_{t,s} &= \mathfrak{c}_{\alpha,\beta}\, t^{\alpha-\frac{1}{2}}\int_0^t (t-v)^{-\alpha-\frac{1}{2}} v^{\frac{1}{2}-\alpha} \\
&\quad\times\left[s^{\frac{1}{2}-\beta} b(X_{v,s}) + \mathfrak{c}_{\alpha,\beta}\, s^{\beta-\frac{1}{2}} \int_0^s \frac{b(X_{v,s})s^{\frac{1}{2}-\beta} - b(X_{v,u})u^{\frac{1}{2}-\beta}}{(s-u)^{\frac{1}{2}+\beta}}\,\mathrm{d}u \right] \mathrm{d}v.
\end{aligned}
$$

Clearly, this case can be handled by combining the methods used in the first two cases. The only thing we need to note here is that the Fernique theorem holds for the sub-Gaussian process X (actually, here we can use the fact that for each v, the process $s\mapsto X_{v,s}$ is sub-Gaussian with respect the metric $|s'-s|^{\beta}$, see the proof of Lemma 2). □

5 Alternative form of the estimator

We will try to relate here our approach with the one considered by [7] in the one-parameter scale.

As in the one -parameter case (see [10]) one can associate to the fractional Brownian sheet a two-parameter martingale (the so-called *fundamental martingale*). We refer to [16] for the two-parameter case. More precisely, let us define the deterministic function

$$
k_\alpha(t,u) = c_\alpha^{-1} u^{\frac{1}{2}-\alpha}(t-u)^{\frac{1}{2}-\alpha}, \qquad c_\alpha = 2\alpha\Gamma(\frac{3}{2}-\alpha)\Gamma(\alpha+\frac{1}{2})
$$

and

$$
\omega_t^\alpha = \lambda_\alpha^{-1} t^{2-2\alpha}, \qquad \lambda_\alpha = \frac{2\Gamma(3-2\alpha)\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{3}{2}-\alpha)}.
$$

Then the process

$$
M_{t,s}^{\alpha,\beta} = \int_0^t \int_0^s k_\alpha(t,v)k_\beta(s,u)\,\mathrm{d}W_{u,v}^{\alpha,\beta} \tag{5.1}
$$

is a two-parameter Gaussian (strong) martingale with quadratic variation equal to $\omega_t^\alpha \omega_s^\beta$ (the stochastic integral in (5.1) can be defined in a Wiener sense with respect to the fractional Brownian sheet). The filtration generated by $M^{\alpha,\beta}$ coincides to the one generated by $W^{\alpha,\beta}$.

Let us integrate the deterministic kernel $k_\alpha(t,v)k_\beta(s,u)$ with respect to both sides of (3.1). We get

$$Z_{t,s} := \int_0^t \int_0^s k_\alpha(t,v) k_\beta(s,u) \; dX_{v,u}$$

$$(5.2) \qquad = \int_0^t \int_0^s k_\alpha(t,v) k_\beta(s,u) b(X_{v,u}) \, \mathrm{d}u\mathrm{d}v + M_{t,s}^{\alpha,\beta}.$$

5.3 Remark. Moreover, for $\alpha, \beta > \frac{1}{2}$, it follows from [6] and [16] that if we denote

$$K_\alpha(t,v) = \alpha(2\alpha - 1) \int_v^t r^{2\alpha-1}(r-v)^{\alpha-\frac{3}{2}} \, \mathrm{d}r$$

then it holds that

$$(5.4) \qquad X_{t,s} = \int_0^t \int_0^s K_\alpha(t,v) K_\beta(s,u) \, \mathrm{d}Z_{u,v}.$$

Denote

$$(5.5) \qquad R_{t,s} = \frac{\mathrm{d}}{\mathrm{d}\omega_t^\alpha} \frac{\mathrm{d}}{\mathrm{d}\omega_s^\beta} \int_0^t \int_0^s k_\alpha(t,v) k_\beta(s,u) b(X_{v,u}) \, \mathrm{d}u\mathrm{d}v.$$

Then we have the following:

- For every $(\alpha, \beta) \in (0,1)^2$ and if b is Lipschitz, the sample paths of the process R given by (5.5) belong to $L^2([0,1]^2, \omega^\alpha \otimes \omega^\beta)$. This can be viewed in the same way as in Theorem 2.
- Clearly, the process R is related to the process Q (4.9) by

$$R_{t,s} = \mathfrak{c}_{\alpha,\beta} \, t^{\alpha-\frac{1}{2}} s^{\beta-\frac{1}{2}} Q_{t,s}.$$

From (5.2) and (5.5) we obtain that

$$(5.6) \qquad Z_{t,s} = \theta \int_0^t \int_0^s R_{v,u} \, \mathrm{d}\omega_u^\beta \mathrm{d}\omega_v^\alpha + M_{t,s}^{\alpha,\beta}.$$

and then the MLE for the parameter θ in (3.1) can be written as

$$(5.7) \qquad \theta_t = -\frac{\int_0^t \int_0^t R_{v,u} \, \mathrm{d}M_{u,v}^{\alpha,\beta}}{\int_0^t \int_0^t R_{v,u}^2 \, \mathrm{d}\omega_u^\beta \mathrm{d}\omega_v^\alpha}.$$

As a final remark, we derive an easier expression for the process R (or, equivalently, for the process Q) appearing in the expression of the MLE in the linear case. In this case we have

$$(5.8) \qquad R_{t,s} = \frac{\mathrm{d}}{\mathrm{d}\omega_t^\alpha} \frac{\mathrm{d}}{\mathrm{d}\omega_s^\beta} \int_0^t \int_0^s k_\alpha(t,v) k_\beta(s,u) X_{v,u} \, \mathrm{d}u\mathrm{d}v.$$

We need first a more suitable expression of the process R given by (5.8).

5.9 Proposition. *For every* α, β *it holds that*

$$R_{t,s} = \frac{\lambda^*_\alpha}{2}\frac{\lambda^*_\beta}{2}\int_0^t\int_0^s \left(t^{2\alpha-1}+v^{2\alpha-1}\right)\left(s^{2\beta-1}+u^{2\beta-1}\right) \mathrm{d}Z_{u,v} \tag{5.10}$$

with $\lambda^*_\alpha = \frac{\lambda^\alpha}{2(1-\alpha)}$.

Proof. We will restrict ourselves to the case $\alpha, \beta \leq \frac{1}{2}$. By (5.4) and (5.8) it holds that

$$\begin{aligned}\int_0^t\int_0^s R_{v,u}\,\mathrm{d}\omega^\beta_u\,\mathrm{d}\omega^\alpha_v &= \int_0^t\int_0^s k_\alpha(t,v)k_\beta(s,u)\left(\int_0^v\int_0^u K_\alpha(v,b)K_\beta(u,a)\,\mathrm{d}Z_{a,b}\right)\mathrm{d}u\mathrm{d}v\\ &= \int_0^t\int_0^s A_\alpha(b,t)A_\beta(b,s)\,\mathrm{d}Z_{a,b}\end{aligned}$$

where

$$A_\alpha(b,t) = \int_b^t k_\alpha(t,v)K_\alpha(v,b)\,\mathrm{d}v.$$

Let us suppose a function $\Phi_\alpha(b,r)$ such that for every $b<t$,

$$\int_b^t \Phi_\alpha(b,r)\,\mathrm{d}\omega^\alpha_r = A_\alpha(b,t).$$

Then it holds that

$$\begin{aligned}\int_0^t\int_0^s R_{v,u}\,\mathrm{d}\omega^\beta_u\mathrm{d}\omega^\alpha_v &= \int_0^t\int_0^s\left(\int_b^t \Phi_\alpha(b,r)\,\mathrm{d}\omega^\alpha_r\right)\left(\int_a^s \Phi_\beta(a,r')\,\mathrm{d}\omega^\beta_{r'}\right)\mathrm{d}Z_{a,b}\\ &= \int_0^t\int_0^s\left(\int_0^r\int_0^{r'} \Phi_\alpha(b,r)\Phi_\beta(a,r')\,\mathrm{d}Z_{a,b}\right)\mathrm{d}\omega^\beta_{r'}\mathrm{d}\omega^\alpha_r.\end{aligned}$$

As a consequence

$$R_{t,s} = \int_0^t\int_0^s \Phi_\alpha(t,v)\Phi_\beta(s,u)\,\mathrm{d}Z_{u,v}.$$

On the other hand it has been proved in Lemma 3.1. of [7] that

$$\Phi_\alpha(t,v) = \frac{\lambda^*}{2}\left(t^{2\alpha-1}+v^{2\alpha-1}\right).$$

The conclusion follows easily. □

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