SIMULATION OF WEAKLY SELF-SIMILAR STATIONARY INCREMENT $\mathrm{Sub}_\varphi(\Omega)$-PROCESSES: A SERIES EXPANSION APPROACH

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ABSTRACT. We consider simulation of $\mathrm{Sub}_\varphi(\Omega)$-processes that are weakly self-similar with stationary increments in the sense that they have the covariance function

$$R(t,s) \;=\; \frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right)$$

for some $H\in(0,1)$. This means that the second order structure of the processes is that of the fractional Brownian motion. Also, if $H>\frac{1}{2}$ then the process is long-range dependent.

The simulation is based on a series expansion of the fractional Brownian motion due to Dzhaparidze and van Zanten. We prove an estimate of the accuracy of the simulation in the space $C([0,1])$ of continuous functions equipped with the usual sup-norm. The result holds also for the fractional Brownian motion which may be considered as a special case of a $\mathrm{Sub}_{x^2/2}(\Omega)$-process.

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1. Introduction

We consider simulation of centred second order processes defined on the interval $[0,1]$ whose covariance function is

$$R(t,s) \;=\; \frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right),$$

and belong to the space $\mathrm{Sub}_\varphi(\Omega)$. This space is defined later in Section 2. The parameter H takes values in the interval $(0,1)$ the other cases being either uninteresting or impossible.

The motivation to study processes with the second order structure given by R comes from the notions of statistical self-similarity and long-range dependence. A stationary square integrable process is *long-range dependent* if its autocorrelation function is not summable. A process Z is *self-similar* with index H if it satisfies the scaling property

$$(Z_t)_{t\geq 0} \;\stackrel{d}{=}\; \left(a^{-H}Z_{at}\right)_{t\geq 0}$$

for all $a>0$. Here d means equality in distributions. The self-similarity parameter $H\in(0,1)$, or *Hurst index*, has also the following role. If $H\neq\frac{1}{2}$ then Z is a process with dependent increments. There are $\frac{1}{2}$-self-similar processes with independent increments, but these are processes with no variance. If

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$H > \frac{1}{2}$ then the increments of the process Z are long-range dependent. The case $H < \frac{1}{2}$ corresponds to short-range dependence. These properties, self-similarity and long-range dependence, have been shown to be charateristic in e.g. teletraffic and financial time series. See [1, 3, 6, 12] for references to these studies and for self-similarity and long-range dependence in general.

Assume now that a process Z is H-self-similar, has stationary increments, and is centred and square integrable. Then it is easy to see that Z has R as the covariance function. So, if a process has the covariance function R we say that it is *weakly self-similar with stationary increments*, or second order self-similar with stationary increments. In the Gaussian case the properties of the weak self-similarity and the proper one coincide. In this case Z is called the fractional Brownian motion, and, in particular, the Brownian motion if $H = \frac{1}{2}$. The fractional Brownian motion was originally defined and studied by Kolmogorov [8] under the name "Wiener helix". The name "fractional Brownian motion" comes from Mandelbrot and Van Ness [11].

Recently Dzhaparidze and van Zanten [5] proved a series representation for the fractional Brownian motion B:

$$B_t = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n. \tag{1.1}$$

Here the X_n's and the Y_n's are independent zero mean Gaussian random variables with certain variances depending on H and n. The x_n's are the positive real zeros of the Bessel function J_{-H} of the first kind and the y_n's are the positive real zeros of the Bessel function J_{1-H}. The series in (1.1) converge in mean square as well as uniformly on $[0,1]$ with probability 1. Details of representation (1.1) are recalled later in Section 3.

In this paper we study the use of the expansion (1.1) in simulating processes with the covariance function R. In particular, we study processes of the form (1.1) where the X_n's and Y_n's are replaced by independent random variables from the space $\mathrm{Sub}_\varphi(\Omega)$. The fractional Brownian motion may be obtained as a special case with $\varphi(x) = x^2/2$.

Let us end this introduction by saying a few words of the pros and cons of using the series expansion (1.1). The Hurst parameter H is roughly the Hölder index of the process. This means that, especially in the case of small H, the sample paths of the process are very erractic. However, the coefficient functions in (1.1) are smooth. So, in order to have good accuracy in simulation one needs a large truncation point in the expansion. This is the bad news. The good news is that once the coefficient functions are calculated we are in now way restricted to any pregiven time grid. Indeed, unlike in some traditional simulation methods, to calculate the value of the sample path in a new time point one does not have to condition on the already calculated time points. The computational effort in adding a new time point is always constant.

2. Space $\mathrm{Sub}_\varphi(\Omega)$

We recall some basic facts about the space $\mathrm{Sub}_\varphi(\Omega)$ of *φ-sub-Gaussian* (or *generalised sub-Gaussian*) random variables. For details and proofs we refer to Buldygin and Kozachenko [2] and Krasnoselskii and Rutitskii [10].

Definition 2.1. [10] A continuous even convex function u is an *Orlicz N-function* if it is strictly increasing for $x > 0$, $u(0) = 0$,

$$\frac{u(x)}{x} \to 0 \quad \text{as} \quad x \to 0 \qquad \text{and} \qquad \frac{u(x)}{x} \to \infty \quad \text{as} \quad x \to \infty.$$

Proposition 2.2. [10] *The function u is an Orlicz N-function if and only if*

$$u(x) = \int_0^{|x|} l(u)\,\mathrm{d}u, \quad x \in \mathbb{R},$$

where the density function *l is nondecreasing, right continuous, $l(u) > 0$ as $u > 0$, $l(0) = 0$ and $l(u) \to \infty$ as $u \to \infty$.*

Definition 2.3. Let u be an Orlicz N-function. The even function u^* defined by the formula

$$u^*(x) \;=\; \sup_{y>0} \big(xy - u(y)\big)$$

is the *Young-Fenchel transformation* of the function u.

Proposition 2.4. [10] *The function u^* is an Orlicz N-function and for $x > 0$*

$$u^*(x) \;=\; xy_0 - u(y_0) \qquad \textit{if} \quad y_0 = l^{-1}(x).$$

Here l^{-1} is the generalised inverse function *of l, i.e.*

$$l^{-1}(x) \;:=\; \sup\{v \geq 0 : l(v) \leq x\}.$$

Definition 2.5. The *assumption Q* holds for an Orlicz N-function u if it is quadratic around the origin, i.e. there exist such constants $x_0 > 0$ and $C > 0$ that $\varphi(x) = Cx^2$ for $|x| \leq x_0$.

Example 2.6. The assumption Q holds for the following Orlicz N-functions

$$\varphi(x) \;=\; \begin{cases} \frac{|x|^p}{p} & \text{if} \quad |x| > 1, \\ \frac{x^2}{p} & \text{if} \quad |x| \leq 1, \end{cases} \qquad p > 1;$$

$$\varphi(x) \;=\; \begin{cases} \left(\frac{e\alpha}{2}\right)^{\frac{2}{\alpha}} x^2 & \text{if} \quad |x| \leq \left(\frac{2}{\alpha}\right)^{1/\alpha}, \\ \exp\{|x|^\alpha\} & \text{if} \quad |x| > \left(\frac{2}{\alpha}\right)^{1/\alpha}, \end{cases} \qquad 0 < \alpha < 1.$$

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a standard probability space.

Definition 2.7. Let φ be an Orlicz N-function satisfying the assumption Q. A zero mean random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, the space of φ-*sub-Gaussian random variables*, if there exists a positive constant a such that the inequality

$$\mathbf{E} \exp\{\lambda\xi\} \;\leq\; \exp\{\varphi(a\lambda)\}$$

holds for all $\lambda \in \mathbb{R}$.

A stochastic process $X = (X_t)_{t\in[0,1]}$ is a $\mathrm{Sub}_\varphi(\Omega)$*-process* if it is a bounded family of $\mathrm{Sub}_\varphi(\Omega)$-processes: $X_t \in \mathrm{Sub}_\varphi(\Omega)$ for all $t \in [0,1]$ and

$$\sup_{t\in[0,1]} \tau_\varphi(X_t) \;<\; \infty.$$

Remark 2.8. Note that like the Gaussian variables the φ-sub-Gaussian random variables also have light tails. In particular, they have moments of all orders.

Proposition 2.9. [2] *The space* $\mathrm{Sub}_\varphi(\Omega)$ *is a Banach space with respect to the norm*

$$\tau_\varphi(\xi) \;=\; \inf\Big\{a \geq 0 \;:\; \mathbf{E}\exp\{\lambda\xi\} \,\leq\, \exp\big\{\varphi(a\lambda)\big\},\ \lambda \in \mathbb{R}\Big\}.$$

Moreover, for any $\lambda \in \mathbb{R}$ *we have*

$$\mathbf{E}\exp\{\lambda\xi\} \;\leq\; \exp\big\{\varphi\big(\lambda\tau_\varphi(\xi)\big)\big\},$$

$$\big(\mathbf{E}\xi^2\big)^{\frac{1}{2}} \;\leq\; (2C)^{\frac{1}{2}}\tau_\varphi(\xi),$$

where C *is the constant from the assumption Q.*

The properties of random variables from the spaces $\mathrm{Sub}_\varphi(\Omega)$ were studied in the book [2].

Remark 2.10. When $\varphi(x) = \frac{x^2}{2}$ the space $\mathrm{Sub}_\varphi(\Omega)$ is called the space of *sub-Gaussian* random variables and is denoted by $\mathrm{Sub}(\Omega)$. Centred Gaussian random variables ξ belong to the space $\mathrm{Sub}(\Omega)$, and in this case $\tau_\varphi(\xi)$ is just the standard deviation: $(\mathbf{E}\xi^2)^{1/2}$. Also, if ξ is bounded, i.e. $|\xi| \leq c$ a.s. then $\xi \in \mathrm{Sub}(\Omega)$ and $\tau_\varphi(\xi) \leq c$.

Proposition 2.11. *Let* φ *be an Orlicz N-function satisfying the assumption Q. Assume further that the function* $\varphi(\sqrt{\cdot})$ *is convex. Let* $\xi_1, \xi_2, \ldots, \xi_n$ *be independent random variables from the space* $\mathrm{Sub}_\varphi(\Omega)$. *Then*

$$\tau_\varphi^2\left(\sum_{i=1}^n \xi_i\right) \;\leq\; \sum_{i=1}^n \tau_\varphi^2(\xi_i).$$

3. Series representation

Let us now recall the Dzhaparidze–van Zanten series representation (1.1) in detail. Let J_ν be the Bessel function of the first kind of order ν, i.e.

$$J_\nu(x) \;=\; \sum_{n=0}^\infty \frac{(-1)^n (x/2)^{\nu+2n}}{\Gamma(n+1)\Gamma(\nu+n+1)}.$$

Here $x > 0$, $\nu \neq -1, -2, \ldots$ and Γ denotes the Euler Gamma function

$$\Gamma(z) \;=\; \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t.$$

It is well-known that for $\nu > -1$ the Bessel function J_ν has countable number of real positive zeros tending to infinity. Denote by x_n the nth positive real zero of the Bessel function J_{-H}; y_n is the nth positive real zero of J_{1-H}.

Let B be the fractional Brownian motion with index H. Then it may be represented as the mean square convergent series

$$B_t \;=\; \sum_{n=1}^\infty c_n \sin(x_n t)\,\tilde{X}_n \;+\; \sum_{n=1}^\infty d_n\big(1-\cos(y_n t)\big)\,\tilde{Y}_n.$$

Here $\tilde{X}_n,\ \tilde{Y}_n,\ n = 1, 2, \ldots,$ are independent zero mean Gaussian random variables with $\mathbf{E}\tilde{X}_n^2 = \mathbf{E}\tilde{Y}_n^2 = 1$ and

$$c_n = \frac{\sqrt{2c}}{x_n^{H+1}\pi^H J_{1-H}(x_n)}, \qquad n = 1, 2, \ldots, \tag{3.1}$$

$$d_n = \frac{\sqrt{2c}}{y_n^{H+1}\pi^H J_{-H}(y_n)}, \qquad n = 1, 2, \ldots, \tag{3.2}$$

$$c = \frac{\Gamma(2H+1)\sin(\pi H)}{\pi^{2H+1}}. \tag{3.3}$$

We shall generalise the setting above in the following way: Define a process $Z = (Z_t)_{t\in[0,1]}$ by the expansion

$$Z_t = \sum_{n=1}^{\infty} c_n \sin(x_n t)\, \xi_n + \sum_{n=1}^{\infty} d_n \big(1 - \cos(y_n t)\big)\, \eta_n, \tag{3.4}$$

where c_n and d_n are given by (3.1) and (3.2), $\xi_n,\ \eta_n,\ n = 1, 2, \ldots,$ are independent identically distributed centred random variables from the space $\mathrm{Sub}_\varphi(\Omega)$ with $\mathbf{E}\xi_n^2 = \mathbf{E}\eta_n^2 = 1,\ n = 1, 2, \ldots.$ Furthermore, we shall assume that the function $\varphi(\sqrt{\cdot})$ is convex.

Since φ-sub-Gaussian random variables are square integrable we have the following.

Proposition 3.1. *The series (3.4) converge in mean square and the covariance function of the process Z is R.*

In addition to the L^2-convergence the spaces $\mathrm{Sub}_\varphi(\Omega)$ are nice enough to allow uniform ω-by-ω convergence.

Theorem 3.2. *The series (3.4) converge uniformly with probability one and the process Z is almost surely continuous on $[0, 1]$. Moreover, if Z is strongly self-similar with stationary increments then it is β-Hölder continuous with any index $\beta < H$.*

The continuity in Theorem 3.2 follows by using the Hunt's theorem [7]. The Hölder continuity comes from the Kolmogorov's criterion. Let us also note that from the case of fractional Brownian motion we know that in general we cannot have Hölder continuity with index $\beta = H$, cf. [4].

For the reader's convenience we now recite a modification of the Hunt's theorem as a lemma (cf. [2], Example 3.5.2).

Lemma 3.3. *Suppose that $(\xi_n)_{n\geq 1}$ is a sequence of independent centred random variables with $\mathbf{E}\xi_n^2 = 1, n = 1, 2, \ldots.$ Let $(\lambda_n)_{n\geq 1}$ be a sequence such that $\lambda_n \leq \lambda_{n+1}$ and $\lambda_n \to \infty$ as $n \to \infty$.*

If

$$\sum_{n=1}^{\infty} a_n^2 \left(\ln(1+\lambda_n)\right)^{1+\beta} < \infty$$

for some $\beta > 0$ *then the series*

$$\sum_{n=1}^{\infty} a_n \cos(\lambda_n t)\, \xi_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n \sin(\lambda_n t)\, \xi_n$$

converge uniformly on $[0,1]$ *with probability one.*

Proof of Theorem 3.2. Let us consider the almost sure uniform convergence first. Now, from Watson [13], p. 506, we have$x_n \sim y_n \sim \pi n$ as $n \to \infty$. Also from [13], p. 200, we have the following asymptotic relation for the Bessel function J_ν for $\nu > -1$:

$$J_\nu^2(x) + J_{\nu+1}^2(x) \ \sim \ \frac{2}{\pi x}$$

for large $|x|$. Since the zeros x_n of J_ν tend to infinity this yields

$$J_{1+\nu}^2(x_n) \ \sim \ \frac{2}{\pi x_n}$$

as $n \to \infty$. Therefore,

$$c_n^2 \ \sim \ \frac{c}{n^{2H+1}} \qquad \text{and} \qquad d_n^2 \ \sim \ \frac{c}{n^{2H+1}} \tag{3.5}$$

(see (3.1)–(3.3)). Consequently, the series

$$\sum_{n=1}^{\infty} c_n^2 (\ln(1+x_n))^{1+\varepsilon} \qquad \text{and} \qquad \sum_{n=1}^{\infty} d_n^2 (\ln(1+y_n))^{1+\varepsilon}$$

converge for all $\varepsilon > 0$. The almost sure uniform convergence and he continuity of the process follow now from the Hunt's theorem (Lemma 3.3).

To see the Hölder continuity of Z just use strong self-similarity and the stationarity of the increments together with the fact that Z has all moments. Indeed, for all $n \in \mathbb{N}$ we have

$$\mathbf{E}\big|Z_t - Z_s\big|^n \ = \ \mathbf{E}\big|Z_{t-s}\big|^n \ = \ |t-s|^{Hn}\mathbf{E}|Z_1|^n,$$

and the claim follows from the Kolmogorov's criterion. □

4. Simulation, accuracy and reliability

We want to construct a model $\tilde{Z}$ of the process Z, such that $\tilde{Z}$ approximates Z with given reliability and accuracy in the norm of some Banach space. In this paper we consider the space $C([0,1])$ equipped with the usual sup-norm.

Definition 4.1. The model $\tilde{Z}$ approximates the process Z with given *reliability* $1-\nu$, $0 < \nu < 1$, and *accuracy* $\delta > 0$ in $C([0,1])$ if

$$\mathbf{P}\left(\sup_{t\in[0,1]} |Z_t - \tilde{Z}_t| > \delta \right) \ \le \ \nu.$$

A natural model for Z, defined by the expansion (3.4), would be the truncated series

$$\sum_{n=1}^{N} \left(c_n \sin(x_n t)\,\xi_n \;+\; d_n\big(1-\cos(y_n t)\big)\,\eta_n\right).$$

However, it is realistic to assume that the constants c_n and d_n and the zeros x_n, y_n are only calculated approximately, especially since there are fast-to-compute asymptotic formulas for the zeros x_n and y_n (cf. Watson [13], p. 506). Note that the constants c_n and d_n depend on the zeros.

Let $\tilde{c}_n$ and $\tilde{d}_n$ be the approximated values of the c_n and d_n, respectively. Let

$$\begin{aligned} |\tilde{c}_n - c_n| &\le \gamma_n^c, \\ |\tilde{d}_n - d_n| &\le \gamma_n^d, \end{aligned}$$

$n = 1, \ldots, N$. The errors γ_n^c and γ_n^d are assumed to be known. Let $\tilde{x}_n$ and $\tilde{y}_n$ be approximations of the corresponding zeros x_n and y_n with error bounds

$$\begin{aligned} |\tilde{x}_n - x_n| &\le \gamma_n^x, \\ |\tilde{y}_n - y_n| &\le \gamma_n^y. \end{aligned}$$

The error bounds γ_n^x and γ_n^y are also assumed to be known.

Then, the model of the process Z is

$$\tilde{Z}_t \;=\; \sum_{n=1}^{N} \Big(\tilde{c}_n \sin(\tilde{x}_n t)\,\xi_n \;+\; \tilde{d}_n\big(1-\cos(\tilde{y}_n t)\big)\,\eta_n\Big). \tag{4.1}$$

The error in the simulation (model) is

$$\begin{aligned} \Delta_t \;&:=\; Z_t - \tilde{Z}_t \\ &= \sum_{n=1}^{N} \Big\{ \Big(c_n \sin(x_n t) - \tilde{c}_n \sin(\tilde{x}_n t)\Big)\,\xi_n \\ &\qquad\qquad + \Big(d_n\big(1-\cos(y_n t)\big) - \tilde{d}_n\big(1-\cos(\tilde{y}_n t)\big)\Big)\,\eta_n \Big\} \\ &\quad + \sum_{n=N+1}^{\infty} \Big\{ c_n \sin(x_n t)\,\xi_n \;+\; d_n\big(1-\cos(y_n t)\big)\eta_n \Big\} \\ &=:\; \Delta_t^{\text{appr}} + \Delta_t^{\text{cut}}. \end{aligned}$$

In order to bound the error Δ in $C([0,1])$ we need estimates for $\tau_\varphi(\Delta_t)$ and $\tau_\varphi(\Delta_t - \Delta_s)$ for all $s, t \in [0,1]$. The estimates are given in the following proposition.

Proposition 4.2. *Denote* $a_\varphi := \tau_\varphi(\xi_n) = \tau_\varphi(\eta_n)$ *and*

$$\begin{aligned}
\gamma^{\mathrm{cut}} &:= a_\varphi^2 \sum_{n=N+1}^{\infty} \Big(c_n^2 + 4d_n^2\Big), \\
\gamma^{\mathrm{appr}} &:= a_\varphi^2 \sum_{n=1}^{N} \left\{ \Big(c_n \gamma_n^x + \gamma_n^c\Big)^2 + \Big(d_n \gamma_n^y + 2\gamma_n^d\Big)^2 \right\}.
\end{aligned}$$

Let $\alpha \in (0, H)$ *and denote*

$$\begin{aligned}
\gamma_\alpha^{\mathrm{cut}} &:= 2^{2-2\alpha} a_\varphi^2 \sum_{n=N+1}^{\infty} \left(c_n^2 x_n^{2\alpha} + d_n^2 y_n^{2\alpha}\right), \\
\gamma_\alpha^{\mathrm{appr}} &:= 2^{3-2\alpha} a_\varphi^2 \sum_{n=1}^{N} \left(c_n^2 \alpha^2 (\gamma_n^x)^2 + \tilde{x}_n^{2\alpha} (\gamma_n^c)^2 + d_n^2 \alpha^2 (\gamma_n^y)^2 + \tilde{y}_n^{2\alpha} (\gamma_n^d)^2\right).
\end{aligned}$$

Then we have for all $s, t \in [0, 1]$

$$\tau_\varphi^2(\Delta_t) \leq \gamma^{\mathrm{appr}} + \gamma^{\mathrm{cut}}, \tag{4.2}$$

$$\tau_\varphi^2(\Delta_t - \Delta_s) \leq \left(\gamma_\alpha^{\mathrm{appr}} + \gamma_\alpha^{\mathrm{cut}}\right) |t - s|^{2\alpha}. \tag{4.3}$$

Proof. By Proposition 2.11 we know that

$$\tau_\varphi^2(\Delta_t) \leq \tau_\varphi^2(\Delta_t^{\mathrm{appr}}) + \tau_\varphi^2(\Delta_t^{\mathrm{cut}}).$$

For $\tau_\varphi^2(\Delta_t^{\mathrm{appr}})$ we obtain by using Proposition 2.11 and the mean value theorem that

$$\begin{aligned}
\tau_\varphi^2(\Delta_t^{\mathrm{appr}}) &\le \sum_{n=1}^{N}\Big(c_n\sin(x_nt)-\tilde{c}_n(\sin\tilde{x}_nt)\Big)^2\tau_\varphi^2(\xi_n)\\
&\quad+\sum_{n=1}^{N}\Big(d_n\big(1-\cos(y_nt)\big)-\tilde{d}_n\big(1-\cos(\tilde{y}_nt)\big)\Big)^2\tau_\varphi^2(\eta_n)\\
&\le a_\varphi^2\Bigg\{\sum_{n=1}^{N}\Big(c_n\big(\sin(x_nt)-\sin(\tilde{x}_nt)\big)+(c_n-\tilde{c}_n)\sin(\tilde{x}_nt)\Big)^2\\
&\qquad+\sum_{n=1}^{N}\Big(d_n\big(\cos(\tilde{y}_nt)-\cos(y_nt)\big)+(d_n-\tilde{d}_n)\big(1-\cos(\tilde{y}_nt)\big)\Big)^2\Bigg\}\\
&\le a_\varphi^2\Bigg\{\sum_{n=1}^{N}\Big(c_n|x_n-\tilde{x}_n|t+(c_n-\tilde{c}_n)\sin(\tilde{x}_nt)\Big)^2\\
&\qquad+\sum_{n=1}^{N}\Big(d_n|\tilde{y}_n-y_n|t+(d_n-\tilde{d}_n)\big(1-\cos(\tilde{y}_nt)\big)\Big)^2\Bigg\}\\
&\le a_\varphi^2\sum_{n=1}^{N}\Bigg\{\Big(c_n\gamma_n^x+\gamma_n^c\Big)^2+\Big(d_n\gamma_n^y+2\gamma_n^d\Big)^2\Bigg\}.\\
&= \gamma^{\mathrm{appr}}.
\end{aligned}$$

Similarly we obtain

$$\tau_\varphi^2(\Delta_t^{\mathrm{cut}}) \le a_\varphi^2\sum_{n=N+1}^{\infty}\Big(c_n^2+4d_n^2\Big) = \gamma^{\mathrm{cut}}. \tag{4.4}$$

Recall the asymptotics of c_n^2 and d_n^2 (3.5) to see that the series (4.4) above converges. The estimate (4.2) follows.

Now we shall estimate the incremental error $\tau_\varphi^2(\Delta_t - \Delta_s)$. For the "cut-off" part we have

$$\begin{aligned}
\tau_\varphi^2(\Delta_t^{\text{cut}} - \Delta_s^{\text{cut}}) &= \tau_\varphi^2\Bigg(\sum_{n=N+1}^{\infty} c_n\big(\sin(x_n t) - \sin(x_n s)\big)\xi_n \\
&\qquad + \sum_{n=N+1}^{\infty} d_n\big(\cos(y_n s) - \cos(y_n t)\big)\eta_n\Bigg) \\
&\le 2^{2(1-\alpha)} a_\varphi^2 \sum_{n=N+1}^{\infty}\Big(c_n^2(x_n|t-s|)^{2\alpha} + d_n^2(y_n|t-s|)^{2\alpha}\Big) \\
&= 2^{2(1-\alpha)} a_\varphi^2 \sum_{n=N+1}^{\infty}\Big(c_n^2 x_n^{2\alpha} + d_n^2 y_n^{2\alpha}\Big)|t-s|^{2\alpha} \qquad (4.5) \\
&= \gamma_\alpha^{\text{cut}}|t-s|^{2\alpha}.
\end{aligned}$$

Due to the asymptotics (3.5) and $x_n \sim y_n \sim \pi n$ the series in (4.5) converge if $\alpha < H$.

For the "approximating part" we have

$$\begin{aligned}
&\tau_\varphi^2(\Delta_t^{\text{appr}} - \Delta_s^{\text{appr}}) \\
&\quad\le a_\varphi^2\Bigg\{\sum_{n=1}^{N}\Big(c_n\big(\sin(x_n t) - \sin(x_n s)\big) - \tilde{c}_n\big(\sin(\tilde{x}_n t) - \sin(\tilde{x}_n s)\big)\Big)^2 \\
&\qquad + \sum_{n=1}^{N}\Big(d_n\big(\cos(y_n s) - \cos(y_n t)\big) - \tilde{d}_n\big(\cos(\tilde{y}_n s) - \cos(\tilde{y}_n t)\big)\Big)^2\Bigg\} \\
&\quad\le 2^{2(1-\alpha)} a_\varphi^2 \sum_{n=1}^{N}\Bigg(\Big(c_n x_n^\alpha|t-s|^\alpha - \tilde{c}_n\tilde{x}_n^\alpha|t-s|^\alpha\Big)^2 \\
&\qquad + \Big(d_n y_n^\alpha|t-s|^\alpha - \tilde{d}_n\tilde{y}_n^\alpha|t-s|^\alpha\Big)^2\Bigg) \\
&\quad= 2^{2-2\alpha} a_\varphi^2 \sum_{n=1}^{N}\Big((c_n x_n^\alpha - \tilde{c}_n\tilde{x}_n^\alpha)^2 + (d_n y_n^\alpha - \tilde{d}_n\tilde{y}_n^\alpha)^2\Big)|t-s|^{2\alpha}. \qquad (4.6)
\end{aligned}$$

For the summand in (4.6) we have

$$\begin{aligned}
&(c_n x_n^\alpha - \tilde{c}_n \tilde{x}_n^\alpha)^2 + (d_n y_n^\alpha - \tilde{d}_n \tilde{y}_n^\alpha)^2 \\
&\quad = (c_n x_n^\alpha - c_n \tilde{x}_n^\alpha + c_n \tilde{x}_n^\alpha - \tilde{c}_n \tilde{x}_n^\alpha)^2 + (d_n y_n^\alpha - d_n \tilde{y}_n^\alpha + d_n \tilde{y}_n^\alpha - \tilde{d}_n \tilde{y}_n^\alpha)^2 \\
&\quad = (c_n (x_n^\alpha - \tilde{x}_n^\alpha) + (c_n - \tilde{c}_n)\tilde{x}_n^\alpha)^2 + (d_n (y_n^\alpha - \tilde{y}_n^\alpha) + (d_n - \tilde{d}_n)\tilde{y}_n^\alpha)^2 \\
&\quad \le 2c_n^2 (x_n^\alpha - \tilde{x}_n^\alpha)^2 + 2(c_n - \tilde{c}_n)^2 \tilde{x}_n^{2\alpha} + 2d_n^2 (y_n^\alpha - \tilde{y}_n^\alpha)^2 + 2(d_n - \tilde{d}_n)^2 \tilde{y}_n^{2\alpha} \\
&\quad = c_n^2 (x_n^\alpha - \tilde{x}_n^\alpha)^2 + (\gamma_n^c)^2 \tilde{x}_n^{2\alpha} + d_n^2 (y_n^\alpha - \tilde{y}_n^\alpha)^2 + (\gamma_n^d)^2 \tilde{y}_n^{2\alpha} \\
&\quad \le c_n^2 \alpha^2 (\gamma_n^x)^2 + \tilde{x}_n^{2\alpha} (\gamma_n^c)^2 + d_n^2 \alpha^2 (\gamma_n^y)^2 + \tilde{y}_n^{2\alpha} (\gamma_n^d)^2 \qquad (4.7) \\
&\quad = \gamma_\alpha^{\mathrm{appr}} \cdot 2^{2\alpha - 2} a_\varphi^{-2}.
\end{aligned}$$

In (4.7) we used the mean value theorem together with the fact that the y_n's and x_n's are bigger than one.

Estimate (4.3) follows now by collecting the estimates above and by using Proposition 2.11. □

Now we are ready to state, although not yet to prove, our main result.

Theorem 4.3. *Let b and α be such that $0 < b < \alpha < H$. Let γ^{appr}, $\gamma_\alpha^{\mathrm{appr}}$, γ^{cut} and $\gamma_\alpha^{\mathrm{cut}}$ be as in Proposition 4.2. Denote*

$$\begin{aligned}
\gamma_0 &= \sqrt{\gamma^{\mathrm{appr}} + \gamma^{\mathrm{cut}}}, \\
\gamma_\alpha &= \sqrt{\gamma_\alpha^{\mathrm{appr}} + \gamma_\alpha^{\mathrm{cut}}}, \\
\beta &= \min\left\{\gamma_0, \frac{\gamma_\alpha}{2^\alpha}\right\}.
\end{aligned}$$

Let l be the density of φ.

The model $\tilde{Z}$, defined by (4.1), approximates the separable process Z, defined by (3.4), with given reliability $1-\nu$, $0 < \nu < 1$, and accuracy $\delta > 0$ in $C([0,1])$ if the following three inequalities are satisfied:

$$\gamma_0 < \delta, \tag{4.8}$$

$$\frac{\beta\gamma_0}{\gamma_\alpha} < \frac{\delta}{2^\alpha(\exp\{\varphi(1)\} - 1)^\alpha}, \tag{4.9}$$

$$2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0} - 1\right)\right\}\left(\frac{\left(\frac{\gamma_\alpha\delta}{\beta\gamma_0}\right)^{\frac{1}{\alpha}}}{2\left(1 - \frac{b}{\alpha}\right)^{\frac{1}{b}}} l^{-1}\left(\frac{\delta}{\gamma_0} - 1\right)^{\frac{1}{b}} + 1\right)^2 \le \nu. \tag{4.10}$$

The following lemma is our main tool for proving Theorem 4.3. For the proof of it we refer to Kozachenko and Vasylyk [9], Lemma 3.3.

Lemma 4.4. *Let $X = (X_t)_{t\in[0,1]}$ be a separable random process from the space* $\mathrm{Sub}_\varphi(\Omega)$. *Let $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing continuous function such that $\sigma(h) \to 0$ as $h \to 0$ and*

$$\sup_{|t-s|\le h} \tau_\varphi(X_t - X_s) \;\le\; \sigma(h).$$

Denote $\gamma_0 = \sup_{t\in[0,1]} \tau_\varphi(X_t)$ and let β be such a number that $\beta \le \sigma(\frac{1}{2})$. Let $r : [1,\infty) \to \mathbb{R}_+$ be a nondecreasing continuous function such that $r(1) = 0$ and the mapping $u \mapsto r(e^u)$ is convex. Suppose that

$$\int_0^\beta \theta(u)\,\mathrm{d}u \;<\; \infty,$$

where

$$\theta(u) \;=\; \theta(\varphi,\sigma,r;u) \;=\; \frac{r(N(\sigma^{-1}(u)))}{\varphi^{-1}(\ln N(\sigma^{-1}(u)))},$$

and $N(\varepsilon)$ is the minimum number of closed intervals of the radius ε that is needed to cover the interval $[0,1]$ (note that $N(\varepsilon) \le \frac{1}{2\varepsilon} + 1$).

Then for all $\lambda \in \mathbb{R}$ and $p \in (0,1)$ we have

$$\mathbf{E}\exp\left\{\lambda \sup_{t\in[0,1]} |X_t|\right\} \;\le\; 2\exp\left\{\varphi\left(\frac{\lambda\gamma_0}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{1-p}\right)p\right\} \times$$
$$r^{-1}\left(\lambda\gamma_0\theta(p\beta) + \frac{\lambda}{(1-p)p}\int_0^{\beta p^2} \theta(u)\,\mathrm{d}u\right)^2. \quad (4.11)$$

Let us now reformulate Lemma 4.4 above for our case.

Lemma 4.5. *Let α, β, γ_0 and γ_α be as in Theorem 4.3, and let r and θ be as in Lemma 4.4. Then for all $\lambda \in \mathbb{R}$ and $p \in (0,1)$ we have*

$$\mathbf{P}\left(\sup_{t\in[0,1]} |\Delta_t| > \delta\right) \;\le\; 2\exp\left\{-\lambda\delta + \varphi\left(\frac{\lambda\gamma_0}{1-p}\right)\right\} \times$$
$$r^{-1}\left(\frac{\gamma_0}{\beta}\frac{\lambda}{p(1-p)}\int_0^{\beta p} \theta(u)\,\mathrm{d}u\right)^2. \quad (4.12)$$

Proof. >From Proposition 4.2 it follows that for the error process Δ we may take

$$\gamma_0 \;=\; \sqrt{\gamma^{\mathrm{appr}} + \gamma^{\mathrm{cut}}}.$$

and

$$\sigma(h) \;=\; \gamma_\alpha \cdot h^\alpha \;=\; \sqrt{\gamma_\alpha^{\mathrm{appr}} + \gamma_\alpha^{\mathrm{cut}}} \cdot h^\alpha.$$

In the inequality (4.11) we put $\beta = \min\{\gamma_0, \frac{\gamma_\alpha}{2^\alpha}\}$. Since $\beta \le \gamma_0$ we have

$$\varphi\left(\frac{\lambda\gamma_0}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{1-p}\right)p \;\le\; \varphi\left(\frac{\lambda\gamma_0}{1-p}\right)$$

So, it follows from the Chebyshev inequality and from (4.11) that for any $\delta > 0$ we have

$$\mathbf{P}\left(\sup_{t\in[0,1]} |\Delta_t| > \delta\right) \;\le\; \exp\left\{-\lambda\delta + \varphi\left(\frac{\lambda\gamma_0}{1-p}\right)\right\}\cdot 2I_r^2,$$

where we have used the denotation

$$I_r \;=\; r^{-1}\left(\lambda\gamma_0\theta(p\beta) + \frac{\lambda}{(1-p)p}\int_0^{\beta p^2}\theta(u)\,\mathrm{d}u\right).$$

Since the function $t \mapsto r\left(e^{\varphi(t)}\right)$ is an Orlicz N-function $\frac{r\left(e^{\varphi(t)}\right)}{t}$ increases in $t \ge 0$ (cf. [10]). Therefore, $\psi(\varphi^{-1}(x)) = \frac{r(e^x)}{\varphi^{-1}(x)}$ increases in $x \ge 0$. Consequently, θ is a decreasing function. Thus,

$$\theta(p\beta) \;\le\; \frac{1}{\beta p(1-p)}\int_{\beta p^2}^{\beta p}\theta(u)\,\mathrm{d}u$$

and

$$\lambda\gamma_0\theta(p\beta) \le \frac{\lambda\gamma_0}{\beta p(1-p)}\int_{\beta p^2}^{\beta p}\theta(u)\,\mathrm{d}u.$$

Since $\frac{\gamma_0}{\beta} \ge 1$ we have

$$\lambda\gamma_0\theta(p\beta) + \frac{\lambda}{p(1-p)}\int_0^{\beta p^2}\theta(u)\,\mathrm{d}u \;\le\; \frac{\gamma_0}{\beta}\,\frac{\lambda}{p(1-p)}\int_0^{\beta p}\theta(u)\,\mathrm{d}u.$$

The claim follows now from Lemma 4.4. □

Theorem 4.3 follows now by using the Young–Fenchel transformation and then choosing suitable λ, p and r in the inequality (4.12).

Proof of Theorem 4.3. By Proposition 2.4 we know that $xy = \varphi(x)+\varphi^*(y)$ when $x = l^{-1}(y)$, where l^{-1} is the generalised inverse function of the density l of φ. Since

$$\lambda\delta - \varphi\left(\frac{\lambda\gamma_0}{1-p}\right) \;=\; \frac{\lambda\gamma_0}{1-p}\cdot\frac{\delta(1-p)}{\gamma_0} - \varphi\left(\frac{\lambda\gamma_0}{1-p}\right)$$

we have the equality

$$\lambda\delta - \varphi\left(\frac{\lambda\gamma_0}{1-p}\right) \;=\; \varphi^*\left(\frac{\delta(1-p)}{\gamma_0}\right)$$

when

$$\frac{\lambda\gamma_0}{1-p} \;=\; l^{-1}\left(\frac{\delta(1-p)}{\gamma_0}\right).$$

So, we choose the following λ :

$$\lambda \;=\; \frac{1-p}{\gamma_0}l^{-1}\left(\frac{\delta(1-p)}{\gamma_0}\right).$$

Setting this λ in the inequality (4.12) we obtain

$$
\begin{aligned}
\mathbf{P}\left(\sup_{t\in[0,1]} |\Delta_t| > \delta\right) &\leq 2\exp\left\{-\varphi^*\left(\frac{\delta(1-p)}{\gamma_0}\right)\right\} \times \\
& \quad r^{-1}\left(\frac{\gamma_0}{\beta}\frac{\lambda_0}{p(1-p)}\int_0^{\beta p}\theta(u)\,\mathrm{d}u\right)^2 \\
&= 2\exp\left\{-\varphi^*\left(\frac{\delta(1-p)}{\gamma_0}\right)\right\} \times \\
& \quad r^{-1}\left(\frac{\gamma_0}{\beta}\frac{(1-p)}{\gamma_0}l^{-1}\left(\frac{\delta(1-p)}{\gamma_0}\right)\frac{1}{p(1-p)}\int_0^{\beta p}\theta(u)\,\mathrm{d}u\right)^2 \\
&= 2\exp\left\{-\varphi^*\left(\frac{\delta(1-p)}{\gamma_0}\right)\right\} \times \\
& \quad r^{-1}\left(\frac{1}{\beta p}l^{-1}\left(\frac{\delta(1-p)}{\gamma_0}\right)\int_0^{\beta p}\theta(u)\,\mathrm{d}u\right)^2
\end{aligned}
$$

Let us now consider the integral term above. In our case we have

$$
\begin{aligned}
\int_0^{\beta p}\theta(u)\,\mathrm{d}u &= \int_0^{\beta p}\frac{r(N(\sigma^{-1}(u)))}{\varphi^{-1}(\ln N(\sigma^{-1}(u)))}\,\mathrm{d}u \\
&\leq \int_0^{\beta p}\frac{r\left(\frac{1}{2\sigma^{-1}(u)}+1\right)}{\varphi^{-1}\left(\ln\left(\frac{1}{2\sigma^{-1}(u)}+1\right)\right)}\,\mathrm{d}u \\
&= \int_0^{\beta p}\frac{r\left(\frac{1}{2}\left(\frac{\gamma_\alpha}{u}\right)^{\frac{1}{\alpha}}+1\right)}{\varphi^{-1}\left(\ln\left(\left(\frac{\gamma_\alpha}{u}\right)^{\frac{1}{\alpha}}+1\right)\right)}\,\mathrm{d}u.
\end{aligned}
$$

Now, if the denominator satisfies

$$
\varphi^{-1}\left(\ln\left(\frac{1}{2}\left(\frac{\gamma_\alpha}{u}\right)^{\frac{1}{\alpha}}+1\right)\right) \geq 1
$$

as $u \leq \beta p$, that is

$$
p \leq \frac{\gamma_\alpha}{\beta 2^\alpha\left(\exp\{\varphi(1)\}-1\right)^\alpha}, \tag{4.13}
$$

then we have

$$
\int_0^{\beta p}\theta(u)\,\mathrm{d}u \leq \int_0^{\beta p} r\left(\frac{1}{2}\left(\frac{\gamma_\alpha}{u}\right)^{\frac{1}{\alpha}}+1\right)\,\mathrm{d}u.
$$

Let us choose $r(x) = x^b - 1$, where $0 < b < \alpha$. Then, by using the estimate above and the fact that $(x+1)^b - x^b \le 1$, we obtain

$$\int_0^{\beta p} \theta(u)\,\mathrm{d}u \;\le\; \int_0^{\beta p} \left(\frac{1}{2}\left(\frac{\gamma_\alpha}{u}\right)^{\frac{1}{\alpha}}\right)^b \mathrm{d}u \;=\; \frac{\gamma_\alpha^{\frac{b}{\alpha}}}{2^b}\frac{(\beta p)^{1-\frac{b}{\alpha}}}{1-\frac{b}{\alpha}}.$$

Thus, we have obtained the estimate

$$\begin{aligned}
\mathbf{P}\left(\sup_{t\in[0,1]} |\Delta_t| > \delta\right) &\le 2\exp\left\{-\varphi^*\left(\frac{\delta(1-p)}{\gamma_0}\right)\right\} \times \\
&\quad r^{-1}\left(\frac{1}{\beta p} l^{-1}\left(\frac{\delta(1-p)}{\gamma_0}\right)\frac{\gamma_\alpha^{\frac{b}{\alpha}}}{2^b}\frac{(\beta p)^{1-\frac{b}{\alpha}}}{1-\frac{b}{\alpha}}\right)^2 \\
&= 2\exp\left\{-\varphi^*\left(\frac{\delta(1-p)}{\gamma_0}\right)\right\} \times \\
&\quad r^{-1}\left(\frac{\gamma_\alpha^{\frac{b}{\alpha}}(\beta p)^{-\frac{b}{\alpha}}}{2^b(1-\frac{b}{\alpha})} l^{-1}\left(\frac{\delta(1-p)}{\gamma_0}\right)\right)^2.
\end{aligned}$$

For p we choose

$$p = \frac{\gamma_0}{\delta} \tag{4.14}$$

(recall that $\gamma_0 < \delta$) and we obtain the inequality

$$\begin{aligned}
\mathbf{P}\left(\sup_{t\in[0,1]} |\Delta_t| > \delta\right) &\le 2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0}-1\right)\right\} \times \\
&\quad r^{-1}\left(\frac{\gamma_\alpha^{\frac{b}{\alpha}}(\beta\frac{\gamma_0}{\delta})^{-\frac{b}{\alpha}}}{2^b(1-\frac{b}{\alpha})} l^{-1}\left(\frac{\delta}{\gamma_0}-1\right)\right)^2 \\
&= 2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0}-1\right)\right\} \times \\
&\quad r^{-1}\left(\frac{1}{2^b(1-\frac{b}{\alpha})}\left(\frac{\gamma_\alpha\delta}{\beta\gamma_0}\right)^{\frac{b}{\alpha}} l^{-1}\left(\frac{\delta}{\gamma_0}-1\right)\right)^2 \\
&= 2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0}-1\right)\right\} \times \\
&\quad \left(\frac{1}{2(1-\frac{b}{\alpha})^{\frac{1}{b}}}\left(\frac{\gamma_\alpha\delta}{\beta\gamma_0}\right)^{\frac{1}{\alpha}} l^{-1}\left(\frac{\delta}{\gamma_0}-1\right)^{\frac{1}{b}}+1\right)^2
\end{aligned} \tag{4.15}$$

The claim follows from the inequalities (4.13), (4.14) and (4.15) and Lemma 4.5. □

Let us now assume that the constants c_n and d_n and the zeros x_n and y_n are actually correctly calculated.

Corollary 4.6. *Suppose that there is no approximation error, i.e.* $\gamma_n^c = \gamma_n^d = \gamma_n^x = \gamma_n^y = 0$*. Then the conditions (4.8) – (4.10) of Theorem 4.3 are satisfied if*

$$N \;\geq\; \max\left\{\left(\frac{A_0}{\delta}\right)^{1/H}+1;\ \left(\frac{A_0(\exp\{\varphi(1)\}-1)^{\alpha}}{\delta}\right)^{1/H}+1;\ 2\left(\frac{A_0}{A_\alpha}\right)^{\frac{1}{\alpha}}\right\} \tag{4.16}$$

and

$$\nu \;\geq\; 2\exp\left\{-\varphi^*\left(\frac{\delta N^H}{A_0}-1\right)\right\}\times \left(\frac{(\delta A_\alpha)^{\frac{1}{\alpha}}(N+1)^{2H/\alpha}}{2\left(1-\frac{b}{\alpha}\right)^{\frac{1}{b}}A_0^{\frac{2}{\alpha}}N^{\frac{H-\alpha}{\alpha}}}l^{-1}\left(\frac{\delta(N+1)^H}{A_0}-1\right)^{\frac{1}{b}}+1\right)^2, \tag{4.17}$$

where

$$A_0 \;=\; a_\varphi\sqrt{\frac{5c}{2H}} \qquad \textit{and} \qquad A_\alpha \;=\; 2^{1-\alpha}a_\varphi\pi^\alpha\sqrt{\frac{c}{H-\alpha}}.$$

Proof. Note that now $\gamma^{\mathrm{appr}} = \gamma_\alpha^{\mathrm{appr}} = 0$.

We shall use the asymptotics $x_n \sim y_n \sim n\pi$ and $c_n^2 \sim d_n^2 \sim \frac{c}{n^{2H+1}}$ in the expressions for γ^{cut} and $\gamma_\alpha^{\mathrm{cut}}$.

For γ^{cut} we get the upper bound

$$\begin{aligned}
\gamma^{\mathrm{cut}} &= a_\varphi^2\sum_{n=N+1}^{\infty}(c_n^2+4d_n^2) \\
&\sim a_\varphi^2\sum_{n=N+1}^{\infty}\frac{5c}{n^{2H+1}} \\
&\leq 5ca_\varphi^2\sum_{n=N}^{\infty}\int_n^{n+1}\frac{\mathrm{d}x}{x^{2H+1}} \\
&= \frac{5ca_\varphi^2}{2HN^{2H}}.
\end{aligned}$$

For $\gamma_\alpha^{\mathrm{cut}}$ we obtain

$$\begin{aligned}
\gamma_\alpha^{\mathrm{cut}} &= 2^{2-2\alpha} a_\varphi^2 \sum_{n=N+1}^{\infty} (c_n^2 x_n^{2\alpha} + d_n^2 y_n^{2\alpha}) \\
&\sim 2^{2-2\alpha} a_\varphi^2 \sum_{n=N+1}^{\infty} \left(\frac{c(\pi n)^{2\alpha}}{n^{2H+1}} + \frac{c(\pi n)^{2\alpha}}{n^{2H+1}} \right) \\
&\leq 2^{2-2\alpha} a_\varphi^2 \cdot 2c\pi^{2\alpha} \sum_{n=N}^{\infty} \int_n^{n+1} \frac{\mathrm{d}x}{x^{2(H-\alpha)+1}} \\
&= \frac{2^{2-2\alpha} a_\varphi^2 c\pi^{2\alpha}}{(H-\alpha)N^{2(H-\alpha)}}.
\end{aligned}$$

In the same way we get the lower bounds

$$\begin{aligned}
\gamma^{\mathrm{cut}} &\geq \frac{5ca_\varphi^2}{2H(N+1)^{2H}}, \\
\gamma_\alpha^{\mathrm{cut}} &\geq \frac{2^{2-2\alpha} a_\varphi^2 c\pi^{2\alpha}}{(H-\alpha)(N+1)^{2(H-\alpha)}}.
\end{aligned}$$

Therefore, we have the following bounds for γ_0 and γ_α of Theorem 4.3:

$$\begin{aligned}
\frac{A_0}{(N+1)^H} &\leq \gamma_0 \leq \frac{A_0}{N^H}, \\
\frac{A_\alpha}{(N+1)^{H-\alpha}} &\leq \gamma_\alpha \leq \frac{A_\alpha}{N^{H-\alpha}}.
\end{aligned}$$

If

$$N \geq 2\left(\frac{A_0}{A_\alpha}\right)^{\frac{1}{\alpha}}$$

then in Theorem 4.3 we have $\beta = \gamma_0$. Now we see that the condition (4.8) is satisfied if

$$N \geq \left(\frac{A_0}{\delta}\right)^{1/H} + 1.$$

Similarly, (4.9) is satisfied if

$$N \geq \left(\frac{A_0(\exp\{\varphi(1)\} - 1)^\alpha}{\delta}\right)^{1/H} + 1.$$

Finally, we see that the condition (4.10) is satisfied if (4.17) holds. □

Theorem 4.3 and Corollary 4.6 are still rather general and not readily useful in practice. Indeed, there are still the parameters α and b one has to optimise. If we choose a specific form for the function φ we are able to give an applicable version of Corollary 4.6. The next corollary deals with the sub-Gaussian case, i.e. $\varphi(x) = x^2/2$.

Corollary 4.7. *If the process Z is sub-Gaussian then the conditions (4.16) and (4.17) of Corollary 4.6 are satisfied if*

$$N \geq \max\left\{ \left(\frac{a_\varphi}{\delta}\sqrt{\frac{5c}{2H}}\right)^{1/H} + 1; \quad \frac{2^{2-\frac{4}{H}}5^{\frac{1}{H}}}{\pi} \right\} \tag{4.18}$$

and

$$2\mu \exp\left\{ -\frac{1}{2}\left(\frac{\delta N^H}{a_\varphi\sqrt{\frac{5c}{2H}}} - 1 \right)^2 \right\} N^{14} \leq \nu, \tag{4.19}$$

where

$$\mu = \pi^2 2^{\frac{22}{H}-4} 5^{-\frac{8}{H}} \left(\frac{H}{c}\right)^{\frac{6}{H}} \left(\frac{\delta}{a_\varphi}\right)^{\frac{12}{H}}.$$

Proof. In the sub-Gaussian case we have $\varphi(x) = \frac{x^2}{2}$. So,

$$\varphi^*(x) = \frac{x^2}{2} \qquad \text{and} \qquad l(x) = \varphi'(x) = x = l^{-1}(x).$$

Thus, the conditions (4.16) and (4.17) take the form

$$N \geq \max\left\{ \left(\frac{A_0}{\delta}\right)^{1/H} + 1; \quad 2\left(\frac{A_0}{A_\alpha}\right)^{\frac{1}{\alpha}} \right\} \tag{4.20}$$

and

$$\nu \geq 2\exp\left\{ -\frac{1}{2}\left(\frac{\delta N^H}{A_0} - 1\right)^2 \right\} \times$$
$$\left(\frac{(\delta A_\alpha)^{\frac{1}{\alpha}}(N+1)^{2H/\alpha}}{2\left(1-\frac{b}{\alpha}\right)^{\frac{1}{b}} A_0^{\frac{2}{\alpha}} N^{\frac{H-\alpha}{\alpha}}} \left(\frac{\delta(N+1)^H}{A_0} - 1\right)^{\frac{1}{b}} + 1 \right)^2. \tag{4.21}$$

Let's take $\alpha = \frac{H}{2}$ and $b = \frac{H}{4}$.

In this case $A_0 = a_\varphi\sqrt{\frac{5c}{2H}}$, $A_\alpha = A_{\frac{H}{2}} = a_\varphi \pi^{\frac{H}{2}} 2^{1-\frac{H}{2}}\sqrt{\frac{2c}{H}}$ and from the inequality (4.20) we get

$$N \geq \max\left\{ \left(\frac{a_\varphi}{\delta}\sqrt{\frac{5c}{2H}}\right)^{1/H} + 1; \quad \frac{2^{2-\frac{4}{H}}5^{\frac{1}{H}}}{\pi} \right\}.$$

Since N is large we have in (4.21)

$$\left(\frac{(\delta A_\alpha)^{\frac{1}{\alpha}}(N+1)^{2H/\alpha}}{2\left(1-\frac{b}{\alpha}\right)^{\frac{1}{b}} A_0^{\frac{2}{\alpha}} N^{\frac{H-\alpha}{\alpha}}} \left(\frac{\delta(N+1)^H}{A_0} - 1\right)^{\frac{1}{b}} + 1 \right)^2 \approx \mu N^{14}.$$

The claim follows. □

Remark 4.8. In Corollary 4.7 the condition (4.18) for N is in closed form. Condition (4.19) is still implicit, but it may be solved easily using numerical methods. Corollary 4.7 is readily applicable for the fractional Brownian motion. Indeed, in this case we $a_\varphi = 1$.

Example 4.9. Let

$$\varphi(x) = \begin{cases} \frac{x^p}{p}, & |x| > 1, p > 2; \\ \frac{x^2}{p}, & |x| \le 1. \end{cases}$$

In this case we have:

$$\varphi^*(x) = \frac{x^2}{2}, \quad l(x) = \varphi'(x) = x, \quad l^{-1}(x) = x$$

for $x \in [0,1]$ and

$$\varphi^*(x) = \frac{x^q}{q}, \; (\frac{1}{p} + \frac{1}{q} = 1), \quad l(x) = \varphi'(x) = x^{p-1}, \quad l^{-1}(x) = x^{\frac{1}{p-1}}$$

for $x > 1$.

Then for $0 \le \frac{\delta}{\gamma_0} - 1 \le 1$ the condition (4.10) of Theorem 4.3 takes the form

$$2\exp\left\{-\frac{1}{2}\left(\frac{\delta}{\gamma_0}-1\right)^2\right\}\left(\frac{1}{2\left(1-\frac{b}{\alpha}\right)^{\frac{1}{b}}}\left(\frac{\gamma_\alpha\delta}{\beta\gamma_0}\right)^{\frac{1}{\alpha}}\left(\frac{\delta}{\gamma_0}-1\right)^{\frac{1}{b}}+1\right)^2 \le \nu$$

and for $\frac{\delta}{\gamma_0} - 1 > 1$ we have

$$2\exp\left\{-\frac{1}{q}\left(\frac{\delta}{\gamma_0}-1\right)^q\right\}\left(\frac{1}{2\left(1-\frac{b}{\alpha}\right)^{\frac{1}{b}}}\left(\frac{\gamma_\alpha\delta}{\beta\gamma_0}\right)^{\frac{1}{\alpha}}\left(\frac{\delta}{\gamma_0}-1\right)^{\frac{1}{b(p-1)}}+1\right)^2 \le \nu.$$

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