Independence in Local Abstract Elementary Classes Part I

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Abstract

In this paper we study a specific subclass of abstract elementary classes. We assume disjoint amalgamation, existence of a prime model over $\emptyset$ and a property we call locality. This property is the main novelty of this paper. Almost all examples of AEC's have this property and it allows us to use so called weak types in place of Galois types making it possible to study geometric stability theory in the context of abstract elementary classes. Also ω-stability and $\mathrm{LS}(\mathbb{K}) = \omega$ are assumed. Our goal in the future is to construct a full notion of independence in the style of [6].

In the first section we construct a monster model and introduce an extended language by adding some Skolem functions in the style of [9]. In the second and third sections we introduce our notions of type and independence based on splitting and discuss what assumptions are needed to gain symmetry. Also other basic properties of non-splitting from elementary model theory are proved. In the fourth section we define U-rank and prove that when $\mathscr{A}$ and $\mathscr{B}$ are ω-saturated, $\mathscr{A} \subset \mathscr{B}$, then $U(\bar{a}/\mathscr{A}) = U(\bar{a}/\mathscr{B})$ if and only if $\bar{a}$ is independent from $\mathscr{B}$ over $\mathscr{A}$.

1 Abstract elementary classes

The notion of an abstract elementary class is due to Saharon Shelah, as well as many ideas appearing in this paper and originally from elementary model theory, like splitting, independence and the ideas behind the proof of symmetry for splitting. See [8], [9] and [10]. The notions of Galois type over a model and tameness are also due to Shelah, and they are studied for example in [3] and [4].

Let τ be a countable vocabulary.

Definition 1.1 *A class of τ-structures $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an abstract elementary class if*

1. *Both $\mathbb{K}$ and the binary relation $\preccurlyeq_{\mathbb{K}}$ are closed under isomorphism.*
2. *If $\mathscr{M} \preccurlyeq_{\mathbb{K}} \mathscr{N}$, then $\mathscr{M}$ is a substructure of $\mathscr{N}$.*
3. *$\preccurlyeq_{\mathbb{K}}$ is a partial order on $\mathbb{K}$.*
4. *If $\langle \mathscr{A}_i : i < \delta \rangle$ is an $\preccurlyeq_{\mathbb{K}}$-increasing chain:*
   (a) *$\bigcup_{i<\delta} \mathscr{A}_i \in \mathbb{K}$;*
   (b) *for each $j < \delta$, $\mathscr{A}_j \preccurlyeq_{\mathbb{K}} \bigcup_{i<\delta} \mathscr{A}_i$*
   (c) *if each $\mathscr{A}_i \preccurlyeq_{\mathbb{K}} \mathscr{M} \in \mathbb{K}$, then $\bigcup_{i<\delta} \mathscr{A}_i \preccurlyeq_{\mathbb{K}} \mathscr{M}$.*
5. *If $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{K}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{C}$, $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{C}$ and $\mathscr{A} \subseteq \mathscr{B}$ then $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$.*

When $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$, we say that $\mathscr{B}$ is an AE-extension of $\mathscr{A}$ and $\mathscr{A}$ is an AE-submodel of $\mathscr{B}$.

Definition 1.2 *If $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ and $f : \mathscr{A} \to \mathscr{B}$ an embedding such that $f(\mathscr{A}) \preccurlyeq_{\mathbb{K}} \mathscr{B}$, we say that f is an AE-embedding.*

Assumption 1.3 *$\mathbb{K}$ has arbitrarily large models.*

Assumption 1.4 ($\mathbf{LS}(\mathbb{K}) = \omega$) *If $\mathscr{A} \in \mathbb{K}$ and $B \subset \mathscr{A}$ a subset, there is $\mathscr{A}' \in \mathbb{K}$ such that $B \subset \mathscr{A}' \preccurlyeq_{\mathbb{K}} \mathscr{A}$ and $|\mathscr{A}'| = |B| + \omega$.*

Assumption 1.5 (Prime model) *There is $\mathscr{A}_P \in \mathbb{K}$ such that for each $\mathscr{A} \in \mathbb{K}$ there is an AE-embedding $f : \mathscr{A}_P \to \mathscr{A}$.*

Assumption 1.6 (Disjoint amalgamation) *If $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbb{K}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{C}$ and $\mathscr{B} \cap \mathscr{C} = \mathscr{A}$, there is $\mathscr{D} \in \mathbb{K}$ and a map $f : \mathscr{B} \cup \mathscr{C} \to \mathscr{D}$ such that $f \restriction_{\mathscr{B}}$ and $f \restriction_{\mathscr{C}}$ are AE-embeddings, and $f(\mathscr{B}) \cap f(\mathscr{C}) = f(\mathscr{A})$.*

We need still another assumption to capture the desired properties of the $\preccurlyeq_{\mathbb{K}}$-relation. To define this assumption we use the following concept of $\mathscr{A}$-Galois type.

Definition 1.7 ($\mathscr{A}$-Galois type) *For $\mathscr{A},\mathscr{B} \in \mathbb{K}$ and $\bar{a} \in \mathscr{A}, \bar{b} \in \mathscr{B}$ we say*

$$t^g_{\mathscr{A}}(\bar{a}/\emptyset) = t^g_{\mathscr{B}}(\bar{b}/\emptyset)$$

if there is $\mathscr{C} \in \mathbb{K}$ and AE-embeddings $f : \mathscr{A} \to \mathscr{C}$ and $g : \mathscr{B} \to \mathscr{C}$ such that $f(\bar{a}) = g(\bar{b})$.

Assumption 1.8 (Locality) *If, $\mathscr{A},\mathscr{B} \in \mathbb{K}$, $\mathscr{A} \subset \mathscr{B}$, and for all finite $\bar{a} \in \mathscr{A}$ we have that $t^g_{\mathscr{A}}(\bar{a}/\emptyset) = t^g_{\mathscr{B}}(\bar{a}/\emptyset)$, then $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$.*

Lemma 1.9 *Let $\mathscr{A},\mathscr{B} \in \mathbb{K}$ be such that $A' \subset \mathscr{A}$ and $f : A' \to \mathscr{B}$ a τ-embedding. Then there is $\mathscr{B}' \in \mathbb{K}$ and an isomorphism $h : \mathscr{B} \to \mathscr{B}'$ such that $\mathscr{B}' \cap \mathscr{A} = A' = h(f(A'))$ and $h \circ f = \mathrm{Id}_{A'}$.*

Proof: Because $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is closed under isomorphism, we may take a disjoint copy $\mathscr{B}''$ of $\mathscr{B}$ and an isomorphism $g : \mathscr{B} \to \mathscr{B}''$. Then $g \circ f : A' \to \mathscr{B}''$ is an τ-embedding. Let the universe of $\mathscr{B}'$ be the union of the sets A' and $\mathscr{B}'' \setminus g \circ f(A')$. Define a bijection $F : \mathscr{B}'' \to \mathscr{B}'$

$$F(a) = \begin{cases} (g \circ f)^{-1}(a) & \text{when } a \in g \circ f(A'), \\ a & \text{when } a \in \mathscr{B}'' \setminus g \circ f(A'). \end{cases}$$

Then define the structure in $\mathscr{B}'$ induced by F, so that F becomes an isomorphism. Also remark that the structure of $A' \subset \mathscr{B}'$ becomes identical to $A' \subset \mathscr{A}$, and $F \circ g \circ f : A' \to A'$ the identity. When we denote $h = F \circ g : \mathscr{B} \to \mathscr{B}'$, the claim follows. □

We will mostly use Assumption 1.8 when looking at mappings $f : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$. This assumption gives a sufficient and necessary condition for the mapping to be an AE-embedding.

Lemma 1.10 *Let $\mathscr{A},\mathscr{B} \in \mathbb{K}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$ and $f : \mathscr{A} \to \mathscr{B}$ a mapping. Then the condition that for all $\bar{a} \in \mathscr{A}$*

$$t^g_{\mathscr{B}}(\bar{a}/\emptyset) = t^g_{\mathscr{B}}(f(\bar{a})/\emptyset) \tag{1.1}$$

is equivalent for f being an AE-embedding.

Proof: First we assume that $f : \mathscr{A} \to \mathscr{B}$ has the property 1.1, and then claim that it is also an AE-embedding. We can easily see that from 1.1 it follows that f is an τ-embedding. Thus $f(\mathscr{A}) \in \mathbb{K}$, because $\mathbb{K}$ is closed under isomorphism. Take $\bar{a} \in \mathscr{A}$. Let $\mathscr{C} \in \mathbb{K}, g$ and h be as in the definition of Galois type, i.e. $g : \mathscr{B} \to \mathscr{C}$ and $h : \mathscr{B} \to \mathscr{C}$ AE-embeddings such that $g(\bar{a}) = h(f(\bar{a}))$. Now $g \circ f^{-1} : f(\mathscr{A}) \to \mathscr{C}$ is an AE-embedding, because from $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$ it follows that $(g \circ f^{-1})(f(\mathscr{A})) = g(\mathscr{A}) \preccurlyeq_{\mathbb{K}} g(\mathscr{B}) \preccurlyeq_{\mathbb{K}} \mathscr{C}$ and furthermore $(g \circ f^{-1})(f(\mathscr{A})) \preccurlyeq_{\mathbb{K}} \mathscr{C}$. Also $(g \circ f^{-1})(f(\bar{a})) = g(\bar{a}) = h(f(\bar{a}))$. Hence we get for all $f(\bar{a}) \in f(\mathscr{A})$ that $t^g_{f(\mathscr{A})}(f(\bar{a})/\emptyset) = t^g_{\mathscr{B}}(f(\bar{a})/\emptyset)$, and then from Assumption 1.8 that $f(\mathscr{A}) \preccurlyeq_{\mathbb{K}} \mathscr{B}$. Thus f is an AE-embedding.

Then let $f : \mathscr{A} \to \mathscr{B}$ be an AE-embedding. When we substitute A' for $\mathscr{A}$ and $\mathscr{A}$ for $\mathscr{B}$ in Lemma 1.9, we get $\mathscr{B}' \in \mathbb{K}$ and an isomorphism $h : \mathscr{B} \to \mathscr{B}'$ such that $\mathscr{A} = \mathscr{B}' \cap \mathscr{B}$ and $(h \circ f)(a) = a$ for all $a \in \mathscr{A}$. Again because $f(\mathscr{A}) \preccurlyeq_{\mathbb{K}} \mathscr{B}$, also $\mathscr{A} = h(f(\mathscr{A})) \preccurlyeq_{\mathbb{K}} \mathscr{B}'$. We can use the amalgamation assumption 1.6 to get $\mathscr{C} \in \mathbb{K}$ and $g : \mathscr{B} \cup \mathscr{B}' \to \mathscr{C}$ such that $g \restriction_{\mathscr{B}}$ and $g \restriction_{\mathscr{B}'}$ are AE-embeddings. Now g and $g \circ h$ are AE-embeddings from $\mathscr{B}$ to $\mathscr{C}$ and $g(a) = g((h \circ f)(a)) = (g \circ h)(f(a))$ for all $a \in \mathscr{A}$. We can take $\mathscr{C}, g$ and $g \circ h$ in the definition of Galois type to show that $t^g_{\mathscr{B}}(\bar{a}/\emptyset) = t^g_{\mathscr{B}}(f(\bar{a})/\emptyset)$ for all tuples $\bar{a} \in \mathscr{A}$ simultaneously. $\square$

Finally we define our concept of local abstract elementary class.

Definition 1.11 (Local abstract elementary class) *Abstract elementary class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ satisfying Assumptions 1.3-1.6 and 1.8. is called a local abstract elementary class.*

From now on we will always assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a local abstract elementary class.

1.1 Extended vocabulary τ^*

Sections 1.1 and 1.2 are based on ideas due to Shelah. In this section we first introduce an extended vocabulary with some Skolem-functions. They will be a useful tool especially in section 3.

Definition 1.12 *Let $\tau^* = \tau \cup \{F^k_n : n, k < \omega\}$ and $\mathbb{K}^*$ be τ^*-structures such that for $\mathscr{A} \in \mathbb{K}^*$:*

1. *$\mathscr{A} \restriction_\tau \in \mathbb{K}$,*
2. *For all $\bar{a} \in \mathscr{A}$, $\mathscr{A}_{\bar{a}} = \{(F^{lg(\bar{a})}_n)^{\mathscr{A}}(\bar{a}) : n < \omega\}$, is such that*
    (a) *$\mathscr{A}_{\bar{a}} \in \mathbb{K}$ and $\mathscr{A}_{\bar{a}} \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction_\tau$,*
    (b) *if $\bar{b} \subset \bar{a}$ then $\bar{b} \in \mathscr{A}_{\bar{b}} \subset \mathscr{A}_{\bar{a}}$[1].*

[1]Here $\bar{b} \subset \bar{a}$ means that $lg(\bar{b}) \leq lg(\bar{a})$ and the members of the tuple $\bar{b}$ are contained in the set of members of $\bar{a}$, i.e. when $\bar{b} = (b_0, ..., b_k)$ and $\bar{a} = (a_0, ..., a_n)$, $\{b_0, ..., b_k\} \subset \{a_0, ..., a_n\}$.

(c) *Let* $(a_i)_{i<\omega}$ *be a fixed ordering on* $\mathscr{A}_P$*. The mapping* $f : \mathscr{A}_P \to \mathscr{A}$*, where* $f(a_i) = (F_i^0)^{\mathscr{A}}$*, is an AE-embedding.*

Lemma 1.13 *If* $\mathscr{A} \in \mathbb{K}^*$ *and* $B \subset \mathscr{A}$ *a subset such that* B *is closed under functions* F_n^k*, then* $B \restriction \tau \in \mathbb{K}$ *and* $B \restriction \tau \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction \tau$*.*

Proof: The proof is by induction on the size of B. First we notice that because of the constants $(F_i^0)_{i<\omega}^{\mathscr{A}}$, the prime model $\mathscr{A}_P$ is AE-embeddable in $B \restriction \tau$ and thus $B \neq \emptyset$.

1^o $|B| \leq \omega$. Let $B = (b_i)_{i<\alpha}$, where $\alpha \leq \omega$, and denote $B_i = \mathscr{A}_{\{b_0,\ldots,b_{i-1}\}} \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction \tau$ as in Definition 1.12. By condition $2b$) in Definition 1.12 and condition 5 in Definition 1.1 we get an $\preccurlyeq_{\mathbb{K}}$-increasing chain of models $\mathscr{B}_i$ such that $B = \bigcup_{i<\omega} \mathscr{B}_i$. Now the claim follows from the property 4 in Definition 1.1, that $\mathbb{K}$ is respects unions of $\preccurlyeq_{\mathbb{K}}$-increasing chains.

2^o Assume claim holds for all B' of size less than α and let $\omega < |B| = \alpha$. Because $\mathrm{LS}(\mathbb{K}^*) = \omega$, we may write B as a union of an increasing chain of τ^*-models $(B_i)_{i<\alpha}$, where each B_i is a τ^*-substructure of $\mathscr{A}$, and of size strictly less than α. By induction, $B_i \preccurlyeq_{\mathbb{K}} \mathscr{A}$ for each $i < \alpha$. Again, using the coherence property 5 of Definition 1.1, we get that $(B_i)_{i<\alpha}$ is actually a $\preccurlyeq_{\mathbb{K}}$-increasing chain. The claim follows as in 1^o. □

Remark that if $\mathscr{A}, \mathscr{B} \in \mathbb{K}^*$ and $f : \mathscr{A} \to \mathscr{B}$ is an τ^*-embedding, then $f : \mathscr{A} \restriction_\tau \to \mathscr{B} \restriction_\tau$ is an AE-embedding. This follows from Lemma 1.13, because an image of a model in an embedding is always closed under functions.

Of course from Lemma 1.13 it follows that if $\mathscr{B}$ is a τ^*-submodel of $\mathscr{A} \in \mathbb{K}^*$, then also $\mathscr{B} \restriction_\tau \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction_\tau$. Thus the properties 1.-5. of definition 1.1 hold for $\mathbb{K}^*$ where $\preccurlyeq_{\mathbb{K}}$ is replaced with the τ^*-submodel relation.

Lemma 1.14 *For every* $\mathscr{A} \in \mathbb{K}$ *there is* $\mathscr{A}^* \in \mathbb{K}^*$ *such that* $\mathscr{A}^* \restriction \tau = \mathscr{A}$*.*

Proof: We have to define functions $(F_n^k)^{\mathscr{A}}$ so that they satisfy the conditions in Definition 1.12. We do that by defining functions by induction on $lg(\bar{a})$, and for all $\bar{a} \in \mathscr{A}$ of the same length simultaneously. We notice that $\mathscr{A}_{\bar{a}}$ of Definition 1.12 need not to depend on the ordering of $\bar{a}$, thus we let $(F_i^{\lg(\bar{a})})^{\mathscr{A}}(\bar{a}) = (F_i^{\lg(\bar{a})})^{\mathscr{A}}(\beta(\bar{a}))$, whenever $\beta : \bar{a} \to \bar{a}$ is a bijection. Also if the elements of $\bar{a}$ are already contained in some shorter sequence $\bar{a}'$, we let $\mathscr{A}_{\bar{a}}$ equal $\mathscr{A}_{\bar{a}'}$.

1^o First define constants $(F_i^0)_{i<\omega}^{\mathscr{A}}$. Let f be an AE-embedding of the prime model $\mathscr{A}_P$ into $\mathscr{A}$ and $(b_i)_{i<\omega}$ be the fixed ordering on $\mathscr{A}_P$. We define $(F_i^0)^{\mathscr{A}} = f(b_i)$ for all $i < \omega$.

2^o Assume we have defined $(F_i^{lg(\bar{a})})^{\mathscr{A}}(\bar{a})$ for all $\bar{a}$ of length less or equal to n and for all $i < \omega$. Then define functions for all $\bar{b} \in \mathscr{A}^{n+1}$. We want to check that permutation does not affect to the choice of $\mathscr{A}_{\bar{b}}$, thus we order $\mathscr{A}^{n+1}$ and compare $\bar{b} \in \mathscr{A}^{n+1}$ with the previous ones. Let $\bar{b} \in \mathscr{A}^{n+1}$ and assume we have defined functions for the previous $\bar{b}' \in \mathscr{A}^{n+1}$. If the elements of the sequence $\bar{b}$ are already contained in some shorter sequence $\bar{b}'$ or $\bar{b}$ is a permutation of some previous $\bar{b}' \in \mathscr{A}^{n+1}$, let $((F_i^{n+1})^{\mathscr{A}}(\bar{b})) = ((F_i^{lg(\bar{b}')})^{\mathscr{A}}(\bar{b}'))$ for all $i < \omega$. Otherwise we do the following. Because $\mathrm{LS}(\mathbb{K}) = \omega$, there is $\mathscr{A}_{\bar{b}} \in \mathbb{K}$ such that $|\mathscr{A}_{\bar{b}}| \leq \omega$, $\mathscr{A}_{\bar{b}} \preccurlyeq_{\mathbb{K}} \mathscr{A}$ and $F \subset \mathscr{A}_{\bar{b}}$, where F is the countable set

$$F = \{(F_i^{lg(\bar{a})})^{\mathscr{A}}(\bar{a}) : \bar{a} \subset \bar{b}, lg(\bar{a}) < lg(\bar{b}), i < \omega\} \cup \{\bar{b}\}.$$

We let $((F_i^{n+1})^{\mathscr{A}}(\bar{b}))_{i<\omega}$ enumerate $\mathscr{A}_{\bar{b}}$. When we have defined functions for all $\bar{b} \in \mathscr{A}^{n+1}$, we see that $\mathscr{A}_{\bar{a}} \subset \mathscr{A}_{\bar{b}}$ whenever $\bar{a} \subset \bar{b}$. $\square$

Lemma 1.15 ($\mathbb{K}^*$ -amalgamation) *If $\mathscr{A}, \mathscr{B} \in \mathbb{K}^*$ such that for all $\bar{b} \in \mathscr{A} \cap \mathscr{B}$ and atomic ψ,*

$$\mathscr{A} \models \psi(\bar{b}) \Leftrightarrow \mathscr{B} \models \psi(\bar{b}),$$

then there is $\mathscr{C} \in \mathbb{K}^*$ and $f : \mathscr{A} \cup \mathscr{B} \to \mathscr{C}$ such that $f \restriction_{\mathscr{A}}$ and $f \restriction_{\mathscr{B}}$ are τ^*-embeddings.

Proof: Denote $(\mathscr{A} \cap \mathscr{B})^{\mathscr{A}}$ to be the closure of $(\mathscr{A} \cap \mathscr{B})$ under functions $(F_n^k)^{\mathscr{A}}, k, n \in \omega$, and $(\mathscr{A} \cap \mathscr{B})^{\mathscr{B}}$ respectively. Now because by the assumption $(\mathscr{A} \cap \mathscr{B})^{\mathscr{A}}$ and $(\mathscr{A} \cap \mathscr{B})^{\mathscr{B}}$ are isomorphic over $\mathscr{A} \cap \mathscr{B}$ and by Lemma 1.13 belong to $\mathbb{K}^*$. Let $h' : (\mathscr{A} \cap \mathscr{B})^{\mathscr{A}} \to (\mathscr{A} \cap \mathscr{B})^{\mathscr{B}}$ be an isomorphism such that $h' \restriction_{(\mathscr{A} \cap \mathscr{B})} = \mathrm{Id}_{(\mathscr{A} \cap \mathscr{B})}$. Using Lemma 1.9 we find $\mathscr{B}'$ and an isomorphism $h : \mathscr{B} \to \mathscr{B}'$ such that $h \circ h' \restriction_{(\mathscr{A} \cap \mathscr{B})^{\mathscr{A}}} = \mathrm{Id}_{(\mathscr{A} \cap \mathscr{B})^{\mathscr{A}}}$ and $\mathscr{A} \cap \mathscr{B}' = (\mathscr{A} \cap \mathscr{B})^{\mathscr{A}} = (h((\mathscr{A} \cap \mathscr{B})^{\mathscr{B}})) = (\mathscr{A} \cap \mathscr{B}')^{\mathscr{B}'}$. From Lemma 1.13 we also get that $(\mathscr{A} \cap \mathscr{B}') \restriction \tau \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction \tau$ and $(\mathscr{A} \cap \mathscr{B}') \restriction \tau \preccurlyeq_{\mathbb{K}} \mathscr{B}' \restriction \tau$. We may use the disjoint amalgamation property of $\mathbb{K}$ and find $\mathscr{C} \in \mathbb{K}$, and a map $f' : \mathscr{A} \cup \mathscr{B}' \to \mathscr{C}$ such that $f' \restriction_{\mathscr{A}}$ and $f' \restriction_{\mathscr{B}'}$ are AE-embeddings, and $f'(\mathscr{A}) \cap f'(\mathscr{B}') = f'(\mathscr{A} \cap \mathscr{B}') = f'((\mathscr{A} \cap \mathscr{B})^{\mathscr{A}}) = f'(h((\mathscr{A} \cap \mathscr{B})^{\mathscr{B}}))$. We define functions $(F_n^k)^{\mathscr{C}}$ in $f'(\mathscr{A} \cup \mathscr{B})$ as induced by f'. We can do this, because functions induced by $\mathscr{A}$ on $f'(\mathscr{A})$ and $\mathscr{B}'$ on $f'(\mathscr{B}')$ agree on the intersection. Then we can define functions in $\mathscr{C} \setminus f'(\mathscr{A} \cup \mathscr{B}')$ as in Lemma 1.14. Now $\mathscr{C}$ belongs to $\mathbb{K}^*$ and $f' \restriction_{\mathscr{A}}$ and $f' \restriction_{\mathscr{B}'}$ are τ^*-embeddings. Then look at the mapping $f : \mathscr{A} \cup \mathscr{B} \to \mathscr{C}$, where

$$f(a) = \begin{cases} f'(a) & \text{when } a \in \mathscr{A}, \\ f' \circ h(a) & \text{when } a \in \mathscr{B}. \end{cases}$$

This mapping is well defined, because when $a \in (\mathscr{A} \cap \mathscr{B})$, $h(a) = h \circ h'(a) = a$. Also $f \restriction_{\mathscr{A}} = f' \restriction_{\mathscr{A}}$ and $f \restriction_{\mathscr{B}} = f' \circ h$ are τ-embeddings. $\square$

1.2 Monster model

With $\mathbb{K}^*$-amalgamation and Assumption 1.8 we may construct a **monster model**.

Theorem 1.16 *Let μ be a cardinal. There is $\mathfrak{M}^* \in \mathbb{K}$ such that:*

1. ***μ-Universality:*** *$\mathfrak{M}^*$ is μ-universal, that is for all $\mathscr{A} \in \mathbb{K}^*$, $|\mathscr{A}| < \mu$, there is a τ^*-embedding $f : \mathscr{A} \to \mathfrak{M}^*$.*

2. ***μ-Homogeneity:*** *When $(a_i)_{i<\alpha}, (b_i)_{i<\alpha} \subset \mathfrak{M}^*$, $\alpha < \mu$, and for all $i_0, ..., i_n < \alpha$ and ψ atomic τ^*-formula,*

$$\mathfrak{M}^* \models \psi(a_{i_0}, ..., a_{i_n}) \Leftrightarrow \mathfrak{M}^* \models \psi(b_{i_0}, ..., b_{i_n}),$$

   there is $f \in \mathrm{Aut}(\mathfrak{M}^*)$ such that $f(a_i) = b_i$ for all $i < \alpha$.

3. *For all $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}^* \restriction_\tau$ such that $|\mathscr{A}| < \mu$ and mappings $f : \mathscr{A} \to \mathfrak{M}^*$ such that for all finite tuples $\bar{a} \in \mathscr{A}$*

$$t^g_{\mathfrak{M}^* \restriction \tau}(\bar{a}/\emptyset) = t^g_{\mathfrak{M}^* \restriction \tau}(f(\bar{a})/\emptyset),$$

   there is $g \in \mathrm{Aut}(\mathfrak{M}^* \restriction \tau)$ extending f.

We denote $\mathfrak{M} = \mathfrak{M}^* \restriction \tau$.

Remark 1.17 *By Lemma 1.10 we could also talk about AE-embeddings $f : \mathscr{A} \to \mathfrak{M}$ in condition 3 of Theorem 1.16.*

It is possible to construct such a model for arbitrary μ. For simplicity here we assume that μ is a regular cardinal such that $2^{<\mu} = \mu$. Especially we assume that such a cardinal exists. Then the number of isomorphism types of τ^*-structures of cardinality strictly less that μ is μ. Also the number of partial mappings $f : \mu \to \mu$ with $\mathrm{dom}(f) < \mu$ is μ. With this assumption it is possible to construct a monster model of size μ. Without the assumption the size of the model might be larger.

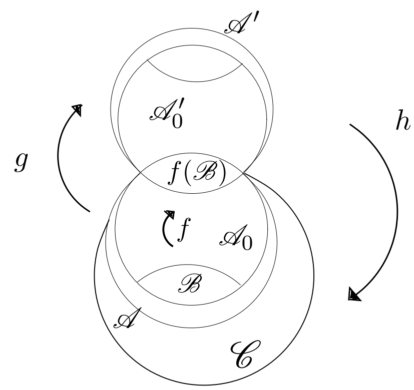
At first we prove some lemmas and finally the theorem.

Lemma 1.18 *Let $2^{<\mu} = \mu$. There is a model $\mathscr{A}_U \in \mathbb{K}^*$ such that $|\mathscr{A}_U| = \mu$ and for every $\mathscr{A}' \in \mathbb{K}$, $|\mathscr{A}'| < \mu$, there is an τ^*-embedding $f_{\mathscr{A}'} : \mathscr{A}' \to \mathscr{A}_U$.*

Proof: Let $(\mathscr{B}_\alpha)_{\alpha<\mu}$ be a sequence of models in $\mathbb{K}^*$, where every isomorphism type of a model in $\mathbb{K}^*$ of size less than μ is represented. For convenience we may assume that $\mathscr{B}_\alpha = \bigcup_{i<\alpha} \mathscr{B}_i$ for every limit ordinal α.

Let $\mathscr{A}_0 = \mathscr{B}_0$ and define $\mathscr{A}_\beta \in \mathbb{K}^*$, $|\mathscr{A}_\beta| \leq \mu$, by induction, where when $\alpha < \beta < \mu$, $\mathscr{A}_\alpha \subset \mathscr{A}_\beta$ a submodel and there is an τ^*-embedding $f_{\alpha+1} : \mathscr{B}_{\alpha+1} \to \mathscr{A}_\beta$ for every $\alpha < \beta$.

Figure 1: Picture for Lemma 1.19.



Remark that then by Lemma 1.13, we also get an increasing $\preccurlyeq_{\mathbb{K}}$-chain of models $\mathscr{A}_\alpha \restriction_\tau \in \mathbb{K}$.

When α is a limit ordinal, we simply take $\mathscr{A}_\alpha = \bigcup_{\beta<\alpha} \mathscr{A}_\beta$. Now $\mathscr{A}_\alpha$ is in $\mathbb{K}^*$ by the remark above and the union property 4 of Definition 1.1. Also $|\mathscr{A}_\alpha| \leq \mu$.

Consider the case where $\alpha = \beta + 1$ is a successor ordinal. We use $\mathbb{K}^*$-amalgamation (1.15) to get $\mathscr{A}_\alpha \in \mathbb{K}$ such that both $\mathscr{A}_\beta$ and $\mathscr{B}_\alpha$ are τ^*-embeddable in $\mathscr{A}_\alpha$. Then by lemma 1.9 we may assume that $\mathscr{A}_\beta \subset \mathscr{A}_\alpha$ is a substructure and also because $\mathrm{LS}(\mathbb{K}^*) = \omega$, we may assume that $|\mathscr{A}_\alpha| = \max\{|\mathscr{A}_\beta|, |\mathscr{B}_\alpha|\} \leq \mu$.

Finally we take $\mathscr{A}_U = \bigcup_{\alpha<\mu} \mathscr{A}_\alpha$. Clearly $|\mathscr{A}_U| = \mu$. □

We remark that now whenever $\mathscr{A} \in \mathbb{K}^*$ is like in Lemma 1.18 and $\mathscr{M} \in \mathbb{K}^*$ such that $\mathscr{A}$ is a submodel of $\mathscr{M}$, then also $\mathscr{M}$ is μ-universal.

Lemma 1.19 *Let $\mathscr{A}_0, \mathscr{A}$ be in $\mathbb{K}^*$, $\mathscr{A}_0 \subset \mathscr{A}$ a submodel and $B \subset \mathscr{A}$ a subset and $f : B \to \mathscr{A}_0$ a mapping such that for all $\bar{b} \in B$ and ψ atomic*

$$\mathscr{A}_0 \models \psi(\bar{b}) \text{ if and only if } \mathscr{A}_0 \models \psi(f(\bar{b})).$$

Then there is $\mathscr{C} \in \mathbb{K}^*$ such that $|\mathscr{C}| = |\mathscr{A}|$, $\mathscr{A} \subset \mathscr{C}$ a submodel and an τ^*-embedding $F : \mathscr{A}_0 \to \mathscr{C}$ such that $f \subset F$.

Proof: At first we use Lemma 1.9 to get $\mathscr{A}' \in \mathbb{K}$ and an isomorphism $g : \mathscr{A} \restriction_\tau \to \mathscr{A}'$ such that $\mathscr{A} \restriction_\tau \cap \mathscr{A}' = f(B)$ and $g \circ f^{-1} \restriction_{f(B)} = Id \restriction_{f(B)}$. Then also $g \restriction_B = f \restriction_B$.

Because $\mathscr{A}_0$ is a submodel of $\mathscr{A}$, they agree on atomic formulas, and we get that if $\bar{b} \in B$ and ψ atomic

$$\mathscr{A} \models \psi(\bar{b}) \Leftrightarrow \mathscr{A}_0 \models \psi(\bar{b}) \Leftrightarrow \mathscr{A}_0 \models \psi(f(\bar{b})) \Leftrightarrow \mathscr{A} \models \psi(f(\bar{b})),$$

and hence

$$\mathscr{A} \models \psi(f(\bar{b})) \Leftrightarrow \mathscr{A} \models \psi(\bar{b}) \Leftrightarrow \mathscr{A}' \models \psi(g(\bar{b})) \Leftrightarrow \mathscr{A}' \models \psi(f(\bar{b})).$$

We may now use $\mathbb{K}^*$-amalgamation 1.15 and get $\mathscr{C} \in \mathbb{K}^*$ and $h : \mathscr{A} \cup \mathscr{A}' \to \mathscr{C}$ such that $h \restriction_{\mathscr{A}}$ and $h \restriction_{\mathscr{A}'}$ are τ^*-embeddings.

Furthermore, by Lemma 1.9 we may assume that $\mathscr{A} \subset \mathscr{C}$ and $h \restriction_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$. Because $\mathrm{LS}(\mathbb{K}^*) = \omega$, we may assume that $|\mathscr{C}| = |\mathscr{A} \cup \mathscr{A}'| = |\mathscr{A}|$.

The mapping $h \circ g : \mathscr{A}_0 \to \mathscr{C}$ is an τ^*-embedding and for $b \in B$, $h \circ g(b) = h \circ f(b) = f(b)$. □

Lemma 1.20 *Let $\mathscr{B}, \mathscr{A}_0$ be in $\mathbb{K}$, $\mathscr{A} \in \mathbb{K}^*$, $\mathscr{A}_0 \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction_\tau, \mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{A}_0$ and $f : \mathscr{B} \to \mathscr{A}_0$ a mapping such that for all $\bar{b} \in \mathscr{B}$*

$$t^g_{\mathscr{A}_0}(\bar{b}/\emptyset) = t^g_{\mathscr{A}_0}(f(\bar{b})/\emptyset).$$

Then there is $\mathscr{C} \in \mathbb{K}^*$ such that $|\mathscr{C}| = |\mathscr{A}|$, $\mathscr{A} \subset \mathscr{C}$ a submodel and an AE-embedding $F : \mathscr{A}_0 \to \mathscr{C} \restriction_\tau$ such that $f \subset F$.

Proof: At first we use Lemma 1.9 to get $\mathscr{A}' \in \mathbb{K}$ and an isomorphism $g : \mathscr{A} \restriction_\tau \to \mathscr{A}'$ such that $\mathscr{A} \restriction_\tau \cap \mathscr{A}' = f(\mathscr{B})$ and $g \circ f^{-1} \restriction_{f(\mathscr{B})} = Id \restriction_{f(\mathscr{B})}$. Then also $g \restriction_{\mathscr{B}} = f \restriction_{\mathscr{B}}$.

Because of the assumption and Lemma 1.10, $f : \mathscr{B} \to \mathscr{A}_0$ is an AE-embedding. Then we have that $f(\mathscr{B}) \preccurlyeq_{\mathbb{K}} \mathscr{A}_0 \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction_\tau$ and because $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction_\tau$, also $f(\mathscr{B}) = g(\mathscr{B}) \preccurlyeq_{\mathbb{K}} g(\mathscr{A}) = \mathscr{A}'$.

Thus we get from amalgamation 1.6 $\mathscr{C} \in \mathbb{K}$ and a mapping $h : \mathscr{A} \cup \mathscr{A}' \to \mathscr{C}$ such that $h \restriction_{\mathscr{A}}$ and $h \restriction_{\mathscr{A}'}$ are AE-embeddings.

By Lemma 1.9 we may assume that $\mathscr{A} \restriction_\tau \subset \mathscr{C}$ and $h \restriction_{\mathscr{A} \restriction \tau} = \mathrm{Id}_{\mathscr{A} \restriction \tau}$. Also because $\mathrm{LS}(\mathbb{K}) = \omega$, we may assume that $|\mathscr{C}| = |\mathscr{A} \cup \mathscr{A}'| = |\mathscr{A}|$.

Now the mapping $h \circ g : \mathscr{A} \restriction_\tau \to \mathscr{C}$ is an AE-embedding. From $\mathscr{A}_0 \preccurlyeq_{\mathbb{K}} \mathscr{A} \restriction_\tau$ we get that $h \circ g \restriction_{\mathscr{A}_0} : \mathscr{A}_0 \to \mathscr{C}$ is also an AE-embedding. When $b \in \mathscr{B}$, $h \circ g(b) = h \circ f(b) = f(b)$.

Finally we may define functions $(F^k_n)^{\mathscr{C}}$ in $\mathscr{A} \subset \mathscr{C}$ as $(F^k_n)^{\mathscr{A}}$ and in $\mathscr{C} \setminus \mathscr{A}$ as in Lemma 1.14. □

Lemma 1.21 *Let $2^{<\mu} = \mu$ and μ be regular. There is a model $\mathscr{M}^* \in \mathbb{K}^*$ such that $|\mathscr{M}^*| = \mu$ and*

1. *$\mathscr{M}^*$ is μ-universal.*

2. *For all submodels* $\mathscr{C} \subset \mathscr{M}^*$*,* $|\mathscr{C}| < \mu$*, and sets* $A, B \subset \mathscr{C}$ *such that* $f : A \to B$ *a bijection and for all* $\bar{a} \in A$*,* ψ *atomic* τ^**-formula*

$$\mathscr{M}^* \models \psi(\bar{a}) \Leftrightarrow \mathscr{M}^* \models \psi(f(\bar{a})),$$

   there is a submodel $\mathscr{D} \subset \mathscr{M}^*$*,* $|\mathscr{D}| < \mu$*, and a* τ^**-embedding* $F : \mathscr{C} \to \mathscr{D}$ *such that* $f \subset F$*.*

3. *For all* $\mathscr{A}, \mathscr{C} \in \mathbb{K}$ *such that* $|\mathscr{C}| < \mu$*,* $\mathscr{C} \preccurlyeq_{\mathbb{K}} \mathscr{M}^* \restriction_\tau$*,* $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{C}$ *and* $f : \mathscr{A} \to \mathscr{C}$ *a mapping such that for all* $\bar{a} \in \mathscr{A}$

$$t^g_{\mathscr{M}^* \restriction_\tau}(\bar{a}/\emptyset) = t^g_{\mathscr{M}^* \restriction_\tau}(f(\bar{a})/\emptyset),$$

   there is a submodel $\mathscr{D} \subset \mathscr{M}^*$*,* $|\mathscr{D}| < \mu$*, and an AE-embedding* $F : \mathscr{C} \to \mathscr{D} \restriction_\tau$ *such that* $f \subset F$*.*

Proof: We define by induction models $\mathscr{M}_i \in \mathbb{K}^*, i < \mu$, $|\mathscr{M}_i| = \mu$ and $\mathscr{M}_i$ a submodel of $\mathscr{M}_j$ for all $i < j < \mu$. Let $\mathscr{M}_0 = \mathscr{A}_U$ the μ-universal model from Lemma 1.18. When $\alpha < \mu$ is a limit ordinal, we take union. When we have defined $\mathscr{M}_\alpha$, we define $\mathscr{M}_{\alpha+1}$ as follows:

Let $(f_i)_{i<\mu}$ enumerate all partial mappings $f_i : \mathscr{M}_\alpha \to \mathscr{M}_\alpha$, where $|\operatorname{dom}(f_i)| < \mu$ and for all $\bar{a} \in \operatorname{dom}(f_i)$

$$\mathscr{M}_\alpha \models \psi(\bar{a}) \Leftrightarrow \mathscr{M}_\alpha \models \psi(f_i(\bar{a})).$$

Let $(g_i)_{i<\mu}$ enumerate all partial mappings $g_i : \mathscr{M}_\alpha \to \mathscr{M}_\alpha$, where $|\operatorname{dom}(g_i)| < \mu$, $\operatorname{dom}(g_i) \preccurlyeq_{\mathbb{K}} \mathscr{M}_\alpha$ and for all $\bar{a} \in \operatorname{dom}(g_i)$

$$t^g_{\mathscr{M}_\alpha \restriction_\tau}(\bar{a}/\emptyset) = t^g_{\mathscr{M}_\alpha \restriction_\tau}(g_i(\bar{a})/\emptyset).$$

Again for convenience we may assume that $f_i = f_{i+1}$ and $g_i = g_{i+1}$ for limit i. Define models $\mathscr{C}_i \in \mathbb{K}^*$, $|\mathscr{C}_i| = \mu$ such that for all $i < \mu$

(a) When $j < i$, $\mathscr{C}_j$ is a submodel of $\mathscr{C}_i$.

(b) There is a τ^*-embedding $F_i : \mathscr{M}_\alpha \to \mathscr{C}_i$ such that $f_i \subset F_i$.

(c) There is an AE-embedding $G_i : \mathscr{M}_\alpha \restriction_\tau \to \mathscr{C}_i \restriction_\tau$ such that $g_i \subset G_i$.

We let $\mathscr{C}_0 = \mathscr{M}_\alpha$ and for limit i, $\mathscr{C}_i = \bigcup_{j<i} \mathscr{C}_j$.

Let $i = j + 1$. First from Lemma 1.19 we get $\mathscr{D} \in \mathbb{K}$ and $F_i : \mathscr{M}_\alpha \to \mathscr{D}$ a τ^*-embedding such that $|\mathscr{D}| = \mu$, $\mathscr{C}_j \subset \mathscr{D}$ a submodel and $f_i \subset F_i$. Then from Lemma 1.20 we get $\mathscr{C}_i \in \mathbb{K}^*$ and an AE-embedding $G_i : \mathscr{M}_\alpha \to \mathscr{C}_i$ such that $|\mathscr{C}_i| = \mu$, $\mathscr{D} \subset \mathscr{C}_i$ a submodel and $g_i \subset G_i$.

$\mathscr{M}_{\alpha+1} = \bigcup_{i<\mu} \mathscr{C}_i$.

Finally take $\mathscr{M}^* = \bigcup_{i<\mu} \mathscr{M}_i$. Property 1 holds for $\mathscr{M}^*$ because the μ-universal model $\mathscr{A}_U$ is a submodel of $\mathscr{M}^*$. We check that properties 2 and 3 hold for $\mathscr{M}^*$. First the less trivial 3. Let $\mathscr{C} \subset \mathscr{M}^*$ be a submodel, $|\mathscr{C}| < \mu$, and $f : \mathscr{C} \to \mathscr{C}$ a partial mapping as in 3. Now because μ is regular, $\mathscr{C} \subset \mathscr{M}_\alpha$ for some $\alpha < \mu$. Because $\mathscr{M}_\alpha \restriction_\tau \preccurlyeq_{\mathbb{K}} \mathscr{M}^* \restriction_\tau$, we get that for all $\bar{a} \in \mathscr{M}_\alpha$

$$t^g_{\mathscr{M}_\alpha \restriction_\tau}(\bar{a}/\emptyset) = t^g_{\mathscr{M}_\alpha \restriction_\tau}(g_i(\bar{a})/\emptyset).$$

Then $f = g_i$ for some $i < \mu$. From the construction of $\mathscr{M}_{\alpha+1}$ we get an AE-embedding $G_i : \mathscr{M}_\alpha \to \mathscr{C}_i$ extending f. Because $\mathscr{C} \restriction_\tau \preccurlyeq_{\mathbb{K}} \mathscr{M}_\alpha \restriction_\tau$ and $\mathscr{C}_i \preccurlyeq_{\mathbb{K}} \mathscr{M}_{\alpha+1} \restriction_\tau$, we get that $G_i \restriction_{\mathscr{C}} : \mathscr{C} \restriction_\tau \to \mathscr{M}_{\alpha+1} \restriction_\tau$ is also an AE-embedding. We can take $\mathscr{D} \in \mathbb{K}^*$ to be a submodel of $\mathscr{M}_{\alpha+1}$ such that $|\mathscr{D}| < \mu$ and $\mathrm{rng}(G_i \restriction_{\mathscr{C}}) \subset \mathscr{D}$. Property 2 follows from the property (b) of the construction similarly. □

Lemma 1.22 *Let μ be regular. Properties 2 and 3 of Lemma 1.16 hold also for the model $\mathscr{M}^*$ of Lemma 1.21. That is, if $\mathscr{M}^*$ satisfies*

2'. *For all submodels $\mathscr{C} \subset \mathscr{M}^*$, $|\mathscr{C}| < \mu$, and sets $A, B \subset \mathscr{C}$ and bijections $f : A \to B$ such that for all $\bar{a} \in A$, ψ atomic τ^*-formula*

$$\mathscr{M}^* \models \psi(\bar{a}) \Leftrightarrow \mathscr{M}^* \models \psi(f(\bar{a})),$$

there is a submodel $\mathscr{D} \subset \mathscr{M}^*$, $|\mathscr{D}| < \mu$, and a τ^*-embedding $F : \mathscr{C} \to \mathscr{D}$ such that $f \subset F$.

3'. *For all $\mathscr{A}, \mathscr{C} \in \mathbb{K}$ such that $|\mathscr{C}| < \mu$, $\mathscr{C} \preccurlyeq_{\mathbb{K}} \mathscr{M}^* \restriction_\tau$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{C}$ and $f : \mathscr{A} \to \mathscr{C}$ a mapping such that for all $\bar{a} \in \mathscr{A}$*

$$t^g_{\mathscr{M}^* \restriction_\tau}(\bar{a}/\emptyset) = t^g_{\mathscr{M}^* \restriction_\tau}(f(\bar{a})/\emptyset),$$

there is a submodel $\mathscr{D} \subset \mathscr{M}^*$, $|\mathscr{D}| < \mu$, and an AE-embedding $F : \mathscr{C} \to \mathscr{D} \restriction_\tau$ such that $f \subset F$.

then it also satisfies

2. *When $(a_i)_{i<\alpha}, (b_i)_{i<\alpha} \subset \mathscr{M}^*$, $\alpha < \mu$, and for all $i_0, ..., i_n < \alpha$ and ψ atomic τ^*-formula,*

$$\mathscr{M}^* \models \psi(a_{i_0}, ..., a_{i_n}) \Leftrightarrow \mathscr{M}^* \models \psi(b_{i_0}, ..., b_{i_n}),$$

there is $f \in \mathrm{Aut}(\mathscr{M}^*)$ such that $f(a_i) = b_i$ for all $i < \alpha$.

3. *For all* $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{M}$ *such that* $|\mathscr{A}| < \mu$ *and* $f : \mathscr{A} \to \mathscr{M}^* \restriction_\tau$ *a mapping such that for all* $\bar{a} \in \mathscr{A}$
$$t^g_{\mathscr{M}^* \restriction \tau}(\bar{a}/\emptyset) = t^g_{\mathscr{M}^* \restriction \tau}(f(\bar{a})/\emptyset)$$
there is $g \in \mathrm{Aut}(\mathscr{M}^* \restriction \tau)$ *extending* f.

Proof: We remark that the mapping $a_i \mapsto b_i$ in condition 2 is an isomorphism from $(a_i)_{i<\alpha}$ onto $(b_i)_{i<\alpha}$. The proof that 2 follows from 2' is very much similar to the proof that 3 follows from 3', thus we only present the latter one.

We denote $\mathscr{M} = \mathscr{M}^* \restriction_\tau$. Let $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ and $f : \mathscr{A} \to \mathscr{B}$ be as in condition 3. Let $\mathscr{M} = \bigcup_{i<\mu} \mathscr{N}_i$, where $\mathscr{A} \cup \mathscr{B} \subset \mathscr{N}_0$, $(\mathscr{N}_i)_{i<\mu}$ is an increasing $\preccurlyeq_{\mathbb{K}}$-chain and $|\mathscr{N}_i| < \mu$ for all $i < \mu$. We can find this chain because $\mathrm{LS}(\mathbb{K}) = \omega$. We may also assume that for limit i, $\mathscr{N}_i = \bigcup_{j<i} \mathscr{N}_j$.

Then define another increasing $\preccurlyeq_{\mathbb{K}}$-chain $(\mathscr{M}_i)_{i<\mu}$, and an increasing chain of partial mappings $f_i : \mathscr{M}_i \to \mathscr{M}_i$ by induction. We want that also $|\mathscr{M}_i| < \mu$, $\mathscr{M}_i \preccurlyeq_{\mathbb{K}} \mathscr{M}$, $\mathscr{N}_i \subset \mathrm{dom}(f_i) \cap \mathrm{rng}(f_i)$, $\mathrm{dom}(f_i) \preccurlyeq_{\mathbb{K}} \mathscr{M}$ and that for all $\bar{a} \in \mathrm{dom}(f_i)$
$$t^g_{\mathscr{M}}(\bar{a}/\emptyset) = t^g_{\mathscr{M}}(f_i(\bar{a})/\emptyset)$$
for all $i < \mu$.

Let $\mathscr{M}_0 = \mathscr{N}_0, f_0 = f$. When i is limit, we take $\mathscr{M}_i = \bigcup_{j<i} \mathscr{M}_j$ and $f_i = \bigcup_{j<i} f_j$. Now because μ is regular, we have $|\mathscr{M}_i| < \mu$.

Let $i = j+1$. Let $\mathscr{C}_1 \in \mathbb{K}$ be an AE-submodel of $\mathscr{M}$ of size strictly less that μ containing both $\mathscr{M}_j$ and $\mathscr{N}_i$. Now f_j is a partial mapping from $\mathscr{C}_1$ to $\mathscr{C}_1$, $\mathrm{dom}(f_j) \preccurlyeq_{\mathbb{K}} \mathscr{C}_1$ and from the property 3' we get $\mathscr{D}_1 \preccurlyeq_{\mathbb{K}} \mathscr{M}$, $|\mathscr{D}_1| < \mu$, and an AE-embedding $g : \mathscr{C}_1 \to \mathscr{D}_1$ extending f_j.

Let $\mathscr{C}_2 \preccurlyeq_{\mathbb{K}} \mathscr{M}$ be a model containing all $\mathscr{N}_i$, $\mathscr{C}_1$ and $\mathscr{D}_1$, $|\mathscr{C}_2| < \mu$. Now $g^{-1} : \mathrm{rng}(g) \to \mathscr{C}_2$ is an AE-embedding. Because $\mathrm{dom}(g^{-1}) = g(\mathscr{C}_1) \preccurlyeq_{\mathscr{M}} \mathscr{M}$, we get from Lemma 1.10 and property 3' a model $\mathscr{D}_2 \preccurlyeq_{\mathbb{K}} \mathscr{M}$, $|\mathscr{D}_2| < \mu$ and an AE-embedding $h : \mathscr{C}_2 \to \mathscr{D}_2$ extending g^{-1}.

Then let $\mathscr{M}_i \preccurlyeq_{\mathbb{K}} \mathscr{M}$, $|\mathscr{M}_i| < \mu$ be a model containing both $\mathscr{C}_2$ and $\mathscr{D}_2$ and $f_i = h^{-1}$. Because $f_j \subset g = (g^{-1})^{-1} \subset h^{-1}$, f_i extends f_j. Also because $\mathscr{N}_i \subset \mathrm{dom}(g)$ and $\mathscr{N}_i \subset \mathrm{dom}(h) = \mathrm{rng}(h^{-1})$, we get that $\mathscr{N}_i \subset \mathrm{dom}(f_i) \cap \mathrm{rng}(f_i)$. As before we also see that because f_i and h are an AE-embeddings and $\mathrm{dom}(f_i) = h(\mathscr{C}_2) \preccurlyeq_{\mathbb{K}} \mathscr{M}$, for all $\bar{a} \in \mathrm{dom}(f_i)$
$$t^g_{\mathscr{M}}(\bar{a}/\emptyset) = t^g_{\mathscr{M}}(f_i(\bar{a})/\emptyset).$$

Finally we take $F = \bigcup_{i<\mu} f_i$. Now $\mathscr{M} \subset \mathrm{dom}(F) \cap \mathrm{rng}(F)$ and because for all $\bar{a} \in \mathrm{dom}(F)$
$$t^g_{\mathscr{M}}(\bar{a}/\emptyset) = t^g_{\mathscr{M}}(F(\bar{a})/\emptyset),$$
F is an automorphism of $\mathscr{M}$. Also F extends f. □

2 Splitting

From now on we will assume that everything takes place in a large enough monster model. If we say that $\mathscr{A}$ is a model, we mean that $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$. We also assume that we can apply the homogeneity and universality properties of Theorem 1.16 to every model and set under discussion.

Definition 2.1 (Galois type) *We write $t^g(\bar{a}/A) = t^g(\bar{b}/A)$ if there is $f \in \mathrm{Aut}(\mathfrak{M})$ such that $f \restriction_A = id$ and $f(\bar{a}) = \bar{b}$.*

Remark 2.2 *For all $\bar{a}$ and $\bar{b}$, $t^g(\bar{a}/\emptyset) = t^g(\bar{b}/\emptyset)$ if and only if $t^g_{\mathfrak{M}}(\bar{a}/\emptyset) = t^g_{\mathfrak{M}}(\bar{b}/\emptyset)$.*

Proof: The other direction is trivial. We prove the direction

$$t^g_{\mathfrak{M}}(\bar{a}/\emptyset) = t^g_{\mathfrak{M}}(\bar{b}/\emptyset) \Rightarrow t^g(\bar{a}/\emptyset) = t^g(\bar{b}/\emptyset).$$

Let $\mathscr{C} \in \mathbb{K}$ and $f : \mathfrak{M} \to \mathscr{C}$, $g : \mathfrak{M} \to \mathscr{C}$ be AE-embeddings such that $f(\bar{a}) = g(\bar{b})$. Now let $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}, \mathscr{B} \preccurlyeq_{\mathbb{K}} \mathfrak{M}, \mathscr{C}' \preccurlyeq_{\mathbb{K}} \mathscr{C}$ be such that $\bar{a} \in \mathscr{A}, \bar{b} \in \mathscr{B}, f(\mathscr{A}) \cup g(\mathscr{B}) \in \mathscr{C}'$ and $\max\{|\mathscr{A}|, |\mathscr{B}|, |\mathscr{C}'|\} < \mu$. Then because $\mathfrak{M}$ is μ-universal, there is an AE-embedding $h : \mathscr{C}' \to \mathfrak{M}$.

Now because $h \circ f : \mathscr{A} \to \mathfrak{M}$ and $h \circ g : \mathscr{B} \to \mathfrak{M}$ are AE-embeddings, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ and $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$, we get from lemma 1.10, that for all $\bar{c} \in \mathscr{A}$

$$t^g_{\mathfrak{M}}(\bar{c}/\emptyset) = t^g_{\mathfrak{M}}(h \circ f(\bar{c})/\emptyset),$$

and similarly for $h \circ g$ and $\mathscr{B}$. Thus by the property 3 of Theorem 1.16, both $h \circ f$ and $h \circ g$ extend to $F, G \in \mathrm{Aut}(\mathfrak{M})$ respectively. Now $G^{-1} \circ F \in \mathrm{Aut}(\mathfrak{M})$ and $(G^{-1} \circ F)(\bar{a}) = ((h \circ g)^{-1} \circ (h \circ f))(\bar{a}) = (g^{-1} \circ f)(\bar{a}) = g^{-1}(f(\bar{a})) = \bar{b}$. $\square$

Definition 2.3 (Weak type) *Let $\mathscr{A} \in \mathbb{K}$ and $\bar{a}, \bar{b}, A$ be in $\mathscr{A}$. We write $t^w_{\mathscr{A}}(\bar{a}/A) = t^w_{\mathscr{A}}(\bar{b}/A)$ if $t^g_{\mathscr{A}}(\bar{a}^\frown\bar{c}/\emptyset) = t^g_{\mathscr{A}}(\bar{b}^\frown\bar{c}/\emptyset)$ for every finite $\bar{c} \in A$.*

When we work inside the monster model $\mathfrak{M}$, we just write $t^w(\bar{a}/A)$ instead of $t^w_{\mathfrak{M}}(\bar{a}/A)$. From Remark 2.2 we get the following:

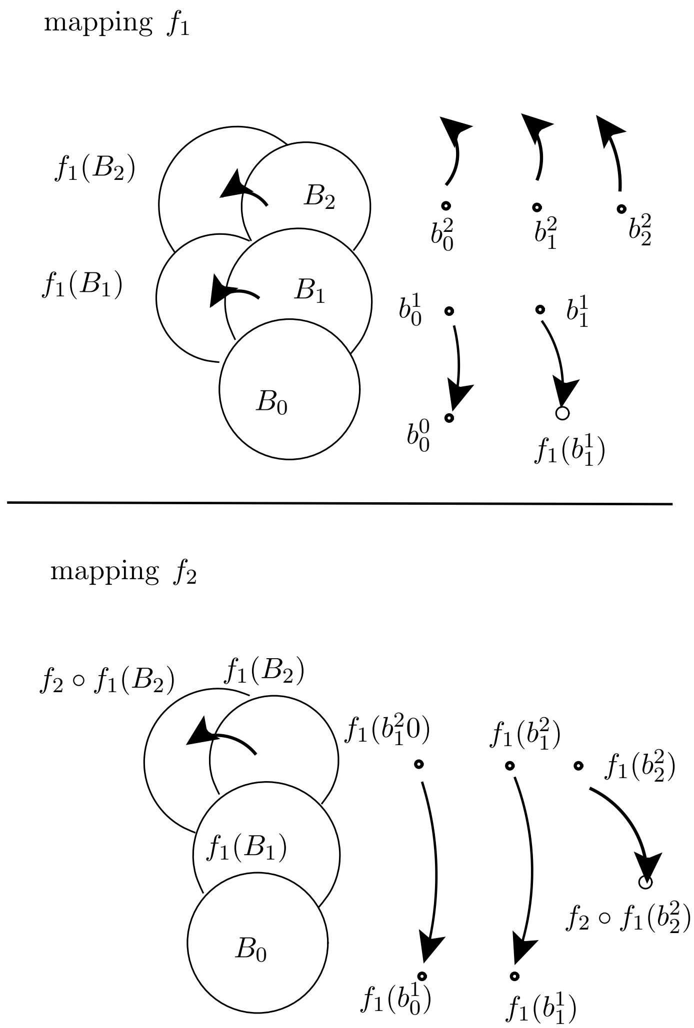
Remark 2.4 *We have that $t^w(\bar{a}/A) = t^w(\bar{b}/A)$ if and only if $t^g(\bar{a}/B) = t^g(\bar{b}/B)$ for every finite $B \subset A$.*

Lemma 2.5 *Assume that $\mathscr{A}$ is a model, $(\bar{b}^i)_{i<\omega}$ is such that $lg(\bar{b}^i) = i + 1$ for all $i < \omega$ and $\mathscr{A} = \bigcup_{i<\omega} B_i$, where $i < j \Rightarrow B_i \subset B_j$ and when we denote $\bar{b}^j = (b^j_0, ..., b^j_j)$,*

$$i < j \Rightarrow t^g((b^j_0, ..., b^j_i)/B_i) = t^g(\bar{b}^i/B_i).$$

Then there is $(a_i)_{i<\omega}$ such that $t^g((a_0, ..., a_i)/B_i) = t^g(\bar{b}^i/B_i)$ for all $i < \omega$.

Figure 2: Picture for Lemma 2.5.



Proof: Define $f_n \in \mathrm{Aut}(\mathfrak{M})$ for $n \in \omega$ such that when we denote $g_n = f_n \circ f_{n-1} \circ ... \circ f_0$

1. $f_{n+1}(g_n(b_i^{n+1})) = g_i(b_i^i)$ for all $0 \leq i \leq n$,

2. $f_{n+1} \restriction_{g_n(B_n)} = \mathrm{Id}_{g_n(B_n)}$.

First let $f_0 = g_0 = \mathrm{Id}_{\mathfrak{M}}$.

Assume we have defined f_i for $i \leq n$. Let $h \in \mathrm{Aut}(\mathfrak{M})$ be such that $h(b_0^{n+1}, ..., b_n^{n+1}) = (b_0^n, ..., b_n^n)$ and $h \restriction_{B_n} = \mathrm{Id}_{B_n}$. Then let

$$f_{n+1} = g_n \circ h \circ {g_n}^{-1}.$$

Clearly $f_{n+1} \restriction_{g_n(B_n)} = (g_n \circ h \circ {g_n}^{-1}) \restriction_{g_n(B_n)} = \mathrm{Id}_{g(B_n)}$. Now if $n = 0$, we have $f_{n+1} = f_1 = h$ and $f_1(b_0^1) = b_0^0 = g_0(b_0^0)$.

If $n > 0$ we have by induction that for $0 \leq i < n$, $f_n(g_{n-1}(b_i^n)) = g_i(b_i^i)$, and hence we may write $f_{n+1}(g_n(b_i^{n+1})) = g_n(h({g_n}^{-1}(g_n(b_i^{n+1})))) = g_n(h(b_i^{n+1})) = g_n(b_i^n) = f_n(g_{n-1}(b_i^n)) = g_i(b_i^i)$. Also $f_{n+1}(g_n(b_n^{n+1})) = g_n \circ h \circ {g_n}^{-1}(g_n(b_n^{n+1})) = g_n(h_n(b_n^{n+1})) = g_n(b_n^n)$.

Now we have defined $f_n \in \mathrm{Aut}(\mathfrak{M}), n < \omega$, such that they satisfy conditions 1 and 2. We see that by condition 2, when $n \leq m$,

$$g_m \restriction_{B_n} = g_n \restriction_{B_n}.$$

Thus when we denote $A' = \bigcup_{n<\omega} g_n(B_n)$,

$$g = \bigcup_{n<\omega} (g_n \restriction_{B_n}) : \mathscr{A} \to A'$$

is a mapping such that for all $\bar{a} \in \mathscr{A}$

$$t^g(\bar{a}/\emptyset) = t^g(g(\bar{a})/\emptyset).$$

Now we can extend g to $G \in \mathrm{Aut}(\mathfrak{M})$. Let

$$a_i = G^{-1}(g_i(b_i^i)) \text{ for all } i < \omega.$$

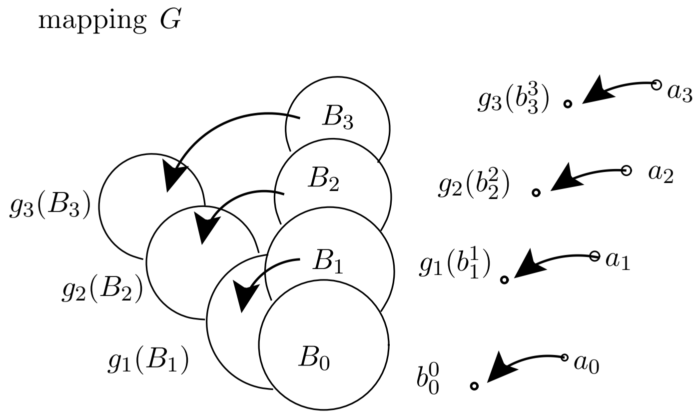
Now when $n \in \omega$, we have $(g - n^{-1} \circ G) \restriction_{B_n} = ({g_n}^{-1} \circ g_n) \restriction_{B_n} = \mathrm{Id}_{B_n}$, $g_0 \circ G(a_0) = g_0({g_0}^{-1} b_0) = b_0$ and when $n > 0$, $0 \leq i < n$, $({g_n}^{-1} \circ G)(a_i) = {g_n}^{-1}(g_i(b_i^i)) = ({g_{n-1}}^{-1} \circ {f_n}^{-1})(g_i(b_i^i)) = {g_{n-1}}^{-1}(g_{n-1}(b_i^n)) = b_i^n$, $({g_n}^{-1} \circ G)(a_n) = {g_n}^{-1}(g_n(b_n^n)) = b_n^n$.

Thus we get

$$t^g((a_0, ..., a_n)/\emptyset) = t^g(\bar{b}^n/\emptyset).$$

□

Figure 3: Picture for Lemma 2.5.



Corollary 2.6 *Assume that $\mathscr{A}$ is a model, $(\bar{b}_i)_{i<\omega}$ a sequence of tuples and $\mathscr{A} = \bigcup_{i<\omega} B_i$, where $i < j \Rightarrow B_i \subset B_j$ and*

$$i < j \Rightarrow t^g(\bar{b}_j/B_i) = t^g(\bar{b}_i/B_i).$$

Then there is $\bar{a}$ such that $t^g(\bar{a}/B_i) = t^g(\bar{b}_i/B_i)$ for all $i < \omega$.

Proof: Let $n = lg(\bar{b_0}) = lg(\bar{b_i})$ for all $i < \omega$. Let $C_i = B_0$ for $0 \leq i \leq n$ and $C_{i+n} = B_i$ for $i+n > n$. When $\bar{b_i} = (b_{0,i}, ...b_{n-1,i})$, we let $\bar{c}^i = (b_{0,0}, ..., b_{i,0})$, when $i \leq n$ and $\bar{c}^{n+i} = (b_{0,i}, ..., b_{n,i}, ..., b_{n,i})$, when $n+i > n$. Now the assumptions of Lemma 2.5 hold for $\mathscr{A} = \bigcup_{i<\omega} C_i$ and $(\bar{c}^i)_{i<\omega}$. We get $(a_i)_{i<\omega}$ such that when $n \leq i < \omega$, $t^g((a_0, ..., a_i)/B_i) = t^g((b_{0,i}, ..., b_{n,i}, ..., b_{n,i})/B_i)$. Thus we get $(a_0, ..., a_n)$ such that $t^g((a_0, ..., a_n)/B_i) = t^g(\bar{b}_i/B_i)$ for all $i < \omega$. $\square$

Definition 2.7 (Splitting) *Let $\bar{a}$ and A be in $\mathfrak{M}$.*[2] *We say that the weak type $t^w(\bar{a}/A)$ splits over finite $B \subset A$ if there are $\bar{c}, \bar{d} \in A$ such that*

$$t^g(\bar{c}/B) = t^g(\bar{d}/B) \text{ but}$$

$$t^g(\bar{c}/B \cup \{\bar{a}\}) \neq t^g(\bar{d}/B \cup \{\bar{a}\}).$$

We say that such $\bar{c}, \bar{d}$ witness the fact.

The next remark is only to note that these definitions are sensible.

[2] In the general definition, where $\bar{a}$ and A are in some $\mathscr{A} \in \mathbb{K}$, we just replace $t^w(\bar{a}/A)$ with $t^w_{\mathscr{A}}(\bar{a}/A)$ and the Galois types with $\mathscr{A}$-Galois types respectively.

Remark 2.8 *If $t^w(\bar{a}/A) = t^w(\bar{b}/A)$ and $B \subset A$ is finite, then $t^w(\bar{a}/A)$ splits over B if and only if $t^w(\bar{b}/A)$ splits over B.*

Proof: Let $\bar{c}$ and $\bar{d}$ witness that $t^w(\bar{a}/A)$ splits over B. Now $\{\bar{c},\bar{d}\} \cup B \subset A$ is finite and thus there is $f \in \mathrm{Aut}(\mathfrak{M})$ such that $f(\bar{a}) = \bar{b}$ and $f \restriction_{\{\bar{c},\bar{d}\} \cup B}$ is the identity. If there would be an automorphism g such that $g(\bar{c}) = \bar{d}$ and $g \restriction_{\{\bar{b}\} \cup B} = \mathrm{Id}_{\{\bar{b}\} \cup B}$, then also $(f^{-1} \circ g \circ f)(\bar{c}) = \bar{d}$ but $(f^{-1} \circ g \circ f) \restriction_{\{\bar{a}\} \cup B}$ is the identity. This contradicts $\bar{c}$ and $\bar{d}$ being witnesses. Thus $\bar{c}$ and $\bar{d}$ also witness that $t^w(\bar{b}/A)$ splits over B. □

Remark 2.9 *If $t^w(\bar{a}/A)$ splits over finite $E \subset A$ and $E' \subset E$, then $t^w(\bar{a}/A)$ splits over E'.*

Proof: Let $\bar{c}, \bar{d} \in A$ witness that $t^w(\bar{a}/A)$ splits over E. Denote by $\bar{c}^\frown E$ the finite tuple where E is indexed after $\bar{c}$ in some chosen order. Now $t^g(\bar{c}^\frown E/E') = t^g(\bar{d}^\frown E/E')$ but $t^g(\bar{c}^\frown E/E' \cup \{\bar{a}\}) \neq t^g(\bar{d}^\frown E/E' \cup \{\bar{a}\})$. Thus the finite tuples $\bar{c}^\frown E$ and $\bar{d}^\frown E$ in A witness that $t^w(\bar{a}/A)$ splits over E'. □

Definition 2.10 (Independence) *Let $\bar{a}, A$ and B be in $\mathfrak{M}$*[3]*. We write that $\bar{a} \downarrow^s_A B$ if there is finite $C \subset A$ such that $t^w(\bar{a}/A \cup B)$ does not split over C.*

Now we introduce a new assumption for $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. From now on we will assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an ω-stable local abstract elementary class.

Assumption 2.11 (ω-stability) *If $A \subset \mathscr{A} \in \mathbb{K}$, A is countable and $\bar{a}_i \in \mathscr{A}$ for $i < \omega_1$, then for some $i < j < \omega_1$ $t^w_{\mathscr{A}}(\bar{a}_i/A) = t^w_{\mathscr{A}}(\bar{a}_j/A)$.*

Lemma 2.12 *Let $\mathscr{A}$ be a model and and $\bar{a}$ a tuple. There is no increasing chain of finite sets $(B_i)_{i<\omega}$ such that $\mathscr{A} = \bigcup_{i<\omega} B_i$ and $t^w(\bar{a}/B_{i+1})$ splits over B_i for all $i < \omega$.*

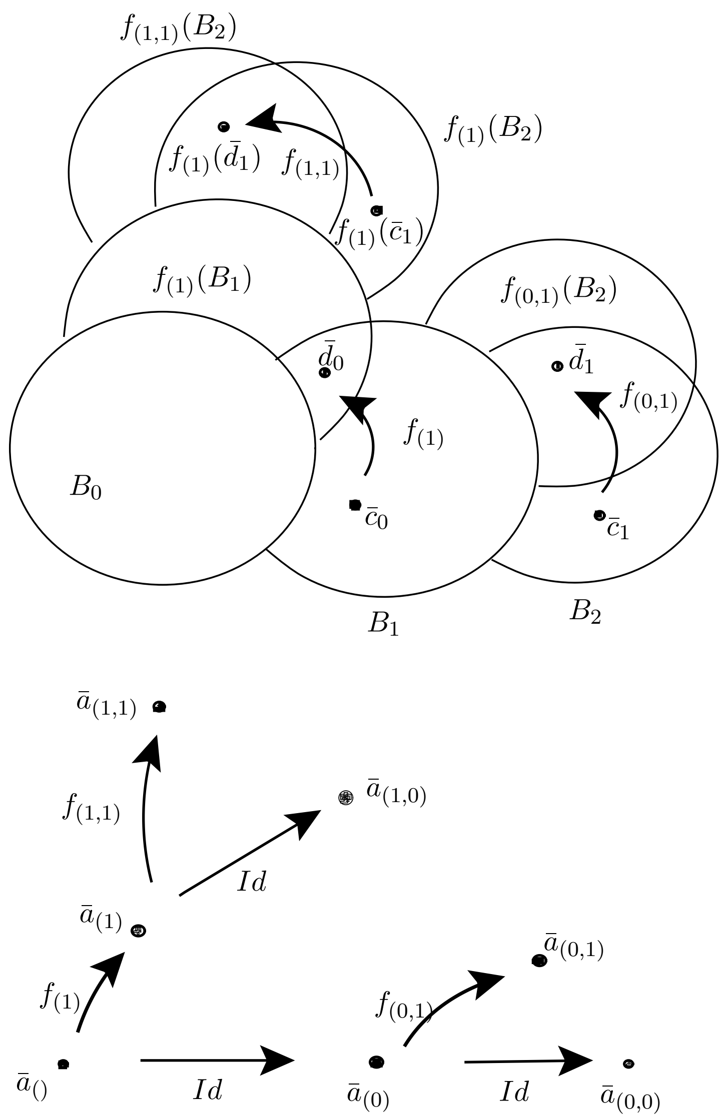
Proof: We assume the contrary. Let $\bar{a}$ and $\mathscr{A} = \bigcup_{n<\omega} B_n$ witness this. Also let $c_n, d_n \in B_{n+1}$ witness that $t^w(\bar{a}/B_{n+1})$ splits over B_n.

Let $\eta : \omega \to 2$ be a mapping. Then for each $n \in \omega$ we define mappings $f_{\eta\restriction_n} \in \mathrm{Aut}(\mathfrak{M})$ such that when we denote $g_{\eta\restriction_n} = f_{\eta\restriction_n} \circ ... \circ f_{\eta\restriction_0}$

1. If $m > n$, $f_{\eta\restriction_m} \restriction_{g_{\eta\restriction_n}(B_n)} = \mathrm{Id}_{g_{\eta\restriction_n}(B_n)}$.

[3]In the general case, where $\bar{a}, A$ and B are in some $\mathscr{A} \in \mathbb{K}$, replace $t^w(\bar{a}/A \cup B)$ with $t^w_{\mathscr{A}}(\bar{a}/A \cup B)$.

Figure 4: Picture for Lemma 2.12.



2. If $\eta(n)=1$, then $f_{\eta\restriction_{n+1}}(g_{\eta\restriction_n}(c_n))=g_{\eta\restriction_n}(d_n)$.

3. If $\eta(n)=1$, then
$$t^g(g_{\eta\restriction_{n+1}}(\bar a)/g_{\eta\restriction_n}(B_n\cup d_n))\neq t^g(g_{\eta\restriction_n}(\bar a)/g_{\eta\restriction_n}(B_n\cup d_n)).$$

4. If $\eta(n)=0$, then $f_{\eta\restriction_{n+1}}=\mathrm{Id}_{\mathfrak{M}}$.

We define such mappings by induction on n as follows:

First let $f_{\eta\restriction_0}=g_{\eta\restriction_0}=\mathrm{Id}_{\mathfrak{M}}$. Also when $\eta(n)=0$, let $f_{\eta\restriction_{n+1}}=\mathrm{Id}_{\mathfrak{M}}$.

Assume $\eta(n)=1$ and we have defined mappings $f_{\eta\restriction_i}$ for $i\le n$. Let $h\in\mathrm{Aut}(\mathfrak{M})$ be such that $h(c_n)=d_n$ and $h\restriction_{B_n}=\mathrm{Id}_{B_n}$. Then let
$$f_{\eta\restriction_{n+1}}=g_{\eta\restriction_n}\circ h\circ {g_{\eta\restriction_n}}^{-1}.$$

Now clearly $f_{\eta\restriction_{n+1}}$ satisfies conditions 1 and 2. To prove that it satisfies also 3, we assume the contrary. Assume that there exists $g\in\mathrm{Aut}(\mathfrak{M})$ such that $g(g_{\eta\restriction_{n+1}}(\bar a))=g_{\eta\restriction_n}(\bar a)$ and $g(x)=x$ for all $x\in g_{\eta\restriction_n}(B_n\cup d_n)$. Then when we denote $g^*={g_{\eta\restriction_n}}^{-1}\circ g\circ g_{\eta\restriction_{n+1}}\in\mathrm{Aut}(\mathfrak{M})$, we see that $g^*(c_n)={g_{\eta\restriction_n}}^{-1}\circ g\circ f_{\eta\restriction_{n+1}}(g_{\eta\restriction_n}(c_n))={g_{\eta\restriction_n}}^{-1}\circ g(g_{\eta\restriction_n}(d_n))={g_{\eta\restriction_n}}^{-1}(g_{\eta\restriction_n}(d_n))=d_n$ and that because $f_{\eta\restriction_{n+1}}(x)=x$ for all $x\in g_{\eta\restriction_n}(B_n)$, $g^*(x)=x$ for all $x\in B_n$. But we have also that $g^*(\bar a)=\bar a$, thus this contradicts c_n and d_n being witnesses.

Denote $\bar a_{\restriction_n}=g_{\eta\restriction_n}(\bar a)$ for all $n\in\omega$. When $m>n$ the mapping $f_{\eta\restriction_m}\circ\ldots\circ f_{\eta\restriction_{n+1}}$ shows that
$$t^g(\bar a_{\eta\restriction_m}/g_{\eta\restriction_n}(B_n))=t^g(\bar a_{\eta\restriction_n}/g_{\eta\restriction_n}(B_n)).$$

Also when $\mathscr{A}_\eta=\bigcup_{n<\omega}g_{\eta\restriction_n}(B_n)$, the mapping $g=\bigcup_{n<\omega}(g_{\eta\restriction_n}\restriction_{B_n}):\mathscr{A}\to\mathscr{A}_\eta$ has the property that, for all $\bar c\in\mathscr{A}$, $t^g(\bar c/\emptyset)=t^g(g(\bar c)/\emptyset)$. We notice that now g is an AE-embedding and $g(\mathscr{A})=\mathscr{A}_\eta\preccurlyeq_{\mathbb{K}}\mathfrak{M}$. Thus from Corollary 2.6 we get $\bar a_\eta$ such that for all $n\in\omega$
$$t^g(\bar a_\eta/g_{\eta\restriction_n}(B_n))=t^g(\bar a_{\eta\restriction_n}/g_{\eta\restriction_n}(B_n)).$$

Denote
$$B=\bigcup_{\eta:\omega\to 2}\mathscr{A}_\eta=\bigcup_{\eta:\omega\to 2}(\bigcup_{n\in\omega}(g_{\eta\restriction_n}(B_n)))=\bigcup_{n\in\omega}(\bigcup_{\eta:\omega\to 2}(g_{\eta\restriction_n}(B_n))).$$

Now because $\bigcup_{\eta:\omega\to 2}(g_{\eta\restriction_n}(B_n))$ is finite for all $n\in\omega$, we get that $B\subset\mathfrak{M}$ is countable.

Let η and η' be two different mappings from ω to 2. Let $n=\min\{i<\omega:\eta(i)\neq\eta'(i)\}$. We may assume that $\eta(n)=1$. Because $\eta\restriction_n=\eta'\restriction_n$ and of conditions 1, 2 and 4
$$g_{\eta\restriction_n}(B_n\cup d_n)=g_{\eta'\restriction_n}(B_n\cup d_n)\subset(g_{\eta\restriction_{n+1}}(B_{n+1})\cap g_{\eta'\restriction_{n+1}}(B_{n+1})).$$

Furthermore, by condition 4,

$$g_{\eta'\restriction_{n+1}}(\bar{a}) = g_{\eta'\restriction_n}(\bar{a}) = g_{\eta\restriction_n}(\bar{a}).$$

Now

$$\begin{aligned} & t^g(\bar{a}_\eta/g_{\eta\restriction_n}(B_n \cup d_n)) \\ = \ & t^g(g_{\eta\restriction_{n+1}}(\bar{a})/g_{\eta\restriction_n}(B_n \cup d_n)) \\ \neq \ & t^g(g_{\eta\restriction_n}(\bar{a})/g_{\eta\restriction_n}(B_n \cup d_n)) \\ = \ & t^g(g_{\eta'\restriction_{n+1}}(\bar{a})/g_{\eta'\restriction_n}(B_n \cup d_n)) \\ = \ & t^g(\bar{a}_{\eta'}/g_{\eta'\restriction_n}(B_n \cup d_n)). \end{aligned}$$

Thus we have that when $\eta \neq \eta'$, $t^w(\bar{a}_\eta/B) \neq t^w(\bar{a}_{\eta'}/B)$. Tuples $(\bar{a}_\eta)_{\eta:\omega\to 2}$ have different weak types over countable B, and there are uncountably many of them. This contradicts the ω-stability assumption 2.11. □

Theorem 2.13 ("$\bar{a} \downarrow^s_{\mathscr{A}} \emptyset$") *For all tuples $\bar{a}$ and models $\mathscr{A}$ there is finite $C \subset \mathscr{A}$ such that $t^w(\bar{a}/\mathscr{A})$ does not split over C.*

Proof: Assume the contrary. We define an increasing chain of finite sets $(B_n)_{n<\omega}$ such that $\bigcup_{n<\omega} B_n \preccurlyeq_{\mathbb{K}} \mathscr{A}$ and that $t^w(\bar{a}/B_{n+1})$ splits over B_n for all $n < \omega$. Then also $\bigcup_{n<\omega} B_n \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ and such a chain contradicts Lemma 2.12.

By Lemma 1.14 we may find $\mathscr{A}^* \in \mathbb{K}^*$ such that $\mathscr{A}^* \restriction_\tau = \mathscr{A}$. Without less of generality we may assume that $\mathscr{A}^*$ is a submodel of $\mathfrak{M}^*$. Let $B_0 = \emptyset$. Assume that we have defined B_i for $i \leq n$ as planned, and that each B_{i+1} is closed under functions $(F^m_k)^{\mathscr{A}^*}$ for $m, k \leq i$. Let $c_n, d_n \in \mathscr{A}$ witness that $t^w(\bar{a}/\mathscr{A})$ splits over finite B_n. We can take $B'_{n+1} = B_n \cup \{c_n, d_n\}$ and then B_{n+1} to be the closure of the finite set B'_{n+1} under finitely many functions $(F^m_k)^{\mathscr{A}^*}$, $k, m \leq n$.

Finally we get that $\bigcup_{n<\omega} B_n \subset \mathscr{A}$ is closed under functions $(F^m_k)^{\mathscr{A}^*}$ for all $m, k \in \omega$ and thus is a $\preccurlyeq_{\mathbb{K}}$-submodel of $\mathscr{A}^* \restriction_\tau = \mathscr{A}$. □

We first define ω-saturation in the class $\mathbb{K}$, but use Definition 2.15 in our context.

Definition 2.14 *We say that a model $\mathscr{A} \in \mathbb{K}$ is ω**-saturated in** $\mathbb{K}$ if for all $\bar{a} \in \mathscr{A}$ and $\bar{b}$ in some AE-elementary extension $\mathscr{B}$ of $\mathscr{A}$ there is $\bar{d} \in \mathscr{A}$ such that $t^g_{\mathscr{A}}(\bar{a}^\frown\bar{d}/\emptyset) = t^g_{\mathscr{B}}(\bar{a}^\frown\bar{b}/\emptyset)$.*

Definition 2.15 *We say that a submodel $A \subset \mathfrak{M}$ is ω**-saturated** if for all $\bar{a} \in \mathfrak{M}$ and finite $B \subset A$ there is $\bar{b} \in A$ such that $t^g(\bar{a}/B) = t^g(\bar{b}/B)$.*

Remark 2.16 *A model* $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$*,* $|\mathscr{A}| < \mu$*, is* ω*-saturated if and only if it is* ω*-saturated in* $\mathbb{K}$*.*

Proof: First assume that $\mathscr{A}$ is ω saturated and show that it is also ω-saturated in $\mathbb{K}$. Let $\mathscr{B}$ be some AE-elementary extension of $\mathscr{A}$, $\bar{b} \in \mathscr{B}$ and $\bar{a} \in \mathscr{A}$. Because $\mathfrak{M}$ is universal, there is an AE-embedding $f : \mathscr{B} \to \mathfrak{M}$. Isomorphism $f^{-1} \restriction_{f(\mathscr{A})} f(\mathscr{A}) \to \mathscr{A}$ extends to an automorphism F. Now $F(f(\bar{b})) \in \mathfrak{M}$ and $F(f(\bar{a})) = \bar{a}$. From ω-saturation we get $G \in \mathrm{Aut}(\mathfrak{M})$ such that $G(\bar{a}) = \bar{a}$ and $G(F(f(\bar{b}))) \in \mathscr{A}$. Now AE-embeddings $F \circ f : \mathscr{B} \to \mathfrak{M}$ and $G^{-1} \restriction_{\mathscr{A}} : \mathscr{A} \to \mathfrak{M}$ show that $t^g_{\mathscr{B}}(\bar{b}^\frown \bar{a}/\emptyset) = t^g_{\mathscr{A}}(G(F(f(\bar{b})))^\frown \bar{a}/\emptyset)$.

Then assume that $\mathscr{A}$ is ω-saturated in $\mathbb{K}$. Let $\bar{b} \in \mathfrak{M}$ and $\bar{a} \in \mathscr{A}$. Let also $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ be such that $\mathscr{A} \cup \{\bar{b}\} \subset \mathscr{B}$ and $|\mathscr{B}| < \mu$. Now $\mathscr{B}$ is an AE-elementary extension of $\mathscr{A}$ and thus there are $\mathscr{C} \in \mathbb{K}$, $\bar{d} \in \mathscr{A}$ and AE-embeddings $f : \mathscr{A} \to \mathscr{C}$ and $g : \mathscr{B} \to \mathscr{C}$ such that $f(\bar{a}^\frown \bar{d}) = g(\bar{a}^\frown \bar{b})$. By the universality of $\mathfrak{M}$ we may assume that $\mathscr{C} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$. Both AE-embeddings extend to automorphisms F and G respectively. Now $F^{-1} \circ G \in \mathrm{Aut}(\mathfrak{M})$ and $F^{-1} \circ G(\bar{a}^\frown \bar{b}) = \bar{a}^\frown \bar{d}$. □

Another remark tells us that ω-saturated models do exists.

Remark 2.17 *Let* A *be a set. There is an* ω*-saturated model* $\mathscr{A}$ *such that* $A \subset \mathscr{A}$ *and* $|\mathscr{A}| \leq |A| + \aleph_0$*.*

Proof: We construct a countable increasing chain of models $\mathscr{A}_n$ such that $A \subset \mathscr{A}_0$, $|\mathscr{A}_n| \leq |A| + \aleph_0$ and for all finite $B \in \mathscr{A}_n$ and $\bar{a}$ there is $\bar{b} \in A_{n+1}$ such that $t^g(\bar{b}/B) = t^g(\bar{a}/B)$. Then we may take $\mathscr{A} = \bigcup_{n<\omega} \mathscr{A}_n$.

First let $\mathscr{A}_0$ be such that $A \subset \mathscr{A}_0$ and $|\mathscr{A}_0| \leq |A| + \aleph_0$. Assume we have defined $\mathscr{A}_m$ for $m \leq n$. When $B \subset \mathscr{A}_n$ is finite, we note that by ω-stability, there are at most countably many $\bar{a}$ with different weak type over B. Because B is finite, Galois type over B agrees with weak type over B. Thus for each finite $B \subset \mathscr{A}_n$ there is a countable set D_B, where every Galois type over B is represented. Then denote $\mathscr{D} = \{B \subset \mathscr{A}_n : B \text{ finite}\}$. We have that $|\mathscr{D}| \leq |\mathscr{A}_n| + \aleph_0 \leq |A| + \aleph_0$. We may take a model $\mathscr{A}_{n+1}$ such that $\bigcup_{B \in \mathscr{D}} D_B \subset \mathscr{A}_{n+1}$ and $|\mathscr{A}_n| \leq |\bigcup_{B \in \mathscr{D}} D_B| \leq |A| + \aleph_0$. □

The next lemma will show that an countable ω-saturated substructure of $\mathfrak{M}$ is actually AE-elementary.

Lemma 2.18 *Assume* A *is a countable set and the following holds: for all* $\bar{a} \in A$ *and* $\bar{b}$ *there is* $\bar{d} \in A$ *such that* $t^g(\bar{a}^\frown \bar{b}/\emptyset) = t^g(\bar{a}^\frown \bar{d}/\emptyset)$*. Then* A *is a model.*

Proof: Let $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ be countable and ω-saturated. Let $A = \{a_n : n < \omega\}$ and $\mathscr{B} = \{b_n : n < \omega\}$. Define inductively sets A_n and B_n and automorphisms f_n such that for all $n < \omega$

1. $f_n(A_n) = B_n$,

2. $\{a_0, ..., a_{n-1}\} \subset A_n \subset A$ and $\{b_0, ..., b_{n-1}\} \subset B_n \subset \mathscr{B}$.

Let $f_0 = Id \restriction_{\mathfrak{M}}$, $A_0 = \emptyset$ and $B_0 = \emptyset$. Then assume we have defined f_m, A_m and B_m for $m \leq n$.

By ω-saturation there exists $g \in \mathrm{Aut}(\mathfrak{M})$ such that $g(f_n(a_n)) \in \mathscr{B}$ and $g \restriction_{B_n} = \mathrm{Id}_{B_n}$. Then by the assumption there exists $h \in \mathrm{Aut}(\mathfrak{M})$ such that $h \restriction_{A_n \cup \{a_n\}}$ is the identity and $h(f_n^{-1} \circ g^{-1}(b_n)) \in A$. Define

$$\begin{aligned} f_{n+1} &= g \circ f_n \circ h^{-1}, \\ A_{n+1} &= A_n \cup \{a_n\} \cup \{(h \circ f_n^{-1} \circ g^{-1})(b_n)\} \text{ and} \\ B_{n+1} &= B_n \cup \{(g \circ f_n)(a_n)\} \cup \{b_n\}. \end{aligned}$$

Then we get that $f_{n+1}(A_{n+1}) = B_{n+1}$.

Finally $f = \bigcup_{n<\omega} (f_n)^{-1} \restriction_{B_n} : \mathscr{B} \to \mathfrak{M}$ is an AE-embedding because it satisfies the property 1.1 of Lemma 1.10. Thus $f(\mathscr{B}) = A \preccurlyeq_{\mathbb{K}} \mathfrak{M}$. □

From the previous construction we can also get the following lemma. The latter part we get by taking $A_0 = B_0 = E$ in the construction.

Lemma 2.19 *Let $\mathscr{A}$, $\mathscr{B}$ be ω-saturated and countable models. Then there is $f \in \mathrm{Aut}(\mathfrak{M})$ such that $f(\mathscr{A}) = \mathscr{B}$. Also if $E \subset \mathscr{A} \cap \mathscr{B}$ is finite, we can take $f \restriction_E = \mathrm{Id}_E$.*

In the following theorem we prove some basic properties for splitting. The version of existence of free extension is in section 3 found to be too restricted, and we then discuss ways to improve it.

Theorem 2.20 *Assume that $A \subset B \subset C \subset D$ and similarly if one of the sets is a model, and thus denoted with a curly letter.*

1. ***Monotonicity*** *If $\bar{a} \downarrow^s_A D$, then $\bar{a} \downarrow^s_B C$.*

2. ***Restricted existence of free extension*** *Assume $\mathscr{A}$ is an ω-saturated model and B is countable. For all $\bar{a}$, if $E \subset \mathscr{A}$ is finite and $t^w(\bar{a}/\mathscr{A})$ does not split over E, then there is $\bar{b}$ such that $t^w(\bar{a}/\mathscr{A}) = t^w(\bar{b}/\mathscr{A})$ and $t^w(\bar{b}/B)$ does not split over E.*

3. ***Uniqueness of free extension*** *Assume $\mathscr{A}$ is an ω-saturated model. If $t^w(\bar{a}/\mathscr{A}) = t^w(\bar{b}/\mathscr{A})$, $\bar{a} \downarrow^s_{\mathscr{A}} B$ and $\bar{b} \downarrow^s_{\mathscr{A}} B$, then $t^w(\bar{a}/B) = t^w(\bar{b}/B)$.*

4. ***Transitivity*** *If $\mathscr{B}$ is an ω-saturated model and C is countable, then $\bar{a} \downarrow^s_A C$ if and only if $\bar{a} \downarrow^s_A \mathscr{B}$ and $\bar{a} \downarrow^s_{\mathscr{B}} C$.*

Proof: **Monotonicity:** Let $C \subset A$ be a finite set such that $t^w(\bar{a}/D)$ does dot split over C. Now $C \subset B$, and if $t^w(\bar{a}/C)$ would split over C, there would be some witnesses $c, d \in B \subset D$, and the same c, d would witness that $t^w(\bar{a}/D)$ splits over C. Thus $t^w(\bar{a}/B)$ does not split over C.

Restricted existence of free extension: Denote $\mathscr{A} = \{a'_i : i < \omega\}$ and $B = E \cup \{b'_i : i < \omega\}$. Define mappings $f_n \in \mathrm{Aut}(\mathfrak{M})$ and elements $a_n, b_n, n < \omega$, such that

1. $f_n \restriction_E = \mathrm{Id}_E$ for all $n < \omega$,
2. $\{b'_0, ..., b'_n\} \subset \mathrm{dom}(f_n)$ and $\{a'_0, ..., a'_n\} \subset \mathrm{rng}(f_n) \subset \mathscr{A}$ for all $n < \omega$.
3. When $n \leq m$, $f_m(b_i) = f_n(b_i) = a_i$ for all $0 \leq i \leq 2n+1$.

Because $\mathscr{A}$ is ω-saturated, there is $f_0 \in \mathrm{Aut}(\mathfrak{M})$ such that $f_0(b'_0) \in \mathscr{A}$ and $f_0 \restriction_E = \mathrm{Id}_E$. Then define $b_0 = b'_0$, $b_1 = f^{-1}(a'_0)$, $a_0 = f_0(b_0)$ and $a_1 = a'_0$. Now $f_0(b_0, b_1) = (a_0, a_1)$ and $b'_0 \in \mathrm{dom}(f_0)$, $a'_0 \in \mathrm{rng}(f_0)$.

Assume we have defined f_i for all $i \leq n$. We use again the ω-saturation of $\mathscr{A}$ to get $g \in \mathrm{Aut}(\mathfrak{M})$ such that $g \restriction_{E \cup \{a_0, ..., a_{2n+1}\}} = \mathrm{Id}_{E \cup \{a_0, ..., a_{2n+1}\}}$ and $g(f_n(b'_{n+1})) \in \mathscr{A}$. We can take $f_{n+1} = g \circ f_n$, $b_{2(n+1)} = b'_{n+1}$, $b_{2(n+1)+1} = {f_{n+1}}^{-1}(a'_{n+1})$, $a_{2(n+1)} = f_{n+1}(b'_{n+1})$ and $a_{2(n+1)+1} = a'_{n+1}$.

Finally we get a mapping

$$f = \bigcup_{i<\omega} ({f_n}^{-1} \restriction_{\{a_0, ..., a_{2n+1}\}}) : \mathscr{A} \to \{b_i : i < \omega\},$$

which has the property that for all $\bar{a} \in \mathscr{A}$ $t^g(\bar{a}/\emptyset) = t^g(f(\bar{a})/\emptyset)$. Now $f : \mathscr{A} \to \mathfrak{M}$ is an AE-embedding and $\{b_i : i < \omega\} = f(\mathscr{A}) \preccurlyeq_{\mathbb{K}} \mathfrak{M}$.

Denote $\bar{c}_n = {f_n}^{-1}(\bar{a})$ for all $n < \omega$. Now when $n \leq m$, ${f_n}^{-1} \circ f_m \restriction_{E \cup \{b_0, ..., b_{2n+1}\}}$ is the identity and ${f_n}^{-1} \circ f_m(\bar{c}_m) = \bar{c}_n$. We can use Corollary 2.6 to get $\bar{b}$ such that for all $n < \omega$

$$t^g(\bar{b}/E \cup \{b_0, ..., b_{2n+1}\}) = t^g(\bar{c}_n/E \cup \{b_0, ..., b_{2n+1}\}).$$

We want to show that this is the $\bar{b}$ we wanted.

Let $c, d \in B$ and $h \in \mathrm{Aut}(\mathfrak{M})$ be such that $h \restriction_E = \mathrm{Id}_E$ and $h(c) = d$. Let n be such that $c, d \in \{b'_0, ..., b'_n\} \subset \mathrm{dom}(f_n) = \{b_0, ..., b_{2n+1}\}$. Then let f be an automorphism such that $f(\bar{b}) = \bar{c}_n$ and $f \restriction_{E \cup \{c,d\}}$ is the identity. Now $(f_n \circ f)(\{c, d\}) = f_n(\{c, d\}) \subset \mathscr{A}$, $(f_n \circ f \circ h \circ (f_n \circ f)^{-1})((f_n \circ f)(c)) = (f_n \circ f)(d)$ and $(f_n \circ f \circ h \circ (f_n \circ f)^{-1}) \restriction_E$ is the identity. Because $t^w(\bar{a}/\mathscr{A})$ does not split over E, we have $h^* \in \mathrm{Aut}(\mathfrak{M})$ such that $h^*((f_n \circ f)(c)) = (f_n \circ f)(d)$ and $h^* \restriction_{E \cup \{\bar{a}\}}$ is the identity. Now $((f_n \circ f)^{-1} \circ h^* \circ (f_n \circ f))(c) = d$, $((f_n \circ f)^{-1} \circ h^* \circ (f_n \circ f)) \restriction_E$ is the identity and $((f_n \circ f)^{-1} \circ h^* \circ (f_n \circ f))(\bar{b}) = f^{-1}({f_n}^{-1}(h^*(f_n(\bar{c}_n)))) = f^{-1}({f_n}^{-1}(h^*(\bar{a}))) = f^{-1}({f_n}^{-1}(\bar{a})) = f^{-1}(\bar{c}_n) = \bar{b}$.

We get that if $c, d \in B$ and $t^g(c/E) = t^g(d/E)$, then also $t^g(c/E\cup\{\bar{b}\}) = t^g(d/E\cup\{\bar{b}\})$. Thus $t^w(\bar{b}/B)$ does not split over E.

Because $f_n((b_0, ..., b_n)) = (a_0, ..., a_n)$, $f_n \restriction_E = Id \restriction_E$, and $t^w(\bar{b}/B)$ does not split over E, there is an automorphism g such that $g(b_0, ..., b_n) = (a_0, ..., a_n)$ and $g \restriction_{E\cup\{\bar{b}\}}$ is the identity. Let $f \in \mathrm{Aut}(\mathfrak{M})$ be such that $f(\bar{b}) = \bar{c}_n$ and $f \restriction_{\{b_0,...,b_n\}}$ is the identity. Now $(f_n \circ f \circ g)(\bar{b}) = f_n(f(\bar{b})) = f_n(\bar{c}_n) = \bar{a}$ and $(f_n \circ f \circ g)(a_i) = f_n(f(b_i)) = f_n(b_i) = a_i$ for all $0 \le i \le n$.

Thus we get for all $n < \omega$ that $t^g(\bar{b}/\{a_0, ..., a_n\}) = t^g(\bar{a}/\{a_0, ..., a_n\})$. Because $\mathscr{A} = \{a_i : i < \omega\}$, we have that $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$.

Uniqueness of free extension: Let $C \subset B$ be an arbitrary finite set. Let $E_{\bar{a}} \subset \mathscr{A}$ be a finite set such that $t^w(\bar{a}/B)$ does not split over $E_{\bar{a}}$ and similarly $E_{\bar{b}} \subset \mathscr{A}$ for $t^w(\bar{b}/B)$. Because $\mathscr{A}$ is ω-saturated, we have $f \in \mathrm{Aut}(\mathfrak{M})$ such that $f(C) \subset \mathscr{A}$ and $f \restriction_{E_{\bar{a}}\cup E_{\bar{b}}}$ is the identity. Now we have that $t^g(C/E_{\bar{a}}) = t^g(f(C)/E_{\bar{a}})$ and then by the choice of $E_{\bar{a}}$ we have also an automorphism $f_{\bar{a}}$ such that $f_{\bar{a}} \restriction_C = f \restriction_C$ and $f_{\bar{a}}(\bar{a}) = \bar{a}$. With similar reasoning we also get $f_{\bar{b}} \in \mathrm{Aut}(\mathfrak{M})$ such that $f_{\bar{b}} \restriction_C = f \restriction_C$ and $f_{\bar{b}}(\bar{b}) = \bar{b}$. Finally we use the assumption that $t^w(\bar{a}/\mathscr{A}) = t^w(\bar{b}/\mathscr{A})$ to get an automorphism g such that $g(\bar{a}) = \bar{b}$ and $g \restriction_{f(C)} = Id \restriction_{f(C)}$. When we combine these mappings we get an automorphism $h = f_{\bar{b}}^{-1} \circ g \circ f_{\bar{a}}$ such that $h(\bar{a}) = \bar{b}$ and that for all $x \in C$, $h(x) = f_{\bar{b}}^{-1}(g(f_{\bar{a}}(x))) = f_{\bar{b}}^{-1}(g(f(x))) = f_{\bar{b}}^{-1}(f(x)) = x$. Thus $t^g(\bar{a}/C) = t^g(\bar{b}/C)$ and because $C \subset B$ was an arbitrary finite set, we get that $t^w(\bar{a}/B) = t^w(\bar{b}/B)$.

Transitivity: The " $\Rightarrow$ "-direction follows from monotonicity. For the other direction, let $E \subset A$ be a finite set such that $t^w(\bar{a}/\mathscr{B})$ does not split over E. Now we use the restricted existence of free extension to get $\bar{b}$ such that $t^w(\bar{b}/C)$ does not split over E and that $t^w(\bar{b}/\mathscr{B}) = t^w(\bar{a}/\mathscr{B})$. Because also $\bar{a} \downarrow^s_{\mathscr{B}} C$ we get from uniqueness that $t^w(\bar{b}/C) = t^w(\bar{a}/C)$. Hence $t^w(\bar{a}/C)$ does not split over $E \subset A$. □

Lemma 2.21 *Assume $\mathscr{B} \subset C$ are countable and $\mathscr{B}$ is an ω-saturated model. Let $A = (a_i)_{i<\omega}$ be a set. There is $A' = (a'_i)_{i<\omega}$ such that for all $n < \omega$ $t^w((a_0, ..., a_n)/\mathscr{B}) = t^w((a'_0, ..., a'_n)/\mathscr{B})$ and $(a'_0, ..., a'_n) \downarrow^s_{\mathscr{B}} C$.*

This we denote $t^w(A/\mathscr{B}) = t^w(A'/\mathscr{B})$ and $A' \downarrow^s_{\mathscr{B}} C$.

Proof: By monotonicity and that $\mathrm{LS}(\mathbb{K}) = \omega$, we may assume that C is actually a model, i.e. $C \preccurlyeq_{\mathbb{K}} \mathfrak{M}$. To emphasize this we denote $C = \mathscr{C}$.

We define an increasing chain of finite sets $E_n \subset \mathscr{B}$ and (d^k_m), $k, m < \omega$, such that for all $n < \omega$,

1. $t^w(a_0, ..., a_n/\mathscr{B}) = t^w(d^n_0, ..., d^n_n/\mathscr{B})$,

2. $t^w(d_0^n,...,d_n^n/\mathscr{C})$ does not split over E_n and

3. when $m \le n$, $t^w(d_0^m,...,d_m^m/\mathscr{C}) = t^w(d_0^n,...,d_m^n/\mathscr{C})$.

First Lemma 2.13 gives us $E_0 \subset \mathscr{B}$ such that $t^w(a_0/\mathscr{B})$ does not split over E_0 Then by Theorem 2.20 we get free extension d_0^0 such that $t^w(a_0/\mathscr{B}) = t^w(d_0^0/\mathscr{B})$ and $t^w(d_0^0/\mathscr{C})$ does not split over E_0.

Assume we have defined E_k and d_m^k for $m,k \le n$. Then we get from Lemma 2.13 such $E'_{n+1} \subset \mathscr{B}$ that $t^w((a_0,...,a_{n+1})/\mathscr{B})$ does not split over E'_{n+1}. Now $t^w(a_0,...,a_{n+1}/\mathscr{B})$ does not split over $E_n \cup E'_{n+1}$ because or Remark 2.9. We define $E_{n+1} = E_n \cup E'_{n+1}$.

Then we use Theorem 2.20 to get a free extension $(d_0^{n+1},...,d_{n+1}^{n+1})$, for which $t^w((d_0^{n+1},...,d_{n+1}^{n+1})/\mathscr{B}) = t^w((a_0,...,a_{n+1})/\mathscr{B})$ and the weak type $t^w((d_0^{n+1},...,d_{n+1}^{n+1})/\mathscr{C})$ does not split over E_{n+1}. We claim that now when $m \le n$, $t^w((d_0^{n+1},...,d_m^{n+1})/\mathscr{C}) = t^w((d_0^m,...,d_m^m)/\mathscr{C})$.

Let $C' \subset \mathscr{C}$ be an arbitrary finite set. Because $\mathscr{B}$ is ω-saturated, there is an automorphism f such that $f(C') \subset \mathscr{B}$ and $f \restriction_{E_{n+1}}$ is the identity. Now because $t^w((d_0^{n+1},...,d_{n+1}^{n+1})/\mathscr{C})$ does not split over E_{n+1} we have $f_1 \in \mathrm{Aut}(\mathfrak{M})$ such that $f_1 \restriction_{C'} = f \restriction_{C'}$ and $f_1(d_i^{n+1}) = d_i^{n+1}$ for all $0 \le i \le n+1$. Similarly, because $E_m \subset E_n$, we get $f_2 \in \mathrm{Aut}(\mathfrak{M})$ such that $f_2 \restriction_{C'} = f \restriction_{C'}$ and $f_2(d_i^m) = d_i^m$ for all $0 \le i \le m$. In addition, because $t^w((d_0^{n+1},...,d_{n+1}^{n+1})/\mathscr{B}) = t^w((a_0,...,a_{n+1})/\mathscr{B})$ and $t^w((d_0^m,...,d_m^m)/\mathscr{B}) = t^w((a_0,...,a_m)/\mathscr{B})$ we have mappings g_1 and g_2 such that $g_1 \restriction_{f(C')} = g_2 \restriction_{f(C')} = \mathrm{Id}_{f(C')}$ and $g_2(d_i^m) = a_i^m = g_1(d_i^{n+1})$ for all $0 \le i \le m$. Now $f_2{}^{-1} \circ g_2{}^{-1} \circ g_1 \circ f_1$ is an automorphism such that $(f_2{}^{-1} \circ g_2{}^{-1} \circ g_1 \circ f_1)(d_i^{n+1}) = d_i^m$ for all $0 \le i \le m$ and $(f_2{}^{-1} \circ g_2{}^{-1} \circ g_1 \circ f_1) \restriction_{C'}$ is the identity.

Let $\mathscr{C} = \bigcup_{i<\omega} C_i$, where $(C_i)_{i<\omega}$ is an increasing chain of finite sets. Because of condition 3 we have that when $m \le n$,

$$t^g((d_0^m,...,d_m^m)/C_m) = t^g((d_0^n,...,d_m^n)/C_m).$$

Thus we may use Lemma 2.5 to get $(a'_i)_{i<\omega}$ such that, for all $n < \omega$,

$$t^g((a'_0,...,a'_n)/C_n) = t^g((d_0^n,...,d_n^n)/C_n).$$

Now we see that actually

$$t^w((a'_0,...,a'_n)/\mathscr{C}) = t^w((d_0^n,...,d_n^n)/\mathscr{C})$$

for all $n < \omega$, because when $n < \omega$ and $C' \subset \mathscr{C}$ is a finite set, there is some $k \ge n$ for which $C' \subset C_k$, and thus from condition 3 we get that $t^g((d_0^n,...,d_n^n)/C') = t^g((d_0^k,...,d_n^k)/C') = t^g((a'_0,...,a'_n)/C')$.

Now also $t^w((a'_0,...,a'_n)/\mathscr{B}) = t^w((d_0^n,...,d_n^n)/\mathscr{B})$ for all $n < \omega$, and because $t^w((d_0^n,...,d_n^n)/\mathscr{C})$ does not split over $E_n \subset \mathscr{B}$, neither does $t^w((a'_0,...,a'_n)/\mathscr{C})$, thus $(a'_0,...,a'_n) \downarrow^s_{\mathscr{B}} \mathscr{C}$ for all $n < \omega$. $\square$

2.1 About weak type over a countable model

In this section we will prove that weak type and Galois type actually agree over countable models.

Lemma 2.22 *Assume $\mathscr{A}$ is a countable model, $\bar{a}, \bar{b}$ tuples and $A \subset \mathscr{A}$ finite. Then there is $\bar{b}'$ and finite $A' \subset \mathscr{A}$ such that*

i) $t^w(\bar{b}'/A \cup \{\bar{a}\}) = t^w(\bar{b}/A \cup \{\bar{a}\})$ and

ii) for all $\bar{c} \in \mathfrak{M}$, if $t^w(\bar{c}/A' \cup \{\bar{a}\}) = t^w(\bar{b}'/A' \cup \{\bar{a}\})$, then $t^w(\bar{c}/\mathscr{A} \cup \{\bar{a}\}) = t^w(\bar{b}'/\mathscr{A} \cup \{\bar{a}\})$.

When ii) holds for $\bar{b}'$ and A' we say that $t^w(\bar{b}'/\mathscr{A} \cup \{\bar{a}\})$ is weakly isolated over $A' \cup \{\bar{a}\}$.

Proof: We assume the contrary, and let $\bar{a}, \bar{b}$ and $A \subset \mathscr{A}$ finite be such that the claim does not hold. Then we derive a contradiction with ω-stability as in 2.12.

Denote $\mathscr{A} = \{a_i : i < \omega\}$. For each $\eta : \omega \to 2$ and $n \in \omega$ we construct finite sets A_n and tuples $\bar{c}_{\eta\restriction n}$ such that

1. $A \cup \{a_n\} \subset A_n \subset A_{n+1} \subset \mathscr{A}$,
2. $t^w(\bar{c}_{\eta\restriction m}/A \cup \{\bar{a}\}) = t^w(\bar{b}/A \cup \{\bar{a}\})$,
3. when $m \geq n$, $t^w(\bar{c}_{\eta\restriction m} {}^\frown \bar{a}/A_n) = t^w(\bar{c}_{\eta\restriction n} {}^\frown \bar{a}/A_n)$,
4. $\eta(n) = 1$ if and only if $t^w(\bar{c}_{\eta\restriction n+1} {}^\frown \bar{a}/A_{n+1}) \neq t^w(\bar{c}_{\eta\restriction n} {}^\frown \bar{a}/A_{n+1})$.

First let $\bar{c}_{\eta\restriction 0} = \bar{b}$ and $A_0 = A \cup \{a_0\}$. Then assume we have defined $\bar{c}_{\eta\restriction m}$ and A_m for $m \leq n$.

Because 2 holds for $\bar{c}_{\eta\restriction n}$, $t^w(\bar{c}_{\eta\restriction n}/\mathscr{A} \cup \{\bar{a}\})$ can't be weakly isolated over $A_n \cup \{\bar{a}\}$. Otherwise $\bar{c}_{\eta\restriction n}$ and A_n would violate the counter-assumption. Thus there exists some $\bar{b}_{\eta\restriction n}$ and some finite $D_{\eta\restriction n} \subset \mathscr{A}$ such that $t^w(\bar{b}_{\eta\restriction n}/A_n \cup \{\bar{a}\}) = t^w(\bar{c}_{\eta\restriction n}/A_n \cup \{\bar{a}\})$ but $t^w(\bar{b}_{\eta\restriction n}/D_{\eta\restriction n} \cup \{\bar{a}\}) \neq t^w(\bar{c}_{\eta\restriction n}/D_{\eta\restriction n} \cup \{\bar{a}\})$.

If $\eta(n) = 0$, we let $\bar{c}_{\eta\restriction n+1} = \bar{c}_{\eta\restriction n}$, and if $\eta(n) = 1$, we let $\bar{c}_{\eta\restriction n+1} = \bar{b}_{\eta\restriction n}$. Then we let

$$A_{n+1} = A_n \cup \{a_n\} \cup \bigcup_{\eta\restriction n : n \to 2} D_{\eta\restriction n}.$$

Finally by 3 and that $\bigcup_{n<\omega} A_n = \mathscr{A}$ we get from corollary 2.6 such $\bar{c}_\eta$ and $\bar{a}_\eta$ that $t^w(\bar{c}_\eta {}^\frown \bar{a}_\eta/A_n) = t^w(\bar{c}_{\eta\restriction n} {}^\frown \bar{a}/A_n)$ for each $n < \omega$. Now if $\eta \neq \eta'$, we claim that $t^w(\bar{c}_\eta {}^\frown \bar{a}_\eta/\mathscr{A}) \neq t^w(\bar{c}_{\eta'} {}^\frown \bar{a}_{\eta'}/\mathscr{A})$. Let n be the least index such that $\eta(n) \neq \eta'(n)$. We may assume that $\eta(n) = 1$. Then $\bar{c}_{\eta\restriction n} = \bar{c}_{\eta'\restriction n}$ and we may conclude that

$$\begin{aligned} & t^w(\bar{c}_{\eta'}{}^\frown\bar{a}_{\eta'}/A_{n+1}) \\ = \; & t^w(\bar{c}_{\eta'\restriction n+1}{}^\frown\bar{a}/A_{n+1}) \\ = \; & t^w(\bar{c}_{\eta'\restriction n}{}^\frown\bar{a}/A_{n+1}) \\ = \; & t^w(\bar{c}_{\eta\restriction n}{}^\frown\bar{a}/A_{n+1}) \\ \neq \; & t^w(\bar{c}_{\eta\restriction n+1}{}^\frown\bar{a}/A_{n+1}) \\ = \; & t^w(\bar{c}_{\eta}{}^\frown\bar{a}_{\eta}/A_{n+1}), \end{aligned}$$

which proves the claim. Now we have continuum-many different weak types over $\mathscr{A}$, a contradiction. □

Theorem 2.23 *Assume that $\mathscr{A}$ is a countable model and $t^w(\bar{a}/\mathscr{A}) = t^w(\bar{b}/\mathscr{A})$. Then also $t^g(\bar{a}/\mathscr{A}) = t^g(\bar{b}/\mathscr{A})$.*

Proof: First we are going to use Lemma 2.22 to define sequences $\bar{a}_i$ and finite sets A_i, $i < \omega$ such that

1. $\bar{a} = \bar{a}_0 \subset \bar{a}_n \subset \bar{a}_{n+1}$ and $A_n \subset A_{n+1} \subset \mathscr{A}$,
2. $t^w(\bar{a}_{n+1}/\mathscr{A} \cup \{\bar{a}_n\})$ is weakly isolated over $A_n \cup \{\bar{a}_n\}$ and
3. $B = \mathscr{A} \cup \bigcup_{i<\omega} \bar{a}_i$ has the following property: for all $\bar{a}' \in B$ and $\bar{b}' \in \mathfrak{M}$ there is $\bar{c} \in B$ such that $t^g(\bar{a}'{}^\frown\bar{c}/\emptyset) = t^g(\bar{a}'{}^\frown\bar{b}'/\emptyset)$.

Let $\bar{a}_0 = \bar{a}$ and $A_0 = \emptyset$. Assume we have defined $\bar{a}_j$ and A_j for $j \leq n$.

By ω-stability, there are only countably many Galois types over $A_i \cup \bar{a}_i$ for specific $i \leq n$. Let $(\bar{c}^i_j)_{j<\omega}$ enumerate representatives for each type and then let $\bar{d}_n$ be a sequence where $\bar{c}^i_j$'s are represented for $i, j \leq n$. By Lemma 2.22 there exists $\bar{a}'$ and finite $A' \subset \mathscr{A}$ such that $t^g(\bar{a}'/A_n \cup \{\bar{a}_n\}) = t^g(\bar{d}_n/A_n \cup \{\bar{a}_n\})$ and $t^w(\bar{a}'/\mathscr{A} \cup \{\bar{a}_n\})$ is weakly isolated over $A' \cup \{\bar{a}_n\}$. Let $A_{n+1} = A_n \cup A'$ and $\bar{a}_{n+1} = \bar{a}_n{}^\frown\bar{a}'$.

Finally we want to claim that 3 holds. Let $\bar{a}' \in \mathscr{A} \cup \bigcup_{i<\omega} \bar{a}_i$ and $\bar{b}' \in \mathfrak{M}$. Now $\bar{a}' \in A_m \cup \{\bar{a}_m\}$ for some $m < \omega$ and $t^g(\bar{b}'/A_m \cup \{\bar{a}_m\}) = t^g(\bar{c}^m_j/A_m \cup \{\bar{a}_m\})$ for some $j < \omega$. Let $n = \max\{m, j\}$. Then $\bar{c}^m_j$ is a subsequence in $\bar{d}_n$ and we get the claim because $t^g(\bar{a}_{n+1}/A_n \cup \{\bar{a}_n\}) = t^g(\bar{d}_n{}^\frown\bar{a}_n/A_n \cup \{\bar{a}_n\})$.

Now we define such $(\bar{b}_i)_{i<\omega}$ that $\bar{b}_0 = \bar{b}$ and

$$t^g(\bar{b}_0{}^\frown ... {}^\frown\bar{b}_n/A_n) = t^g(\bar{a}_0{}^\frown ... {}^\frown\bar{a}_n/A_n). \tag{2.2}$$

We do this so that we always have that

$$t^w(\bar{b}_0{}^\frown ... {}^\frown\bar{b}_n/\mathscr{A}) = t^w(\bar{a}_0{}^\frown ... {}^\frown\bar{a}_n/\mathscr{A}). \tag{2.3}$$

First let $\bar{b}_0 = \bar{b}$. Then 2.3 holds because of the assumption. Assume that we have defined $\bar{b}_m$ for $m \leq n$ and that 2.3 holds. Let $f \in \mathrm{Aut}(\mathfrak{M})$ be such that $f(\bar{a}_0^\frown ...^\frown \bar{a}_n) = \bar{b}_0^\frown ...^\frown \bar{b}_n$ and that $f \restriction_{A_{n+1}} = \mathrm{Id}_{A_{n+1}}$. Then let $\bar{b}_{n+1} = f(\bar{a}_{n+1})$. We claim that 2.3 holds. Assume the contrary and let $B \subset \mathscr{A}$ be a finite set such that $t^g(\bar{a}_0^\frown ...^\frown \bar{a}_{n+1}/B) \neq t^g(\bar{b}_0^\frown ...^\frown \bar{b}_{n+1}/B)$. We may assume that $A_{n+1} \subset B$. Let $g \in \mathrm{Aut}(\mathfrak{M})$ be such that $g(\bar{b}_0^\frown ...^\frown \bar{b}_n) = \bar{a}_0^\frown ...^\frown \bar{a}_n$ and $g \restriction_B = \mathrm{Id}_B$. Then we have that $t^g(\bar{a}_0^\frown ...^\frown \bar{a}_n^\frown g(\bar{b}_{n+1})/A_{n+1}) = t^g(\bar{b}_0^\frown ...^\frown \bar{b}_n^\frown \bar{b}_{n+1}/A_{n+1}) = t^g(\bar{a}_0^\frown ...^\frown \bar{a}_n^\frown \bar{a}_{n+1}/A_{n+1})$ and because $t^w(\bar{a}_{n+1}/\mathscr{A} \cup \{\bar{a}_n\})$ is weakly isolated over $A_{n+1} \cup \{\bar{a}_n\}$, we have also that $\bar{a}_0^\frown ...^\frown \bar{a}_n^\frown g(\bar{b}_{n+1})$ has the same Galois type than $\bar{a}_0^\frown ...^\frown \bar{a}_n^\frown \bar{a}_{n+1}$ over B. But then we get that $t^g(\bar{b}_0^\frown ...^\frown \bar{b}_n^\frown \bar{b}_{n+1}/B) = t^g(\bar{a}_0^\frown ...^\frown \bar{a}_n^\frown g(\bar{b}_{n+1})/B) = t^g(\bar{a}_0^\frown ...^\frown \bar{a}_n^\frown \bar{a}_{n+1}/B)$, a contradiction.

Then because 2.2 we have automorphisms f_n such that when $m \geq n$, $f_m \restriction_{A_n \cup \{\bar{a}_n\}} = f_n \restriction_{A_n \cup \{\bar{a}_n\}}$ and $f_n(A_n \cup \{\bar{a}_n\}) = A_n \cup \{\bar{b}_n\}$. Then because $\mathscr{A} \cup \bigcup_{i<\omega} \bar{a}_i$ is a model by Lemma 2.18, we get that $\bigcup_{i<\omega}(f_n \restriction_{A_n \cup \{\bar{a}_n\}})$ extends to an automorphism F such that $F \restriction_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$ and $F(\bar{a}) = F(\bar{a}_0) = \bar{b}_0 = \bar{b}$. □

Now we can improve the result of Corollary 2.6.

Lemma 2.24 *Assume $\mathscr{A}$ is a model, $|\mathscr{A}| \leq \aleph_1$ and for all finite $A \subset \mathscr{A}$ there is $\bar{a}_A$ such that if $B \subset A$, then $t^g(\bar{a}_B/B) = t^g(\bar{a}_A/B)$. Then there is $\bar{a}$ such that for all finite $A \subset \mathscr{A}$, $t^g(\bar{a}/A) = t^g(\bar{a}_A/A)$.*

Proof: Let $\mathscr{A} = \bigcup_{i<\omega_1} \mathscr{A}_i$, where $(\mathscr{A}_i)_{i<\omega_1}$ is an $\preccurlyeq_{\mathbb{K}}$-increasing chain of countable models such that $\mathscr{A}_\alpha = \bigcup_{i<\alpha} A_i$, when α is a limit ordinal. From Corollary 2.6 we get for each $i < \omega_1$ a sequence $\bar{a}_i$ such that $t^g(\bar{a}_i/A) = t^g(\bar{a}_A/A)$ for all finite $A \subset \mathscr{A}_i$. Now if $j < i$, we have that $t^w(\bar{a}_i/\mathscr{A}_j) = t^w(\bar{a}_j/\mathscr{A}_j)$, because when $A \subset \mathscr{A}_j$ is a finite subset, $t^g(\bar{a}_i/A) = t^g(\bar{a}_A/A) = t^g(\bar{a}_j/A)$. Now we get from Theorem 2.23 that also $t^g(\bar{a}_i/\mathscr{A}_j) = t^g(\bar{a}_j/\mathscr{A}_j)$.

Then we do a similar construction as in Lemma 2.5. We define automorphisms g_i, $i < \omega_1$, such that

1. For $j < i < \omega_1$, $g_i \restriction_{\mathscr{A}_j} = g_j \restriction_{\mathscr{A}_j}$ and
2. $g_i(\bar{a}_i) = \bar{a}_0$.

Let $g_0 = \mathrm{Id}_{\mathfrak{M}}$. Assume we have defined g_i for $i < \alpha$.

Case 1: $\alpha = \beta + 1$. This case is similar to the situation in Lemma 2.5. Because $t^g(\bar{a}_\alpha/\mathscr{A}_\beta) = t^g(\bar{a}_\beta/\mathscr{A}_\beta)$, also $t^g(g_\beta(\bar{a}_\alpha)/g_\beta(\mathscr{A}_\beta)) = t^g(g_\beta(\bar{a}_\beta)/g_\beta(\mathscr{A}_\beta))$, and we have an automorphism f such that $f \restriction_{g_\beta(\mathscr{A}_\beta)}$ is the identity and $f(g_\beta(\bar{a}_\alpha)) = g_\beta(\bar{a}_\beta) = \bar{a}_0$. We can take $g_\alpha = f \circ g_\beta$.

Case 2: α is a limit ordinal. The mapping $\bigcup_{i<\alpha}(g_i \restriction_{\mathscr{A}_i}) : \mathscr{A}_\alpha \to \mathfrak{M}$ extends to an automorphism F. When $A \subset \mathscr{A}_\alpha$ is finite, there is some $i < \alpha$ such that $A \subset \mathscr{A}_i$. Let h witness that $t^g(\bar{a}_i/\mathscr{A}_i) = t^g(\bar{a}_\alpha/\mathscr{A}_i)$. Then $(h \circ {g_i}^{-1} \circ F) \restriction_A = \mathrm{Id}_A$ and $(h \circ {g_i}^{-1} \circ F)(F^{-1}(\bar{a}_0)) = (h \circ {g_i}^{-1})(\bar{a}_0) = h(\bar{a}_i) = \bar{a}_\alpha$. Thus we have that $t^w(F^{-1}(\bar{a}_0)/\mathscr{A}_\alpha) = t^w(\bar{a}_\alpha/\mathscr{A}_\alpha)$ and again, by Theorem 2.23, $t^g(F^{-1}(\bar{a}_0)/\mathscr{A}_\alpha) = t^g(\bar{a}_\alpha/\mathscr{A}_\alpha)$. Let f be an automorphism such that $f(\bar{a}_\alpha) = F^{-1}(\bar{a}_0)$ and $f \restriction_{\mathscr{A}_\alpha} = \mathrm{Id}_{\mathscr{A}_\alpha}$. We can take $g_\alpha = F \circ f$. Now 1 holds and also $g_\alpha(\bar{a}_\alpha) = (F(f(\bar{a}_\alpha)) = F(F^{-1}(\bar{a}_0)) = \bar{a}_0$.

Finally the mapping $\bigcup_{i<\omega_1}(g_i \restriction_{\mathscr{A}_i}) : \mathscr{A} \to \mathfrak{M}$ extends to an automorphism G. We can take $\bar{a} = G^{-1}(\bar{a}_0)$. Then for each $i < \omega_1$, automorphism $G^{-1} \circ g_i$ shows that $t^g(\bar{a}_i/\mathscr{A}_i) = t^g(\bar{a}/\mathscr{A}_i)$. Thus when $A \subset \mathscr{A}$ finite, there is some $i < \alpha$ such that $A \subset \mathscr{A}_i$. Then $t^g(\bar{a}_A/A) = t^g(\bar{a}_i/A) = t^g(\bar{a}/A)$. $\square$

3 Symmetry

For symmetry we need an extra property, namely a non-restricted version of existence of free extension, formulated in 3.2. To be more specific, we need this property in lemma 3.4 and thus also in Theorem 3.13. In this section we first prove symmetry using this property as an assumption, and then discuss what more natural assumptions would imply this property. Note that we now could prove already a stronger result than the one in 2.20.

Remark 3.1 *Let $\mathscr{A}$ be an ω-saturated model, $E \subset \mathscr{A}$ finite such that $t^w(\bar{a}/\mathscr{A})$ does not split over E and B such that $\mathscr{A} \subset B$ and $|B| \leq \aleph_1$. Then there is $\bar{b}$ such that $t^w(\bar{b}/B)$ does not split over E.*

Proof: The proof is identical to the proof of Theorem 3.19. We just use Lemma 2.24 in the place of Lemma 3.18. $\square$

We formulate the new assumption generally in $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$, but it is clear how to interpret it in the context of a monster model.

Assumption 3.2 (Existence of free extension) *Let $\mathscr{A} \in \mathbb{K}$ be ω-saturated in $\mathbb{K}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$, $\mathscr{A} \subset B \subset \mathscr{B}$ and $\bar{a} \in \mathscr{B}$ such that $t^w_{\mathscr{B}}(\bar{a}/\mathscr{A})$ does not split over finite $E \subset \mathscr{A}$. Then there is $\mathscr{C} \in \mathbb{K}$ and $\bar{b} \in \mathscr{C}$ such that $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{C}$, $t^w_{\mathscr{B}}(\bar{a}/\mathscr{A}) = t^w_{\mathscr{C}}(\bar{b}/\mathscr{A})$ and $t^w_{\mathscr{C}}(\bar{b}/\mathscr{B})$ does not split over E.*

Lemma 3.3 *Let $\mathscr{A}$ be an ω-saturated model, $\bar{b} \downarrow^s_{\mathscr{A}} B$ and $\bar{c} \not\downarrow^s_{\mathscr{A}} C$. Then there is a countable ω-saturated $\mathscr{A}' \preccurlyeq_{\mathbb{K}} \mathscr{A}$ such that $\bar{b} \downarrow^s_{\mathscr{A}'} B$ and $\bar{c} \not\downarrow^s_{\mathscr{A}'} C$.*

Proof: Let $E \subset \mathscr{A}$ be a finite set such that $t^w(\bar{b}/\mathscr{A} \cup B)$ does not split over E. Now if $E \subset \mathscr{A}' \subset \mathscr{A}$, also $\bar{b} \downarrow^s_{\mathscr{A}'} B$, because there are no more witnesses in $\mathscr{A}' \cup B$ than in $\mathscr{A} \cup B$. We define an increasing chain of countable models $B_n \preccurlyeq_{\mathbb{K}} \mathscr{A}$ inductively so that

1. $E \subset B_0$,
2. for all finite $D \subset B_n$ $t^w(\bar{c}/B_{n+1} \cup C)$ splits over D,
3. for all finite $D \subset B_n$ and $\bar{a}$ there is $f \in \mathrm{Aut}(\mathfrak{M})$ such that $f(\bar{a}) \in B_{n+1}$ and $f \restriction_D = \mathrm{Id}_D$.

First let $B_0 \preccurlyeq_{\mathbb{K}} \mathscr{A}$ be a countable model such that $E \subset B_0$. Then assume we have defined B_i for $i \leq n$. Denote

$$\mathscr{B} = \{D \subset B_n : D \text{ a finite subset } \}.$$

Because B_n is countable, also $\mathscr{B}$ is countable. Because $\bar{c} \not\downarrow^s_{\mathscr{A}} C$, for every $D \in \mathscr{B}$ there are $\bar{c}_D, \bar{d}_D \in \mathscr{A} \cup C$ witnessing that $t^w(\bar{c}/\mathscr{A} \cup C)$ splits over D. If $\{\bar{c}_D, \bar{d}_D\} \cap \mathscr{A} \subset B_{n+1}$ for all $D \in \mathscr{B}$, 2 holds. Also because $\mathfrak{M}$ is ω-stable, there are only countably many $\bar{a}^D \in \mathfrak{M}$ that have different weak type over D. Because for a finite set D weak type and Galois type over D coinside, we can enumerate such $(\bar{a}_i^D)_{i<\omega}$ that for all $\bar{a}$ exists such i that $t^g(\bar{a}/D) = t^g(\bar{a}_i^D/D)$. Then because $\mathscr{A}$ is ω-saturated, we may take $\bar{a}_i^D \in \mathscr{A}$ for every $i < \omega$. Then let

$$B'_{n+1} = B_n \cup \bigcup_{D \in \mathscr{B}} (\{\bar{c}_D, \bar{d}_D\} \cap \mathscr{A}) \cup \bigcup_{D \in \mathscr{B}} \{\bar{a}_i^D : i < \omega\}.$$

Now $B'_{n+1} \subset \mathscr{A}$ is countable and we can take $B_{n+1} \preccurlyeq_{\mathbb{K}} \mathscr{A}$ such that $B'_{n+1} \subset B_{n+1}$. This B_{n+1} satisfies both 2 and 3.

Finally let $\mathscr{A}' = \bigcup_{n<\omega} B_n$. When $D \subset \mathscr{A}'$ is a finite subset, there is $n < \omega$ such that $D \subset B_n$. Thus $\mathscr{A}'$ is as we wanted. □

Lemma 3.4 *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ satisfies the existence of free extension -property. Assume that $\mathscr{A}$ is a countable ω-saturated model, $\bar{a} \downarrow^s_{\mathscr{A}} \bar{b}$ and $\bar{b} \not\downarrow^s_{\mathscr{A}} \bar{a}$. Then for any ordinal λ there exists a sequence $(\bar{a}_i, \bar{b}_i)_{i<\lambda}$ of length λ such that $\bar{b}_i \downarrow^s_{\mathscr{A}} \bar{a}_j$ if and only if $i > j$.*

Proof: We construct such a sequence by induction. Let $\bar{a}_0 = \bar{a}$ and $\bar{b_0} = \bar{b}$. Assume we have found $\bar{a}_i, \bar{b_i}$ for all $i < \alpha$. Now we use Theorem 2.13 and the existence of free extension to get $\bar{a}_\alpha$ and $\bar{b}_\alpha$ such that

1. $t^w(\bar{a_\alpha}{}^\frown\bar{b}_\alpha/\mathscr{A}) = t^w(\bar{a_0}{}^\frown\bar{b}_0/\mathscr{A})$ and

2. $\bar{a_\alpha}\frown\bar{b}_\alpha \downarrow^s_{\mathscr{A}} (\bigcup_{i<\alpha}\{\bar{a}_i, \bar{b}_i\})$.

From monotonicity we get that $\bar{a_\alpha}\frown\bar{b}_\alpha \downarrow^s_{\mathscr{A}} \bar{a}_i$ for all $i < \alpha$ and thus $\bar{b}_\alpha \downarrow^s_{\mathscr{A}} \bar{a}_i$ for all $i < \alpha$. First we claim that

3. when $\beta \leq \alpha$, $t^w(\bar{a_\alpha}\frown\bar{b}_\beta/\mathscr{A}) = t^w(\bar{a_0}\frown\bar{b}_0/\mathscr{A})$.

The proof of this claim is much similar to the uniqueness proof of 2.20. If $\beta = \alpha$, the claim follows from the definition of $\bar{a}_\alpha$ and $\bar{b}_\alpha$. Thus let $\beta < \alpha$. Because $\bar{a}_0 \downarrow^s_{\mathscr{A}} \bar{b}_0$ and $\bar{a}_\alpha \downarrow^s_{\mathscr{A}} \bar{b}_\beta$, we have finite $E_1, E_2 \subset \mathscr{A}$ such that

(a) $t^w(\bar{a}_0/\mathscr{A} \cup \{\bar{b}_0\})$ does not split over E_1 and

(b) $t^w(\bar{a}_\alpha/\mathscr{A} \cup \{\bar{b}_\beta\})$ does not split over E_2.

Let $C \subset \mathscr{A}$ be an arbitrary finite set. Because $\mathscr{A}$ is ω-saturated, there exists an automorphism f such that $f(\bar{b}_0) \subset \mathscr{A}$ and $f \restriction_{E_1 \cup E_2 \cup C}$ is the identity. From (a) we get an automorphism such that $h_1 \restriction_{C \cup \{\bar{b}_0\}} = f \restriction_{C \cup \{\bar{b}_0\}}$ and $h_1(\bar{a}_0) = \bar{a}_0$.

Now we use the fact that $t^w(\bar{b}_\beta/\mathscr{A}) = t^w(\bar{b}_0/\mathscr{A})$ to get an automorphism f' such that $f'(\bar{b_\beta}) = \bar{b}_0$ and $f' \restriction_{C \cup E_2}$ is the identity. Then $(f \circ f') \restriction_{E_2} = \mathrm{Id}_{E_2}$ and $(f \circ f')(\bar{b_\beta}\frown C) = f(\bar{b}_0)\frown C$. Thus from (b) we get an automorphism h_2 such that $h_2(\bar{a}_\alpha) = \bar{a}_\alpha$, $h_2 \restriction_C = \mathrm{Id}_C$ and $h_2(\bar{b}_\beta) = f(\bar{b}_0)$.

Because $t^w(\bar{a}_0/\mathscr{A}) = t^w(\bar{a}_\alpha/\mathscr{A})$, there is also $h \in \mathrm{Aut}(\mathfrak{M})$ such that $h(\bar{a}_0) = \bar{a}_\alpha$ and $h \restriction_{\{f(\bar{b}_0)\} \cup C}$ is the identity.

Finally we combine these automorphisms to ${h_2}^{-1} \circ h \circ h_1 \in \mathrm{Aut}(\mathfrak{M})$. Now $({h_2}^{-1} \circ h \circ h_1) \restriction_C$ is the identity and $({h_2}^{-1} \circ h \circ h_1)(\bar{a}_0, \bar{b}_0) = {h_2}^{-1}(h(\bar{a}_0, f(\bar{b}_0))) = {h_2}^{-1}(\bar{a}_\alpha, f(\bar{b}_0)) = (\bar{a}_\alpha, \bar{b}_\beta)$. Because $C \subset \mathscr{A}$ was arbitrary, this proves claim 3.

Now we want to show that

4. for all $i \leq \alpha$, $\bar{b}_i \not\downarrow^s_{\mathscr{A}} \bar{a}_\alpha$.

To prove this, we assume the contrary. Let $\beta \leq \alpha$ and $E \subset \mathscr{A}$ be a finite set such that $t^w(\bar{b}_\beta/\mathscr{A} \cup \{\bar{a}_\alpha\})$ does not split over E. We have that $\bar{b}_0 \not\downarrow^s_{\mathscr{A}} \bar{a}_0$ and thus $t^w(\bar{b}_0/\mathscr{A} \cup \{\bar{a}_0\})$ splits over E. Let $\bar{c}, \bar{d} \subset \mathscr{A} \cup \{\bar{a}_0\}$ witness that.

From 3 we get $g \in \mathrm{Aut}(\mathfrak{M})$ such that $g(\bar{a}_0, \bar{b}_0) = (\bar{a}_\alpha, \bar{b}_\beta)$ and $g \restriction_{(\{\bar{c}, \bar{d}\} \cap \mathscr{A}) \cup E}$ is the identity.

Because $g(\bar{c})$ and $g(\bar{d})$ are in $\mathscr{A} \cup \{\bar{a}_\alpha\}$ and $t^g(g(\bar{c})/E) = t^g(g(\bar{d})/E)$ from the choice of E we get $g^* \in \mathrm{Aut}(\mathfrak{M})$ such that $g(\bar{c}) = \bar{d}$ and $g \restriction_{E \cup \{\bar{b}_\beta\}}$ is the identity.

Now $(g^{-1} \circ g^* \circ g)(\bar{c}) = \bar{d}$ and $(g^{-1} \circ g^* \circ g) \restriction_{E \cup \{\bar{b}_0\}}$ is the identity, which contradicts the choice of $\bar{c}$ and $\bar{d}$. This proves 4. □

The proof for the following theorem can be found for example in [2].

Theorem 3.5 (Erdős and Rado) *Let α be an infinite cardinal and let $n < \omega$. Suppose that*

1. $|X| > \beth_n(\alpha)$,
2. $[X]^{n+1} \subset \bigcup_{i \in I} C_i$ *and*
3. $|I| \leq \alpha$.

Then there are a subset $Y \subset X$ and $i \in I$ such that

$$|Y| > \alpha \text{ and } [Y]^{n+1} \subset C_i.$$

Remark 3.6 *Let $A \subset \mathfrak{M}^*$ be a subset and $(a)_{c_a \in A}$ a set of new constants. Then let $\mathfrak{M}^*_A = (\mathfrak{M}, c_a)_{a \in A}$ be the model where the new constants are interpreted as elements of A respectively. The following are equivalent for all subsets $B = (b_i)_{i \in I}$ and $C = (c_i)_{i \in I}$ of $\mathfrak{M}^*$.*

1. *For all first order formulas ϕ of vocabulary $\tau^* \cup \{c_a : a \in A\}$, all $n < \omega$ and all indexes $i_0, ..., i_n \in I$ $\mathfrak{M}^*_A \models \phi(b_{i_0}, ..., b_{i_n})$ if and only if $\mathfrak{M}^*_A \models \phi(c_{i_0}, ..., c_{i_n})$.*
2. *For all atomic formulas ϕ of vocabulary $\tau^* \cup \{c_a : a \in A\}$, all $n < \omega$ and all indexes $i_0, ..., i_n \in I$ $\mathfrak{M}^*_A \models \phi(b_{i_0}, ..., b_{i_n})$ if and only if $\mathfrak{M}^*_A \models \phi(c_{i_0}, ..., c_{i_n})$.*
3. *For all $n < \omega$ and indexes $i_0, ..., i_n \in I$ there is an automorphism f of $\mathfrak{M}^*$ such that $f(b_{i_k}) = c_{i_k}$ for $0 \leq k \leq n$ and $f\restriction_A = \mathrm{Id}_A$.*
4. *There is an automorphism f of $\mathfrak{M}^*$ such that $f(b_i) = c_i$ for all $\in I$ and $f\restriction_A = \mathrm{Id}_A$.*

This remark follows clearly from the homogeneity of $\mathfrak{M}^*$.

Definition 3.7 ($*$-type) *Let $B = (b_i)_{\in I}$ and $C = (c_i)_{i \in I}$ be subsets of $\mathfrak{M}$. We write*

$$t^*(B/A) = t^*(C/A)$$

if one (and all) of the conditions 1, 2, 3 and 4 of Remark 3.6 hold for B and C.

Remark 3.8 *When $(\bar{a}_i)_{i<(2^{\aleph_0})^+} \subset \mathfrak{M}^*$ and A countable, there are $i, j < (2^{\aleph_0})^+$ such that $t^*(\bar{a}_i/A) = t^*(\bar{a}_j/A)$.*

Proof: Denote by Φ the set of all atomic formulas of $\tau^* \cup \{c_a : a \in A\}$, where c_a is a new constant for each $a \in A$. Because τ^* and A are countable, the set Φ is countable. Let $(\phi_i)_{i<\omega}$ enumerate Φ. Let $\mathfrak{M}^*_A$ be the model $(\mathfrak{M}^*, c_a)_{a \in A}$, where constants c_a are interpreted as elements of A respectively. Now for each $\bar{a}$ we have a function $\eta_{\bar{a}} : \omega \to 2$ such that

$$\eta_{\bar{a}}(i) = \begin{cases} 1 & \text{when } \mathfrak{M}^*_A \models \phi_i(\bar{a}), \\ 0 & \text{when } \mathfrak{M}^*_A \not\models \phi_i(\bar{a}). \end{cases}$$

Clearly if $\eta_{\bar{a}} = \eta_{\bar{b}}$, the countable sequences $\bar{a}$ and $\bar{b}$ satisfy exactly the same atomic formulas of $\tau^* \cup \{c_a : a \in A\}$.

There can not be more than $2^{\aleph_0}$ tuples with different $*$-type. □

Lemma 3.9 *Assume that* $(\bar{b}^i)_{i<\omega}$ *is a sequence of tuples such that* $lg(\bar{b}^i) = i+1$ *for all* $i < \omega$ *and* A *a set such that* $A = \bigcup_{i<\omega} B_i$, *where* $i < j \Rightarrow B_i \subset B_j$ *and*

$$i < j \Rightarrow t^*((b^j_0, ..., b^j_i)/B_i) = t^*(\bar{b^i}/B_i).$$

Then there is $(a_i)_{i<\omega}$ *such that* $t^*((a_0, ..., a_i)/B_i) = t^*(\bar{b}^i/B_i)$ *for all* $i < \omega$.

Proof: The proof is similar to the proof of Lemma 2.5. □

Definition 3.10 (Order-indiscernible) *Let* $(I, <)$ *be a linear ordering. We say that a sequence* $(\bar{a}_i)_{i \in I}$ *is n-indiscernible over* A *if for all* $i_0 < ... < i_{n-1} \in I$ *and* $j_0 < ... < j_{n-1} \in I$

$$t^*(\bar{a}_{i_0}, ..., \bar{a}_{i_{n-1}}/A) = t^*(\bar{a}_{j_0}, ..., \bar{a}_{j_{n-1}}/A).$$

We say that the sequence is order-indiscernible if it is n-indiscernible for all $n < \omega$.

Lemma 3.11 *Let* $(\bar{a}_i)_{i<\lambda}$ *be a sequence of tuples,* A *a countable set and* λ *a cardinal such that* $\lambda = \bigcup_{\alpha<((2^{\aleph_0})^+)} \kappa_\alpha$, *where* $\kappa_0 \geq 2^{\aleph_0}$ *and for all* $n \in \omega$ $\beth_n(\kappa_\alpha) < \kappa_{\alpha+1}$. *Then there exists a sequence* $(\bar{a}'_i)_{i<\omega}$ *such that it is order-indiscernible over* A *and for all* $n < \omega$ *there exists* $i_0 < ... < i_n < \lambda$ *such that*

$$t^*(\bar{a}'_0, ..., \bar{a}'_n/A) = t^*(\bar{a}_{i_0}, ..., \bar{a}_{i_n}/A).$$

Proof: For a shorter notation we assume that $\bar{a}_i = a_i$ and $\bar{a'}_j = a'_j$ for all $i < \lambda, j < \omega$. We want to define by induction on $n < \omega$ sets $I^n_\alpha \subset \mathfrak{M}^{*lg(\bar{a}_0)}$, $\alpha < (2^{\aleph_0})^+$ such that

1. $I^n_\alpha \subset (a_i)_{i<\lambda}$,

2. $|I^n_\alpha| \geq \kappa_\alpha$,

3. I^n_α is n-indiscernible over A,

4. When $b_0, ..., b_{n-1} \in I^n_\alpha$ with increasing indexes and $c_0, ..., c_{n-1} \in I^n_\beta$ with increasing indexes,
$$t^*(b_0, ..., b_{n-1}/A) = t^*(c_0, ..., c_{n-1}/A).$$

5. When $b_0, ..., b_{n-1} \in I^n_\alpha$ with increasing indexes and $c_0, ..., c_{n-1} \in I^m_\alpha$ with increasing indexes and $m > n$,
$$t^*(b_0, ..., b_{n-1}/A) = t^*(c_0, ..., c_{n-1}/A).$$

Let $n = 0$. Define $I^0_\alpha = (a_i)_{i<\kappa_\alpha}$. Now 3,4 and 5 are trivial because we are looking at sequences of length 0, i.e. empty sequences.

Assume we have defined I^m_α for all $\alpha < (2^{\aleph_0})^+$ and $m \leq n$. Let $[(a_i)_{i<\lambda}]^{<\omega}$ denote all finite subsets of $\{a_i : i < \lambda\}$. To every $A \in [(a_i)_{i<\lambda}]^{<\omega}$ we may attach a type in a natural way, i.e. the type $t^*(a_{i_0}, ..., a_{i_n}/A)$, where $\{a_{i_0}, ..., a_{i_n}\} = A$ and the indexes $i_0, ..., i_n$ are in an increasing order. By Remark 3.8, there can't be more than $2^{\aleph_0}$ different types for $A \in [(a_i)_{i<\lambda}]^{<\omega}$ and thus $[(a_i)_{i<\lambda}]^{<\omega} = \bigcup_{i\in I} C_i$, where $|I| \leq 2^{\aleph_0}$ and for all $i \in I$, $t^*((a_{i_0}, ..., a_{i_p})/A) = t^*((b_{j_0}, ..., b_{j_m})/A)$ when $\{a_{i_0}, ..., a_{i_p}\}, \{b_{j_0}, ..., b_{j_m}\} \in C_i$ and indexes i_k, j_k are in an increasing order. Then of course $p = m$.

First we define sets J^{n+1}_α, $\alpha < (2^{\aleph_0})^+$ as follows: We have that $|I^n_{\alpha+1}| \geq \kappa_{\alpha+1} > \beth_n(\kappa_\alpha)$, $|I| \leq 2^{\aleph_0} \leq \kappa_\alpha$ and $[I^n_{\alpha+1}]^{n+1} \subset \bigcup_{i\in I} C_i$. Then we get from lemma 3.5 a subset $J^{n+1}_\alpha \subset I^n_{\alpha+1}$ and $i_0 \in I$ such that $|J^{n+1}_\alpha| \geq \kappa_\alpha$ and $[J^{n+1}_\alpha]^{n+1} \subset C_{i_0}$. Thus this J^{n+1}_α is $(n+1)$-indiscernible over A.

Also if we take some $m < n+1$ and tuples $(c_0, ..., c_{m-1}) \in J^{n+1}_\alpha$ and $(b_0, ..., b_{m-1}) \in I^m_\alpha$, because $J^{n+1}_\alpha \subset I^n_{\alpha+1}$, we get from induction that $t^*(c_0, ..., c_{m-1}/A) = t^*(b_0, ..., b_{m-1}/A)$. Thus condition 5 holds for tuples in J^{n+1}_α.

Now again we have that $[J^{n+1}_\alpha]^{n+1} \subset \bigcup_{i\in I} C_i$ for all $\alpha < (2^{\aleph_0})^+$. By the pigeonhole principle there must be an index $i_0 \in I$ such that
$$|\{\alpha < (2^{\aleph_0})^+ : [J^{n+1}_\alpha]^{n+1} \subset C_{i_0}\}| = (2^{\aleph_0})^+.$$

Define
$$\beta_\alpha = \min\{\beta < (2^{\aleph_0})^+ : [J^{n+1}_\beta]^{n+1} \subset C_{i_0} \text{ and } \beta \geq \alpha\} \text{ and}$$
$$I^{n+1}_\alpha = J^{n+1}_{\beta_\alpha}.$$

Now also property 4 holds in I^{n+1}_α for all $\alpha < (2^{\aleph_0})^+$.

Let $\alpha < (2^{\aleph_0})^+$. For all $n \in \omega$ we take some $(n+1)$-tuple $\bar{a}_n \in I_\alpha^{n+1}$ with increasing indexes $i_0^n, ..., i_n^n$. Then we get from condition 5 that when $m \geq n$, $t^*((a_{i_0^n}, ..., a_{i_n^n})/A) = t^*(a_{i_0^m}, ..., a_{i_n^m})/A)$. We get from Lemma 3.9 a sequence $(a_i')_{i<\omega}$ such that $t^*((a_0', ..., a_n')/A) = t^*(a_{i_0^n}, ..., a_{i_n^n})/A)$ for all $n < \omega$. We check that this $(a_i')_{i<\omega}$ is order-indiscernible. For this we take some $(a_{i_0}', ..., a_{i_n}')$ and $(a_{j_0}', ..., a_{j_n}')$ with increasing indexes. Now we can find $(n+1)$-tuples $\bar{a}_n \in I_\alpha^{i_n+1}$ and $\bar{b}_n \in I_\alpha^{j_n+1}$ with increasing indexes such that $t^*((a_{i_0}', ..., a_{i_n}')/A) = t^*(\bar{a}_n/A)$ and $t^*((a_{j_0}', ..., a_{j_n}')/A) = t^*(\bar{b}_n/A)$ by taking suitable subsequences. Thus from condition 5 it follows that $t^*((a_{i_0}', ..., a_{i_n}')/A) = t^*((a_{j_0}', ..., a_{j_n}')/A)$. □

Lemma 3.12 *Let $(\bar{a})_{i<\omega}$ be an order-indiscernible sequence over A and $(I, <')$ a linear ordering. There are tuples $(\bar{c}_i)_{i \in (I,<')}$ in $\mathfrak{M}$ such that for all $n < \omega$ and $i_0 <' ... <' i_n$*

$$t^*(\bar{c}_{i_0}, ..., \bar{c}_{i_n}/A) = t^*(\bar{a}_0, ..., \bar{a}_n/A). \tag{3.4}$$

Proof: We use $<'$ to denote the ordering of I and $<$ to denote a well-order. We prove the claim for all sub-orders $(J, <') \subset (I, <')$ by induction on the size of J.

Assume we have found such $(\bar{c}_i)_{i \in J}$ for all suborders J of size strictly less than a cardinal α and let $(J, <') \subset (I, <')$ be such that $|J| = \alpha$. Then let $(J, <)$ be a α-type well-ordering of $(J, <')$. For $i \in J$ we say that $i < \beta < \alpha$ when we mean that $i < h(\beta)$, where $h : (\alpha, <) \to (J, <)$ is an isomorphism. Then we define the $\bar{c}_i$'s by induction on $<$. Assume we have defined $\bar{c}_i$ for $i < \beta < \alpha$ such that 3.4 holds for all n and all $i_0, ..., i_n < \beta$. Now $(\{i : i \leq \beta\}, <')$ is a suborder of $(I, <')$ and $|\{i : i \leq \beta\}| = |\beta + 1| < \alpha$. By induction there exists elements $(\bar{d}_i)_{i \leq \beta}$ such that 3.4 holds for all n and all $i_0, ..., i_n \leq \beta$. We may use homogeneity to find $f \in \mathrm{Aut}(\mathfrak{M})$ such that $f \restriction_A = \mathrm{Id}_A$ and $f(\bar{c}_i) = \bar{d}_i$ for all $i < \beta$. Then define $\bar{c}_\beta = f^{-1}(\bar{d}_\beta)$. Now $t^*((\bar{c}_i)_{i \leq \beta}/A) = t^*((\bar{d}_i)_{i \leq \beta}/A)$. When we take some $n < \omega$ and indexes $i_0 <' ... <' i_n$ such that $i_k \leq \beta$ for all $0 \leq k \leq n$, we have that $t^*(\bar{c_{i_0}}, ..., \bar{c_{i_n}}/A) = t^*(\bar{d_{i_0}}, ..., \bar{d_{i_n}}/A) = t^*(\bar{a}_0, ..., \bar{a}_n/A)$. Finally we have defined $(\bar{c}_i)_{i<\alpha}$ and thus $(\bar{c}_i)_{i \in (J,<')}$.

□

Now the following theorem finally combines all the previous lemmas.

Theorem 3.13 (Symmetry) *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ satisfies the existence of free extension -property. Let $\mathscr{A}$ be an ω-saturated model. If $\bar{a} \downarrow^s_{\mathscr{A}} \bar{b}$, then $\bar{b} \downarrow^s_{\mathscr{A}} \bar{a}$.*

Proof: We assume the contrary. Let $\bar{a}$ and $\bar{b}$ be such that $\bar{a} \downarrow^s_{\mathscr{A}} \bar{b}$ and $\bar{b} \not\downarrow^s_{\mathscr{A}} \bar{a}$. First by Lemma 3.3 we may assume that $\mathscr{A}$ is countable. Then we get from Lemma 3.4 a sequence $(\bar{a}_i, \bar{b}_i)_{i<\lambda}$ such that λ satisfies the assumptions of Lemma 3.11 and

$$\bar{b}_i \downarrow^s_{\mathscr{A}} \bar{a}_j \text{ if and only if } i > j.$$

Furthermore we use Lemma 3.11 and 3.12 to get a sequence $(\bar{a}_i, \bar{b}_i)_{i \in (\mathbb{R}, <')}$ such that

$$\bar{b}_i \downarrow^s_{\mathscr{A}} \bar{a}_j \text{ if and only if } j <' i.$$

When we denote $B = \mathscr{A} \cup \{(\bar{a}_i, \bar{b}_i) : i \in \mathbb{Q}\}$, B is countable and if $i, j \in \mathbb{R}$ and $i \neq j$, tuples $(\bar{a}_i, \bar{b}_i)$ and $(\bar{a}_j, \bar{b}_j)$ have different weak type over B. Because $\mathbb{R}$ is uncountable, this contradicts the ω-stability assumption. $\square$

3.1 What implies Existence of free extension

Our first candidate for a more natural assumption than 3.2 is tameness. To define tameness we need to define a general concept of Galois type over a model, and then see that under tameness it is equivalent to our notion of weak type over a model.

Definition 3.14 ($\mathscr{B}$-Galois type over a model) *Let $\mathscr{A}, \mathscr{B}, \mathscr{D} \in \mathbb{K}$, $\bar{a} \in \mathscr{B}$, $\bar{b} \in \mathscr{D}$, $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$ and $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{D}$. We say that $t^g_{\mathscr{B}}(\bar{a}/\mathscr{A}) = t^g_{\mathscr{C}}(\bar{b}/\mathscr{A})$ if there is $\mathscr{C} \in \mathbb{K}$ and AE-embeddings $f : \mathscr{B} \to \mathscr{C}$ and $g : \mathscr{D} \to \mathscr{C}$ such that $f(\bar{a}) = g(\bar{b})$ and $f \restriction_{\mathscr{A}} = g \restriction_{\mathscr{A}} = \mathrm{Id}_{\mathscr{A}}$.*

We can see as in Remark 2.2 the following:

Remark 3.15 *Let $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$ and $\bar{a}, \bar{b} \in \mathfrak{M}$. Then $t^g_{\mathfrak{M}}(\bar{a}/\mathscr{A}) = t^g_{\mathfrak{M}}(\bar{b}/\mathscr{A})$ if and only if $t^g(\bar{a}/\mathscr{A}) = t^g(\bar{b}/\mathscr{A})$, where the latter one means that there is $f \in \mathrm{Aut}(\mathfrak{M})$ fixing $\mathscr{A}$ such that $f(\bar{a}) = \bar{b}$.*

Definition 3.16 (Tameness) *Let $LS(\mathbb{K}) \leq \kappa \leq \lambda$. We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is (κ, λ)-tame, if for all $\mathscr{C}, \mathscr{D}, \mathscr{A} \in \mathbb{K}$, $\bar{a} \in \mathscr{C}$ and $\bar{b} \in \mathscr{D}$ such that $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B}$ and $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{D}$, we have that if $t^g_{\mathscr{C}}(\bar{a}/\mathscr{A}) \neq t^g_{\mathscr{D}}(\bar{b}/\mathscr{A})$ and $|A| \leq \lambda$, then there is some $B \preccurlyeq_{\mathbb{K}} \mathscr{A}$ of size κ such that $t^g_{\mathscr{C}}(\bar{a}/\mathscr{B}) \neq t^g_{\mathscr{D}}(\bar{b}/\mathscr{B})$.*

We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is tame if it is $(LS(\mathbb{K}), \lambda)$-tame for all cardinals $\lambda \geq LS(\mathbb{K})$.

From 3.15 it follows that if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is tame and $\mathscr{A}$ is a model, $t^g(\bar{a}/\mathscr{A}) = t^g(\bar{b}/\mathscr{A})$ if and only if $t^g(\bar{a}/\mathscr{B}) = t^g(\bar{b}/\mathscr{B})$ for every countable $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{A}$. The next remark follows from Theorem 2.23.

Remark 3.17 *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is tame. If* $\mathscr{A}$ *is a model, we have that* $t^w(\bar{a}/\mathscr{A}) = t^w(\bar{b}/\mathscr{A})$ *if and only if* $t^g(\bar{a}/\mathscr{A}) = t^g(\bar{b}/\mathscr{A})$*.*

Now similarly as in 2.24, only by induction on $|\mathscr{A}|$, using Remark 3.17 instead of Theorem 2.23, we can prove the following lemma.

Lemma 3.18 *Assume* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is tame,* $\mathscr{A}$ *a model, and that for all finite* $A \subset \mathscr{A}$ *there is* $\bar{a}_A$ *such that if* $B \subset A$*, then* $t^g(\bar{a}_B/B) = t^g(\bar{a}_A/B)$*. Then there is* $\bar{a}$ *such that for all finite* $A \subset \mathscr{A}$*,* $t^g(\bar{a}/A) = t^g(\bar{a}_A/A)$*.*

Theorem 3.19 (Existence of free extension) *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is tame. Let* $\mathscr{A}$ *be* ω*-saturated model and* $E \subset \mathscr{A}$ *finite such that* $t^w(\bar{a}/\mathscr{A})$ *does not split over* E*. Then if* $B \supset \mathscr{A}$*, there is* $\bar{b}$ *such that* $t^w(\bar{a}/\mathscr{A}) = t^w(\bar{b}/\mathscr{A})$ *and* $t^w(\bar{b}/B)$ *does not split over* E*.*

Proof: By monotonicity, we may assume that $B = \mathscr{B}$ is an ω-saturated model.

Let $\mathscr{A}_0 \preccurlyeq_{\mathbb{K}} \mathscr{A}$ be countable such that $E \subset \mathscr{A}_0$. For every finite $B \subset \mathscr{B}$ we get from the restricted existence of free extension we proved in 2.20 some $\bar{b}_B$ such that $t^w(\bar{b}_B/\mathscr{A}_0) = t^w(\bar{a}/\mathscr{A}_0)$ and $t^w(\bar{b}_B/\mathscr{A}_0 \cup B)$ does not split over E. When B and B' are finite and $B \subset B' \subset \mathscr{B}$, we have that $t^w(\bar{b}_B/\mathscr{A}_0) = t^w(\bar{b}_{B'}/\mathscr{A}_0)$, $\bar{b}_B \downarrow^s_{\mathscr{A}_0} B$ and $\bar{b}_{B'} \downarrow^s_{\mathscr{A}_0} B$. Thus $t^w(\bar{b}_B/\mathscr{A}_0 \cup B) = t^w(\bar{b}_{B'}/\mathscr{A}_0 \cup B)$ by uniqueness. Hence we may use lemma 3.18 to get such $\bar{b}$ that $t^g(\bar{b}/B) = t^g(\bar{b}_B/B)$ for every finite $B \subset \mathscr{B}$.

First we see that $t^w(\bar{b}/\mathscr{B})$ does not split over E. That is because if it would split, there would be some witnesses $\bar{c}, \bar{d} \in \mathscr{B}$. But this would contradict the fact that $t^g(\bar{b}/E \cup \{\bar{c}, \bar{d}\}) = t^g(\bar{b}_{E \cup \{\bar{c}, \bar{d}\}}/E \cup \{\bar{c}, \bar{d}\})$ and $t^w(\bar{b}_{E \cup \{\bar{c}, \bar{d}\}}/E \cup \{\bar{c}, \bar{d}\})$ does not split over E.

Then we see that actually $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$. When $A \subset \mathscr{A}$ is a finite subset, we have that $t^w(\bar{a}/\mathscr{A}_0 \cup A)$ does not split over E. Then again from uniqueness we get that $t^w(\bar{a}/\mathscr{A}_0 \cup A) = t^w(\bar{b}_A/\mathscr{A}_0 \cup A)$ and thus $t^g(\bar{a}/A) = t^g(\bar{b}_A/A) = t^g(\bar{b}/A)$. $\square$

We assumed $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ to be ω-stable. When we assume tameness, we gain κ-stability also for every other cardinal κ.

Definition 3.20 *We say that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* κ*-Galois-stable, if for every* $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathscr{B} \in \mathbb{K}$*,* $|\mathscr{A}| \leq \kappa$ *and a sequence* $(\bar{a}_i)_{i<\kappa^+}$*, where* $\bar{a}_i \in \mathscr{B}$ *for every* $i < \kappa^+$*, there are* $i_0, j_0 < \kappa^+$ *such that* $t^g_{\mathscr{B}}(\bar{a}_{i_0}/\mathscr{A}) = t^g_{\mathscr{B}}(\bar{a}_{j_0}/\mathscr{A})$*.*

Theorem 3.21 *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is tame. Then it is also* κ*-Galois-stable for every infinite* κ*.*

Proof: Let $\mathscr{A}$ be a model of size κ. By ω-stability, there is an ω-saturated model A' of size κ such that $\mathscr{A} \subset \mathscr{A}'$. If $t^g(\bar{a}/\mathscr{A}) \neq t^g(\bar{b}/\mathscr{A})$, then $t^g(\bar{a}/\mathscr{A}') \neq t^g(\bar{b}/\mathscr{A}')$. Thus there are more different types over $\mathscr{A}'$ and hence we may assume that $\mathscr{A}$ is ω-saturated.

Let $\bar{a} \in \mathfrak{M}$. By Lemma 2.13 there is finite $A \subset \mathscr{A}$ such that $t^w(\bar{a}/\mathscr{A})$ does not split over A. Let $\mathscr{B}_A \preccurlyeq_{\mathbb{K}} \mathscr{A}$ be ω-saturated and countable such that $A \subset \mathscr{B}_A$. Now if $\bar{b} \in \mathfrak{M}$ is such that $t^w(\bar{b}/\mathscr{B}_A) = t^w(\bar{a}/\mathscr{B}_A)$ and $t^w(\bar{b}/\mathscr{A})$ does not split over A, we get by the uniqueness property proved in Theorem 2.20 that for every countable B such that $\mathscr{B}_A \subset B \subset \mathscr{A}$, $t^w(\bar{b}/B) = t^w(\bar{a}/B)$. Furthermore we get that $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$, and then from Remark 3.17 that $t^g(\bar{b}/\mathscr{A}) = t^g(\bar{a}/\mathscr{A})$.

Let $(\bar{a}_i)_{i<\kappa^+} \subset \mathfrak{M}$. There are only κ-many finite sets $A \subset \mathscr{A}$. Then there is a subsequence $(\bar{a}_{i_j})_{j<\kappa^+}$ such that $t^w(\bar{a}_{i_j}/\mathscr{A})$ does not split over the same finite set A for all $j < \kappa^+$. Then by ω-stability, there are only countably many weak types over $\mathscr{B}_A$ for each A. Thus there are some tuples $\bar{a}_{i_\alpha}, \bar{a}_{i_\beta}$, $\alpha, \beta < \kappa^+$ such that $t^w(\bar{a}_{i_\alpha}/\mathscr{B}_A) = t^w(\bar{a}_{i_\beta}/\mathscr{B}_A)$. Then by previous reasoning, also $t^g(\bar{a}_{i_\alpha}/\mathscr{A}) = t^g(\bar{a}_{i_\beta}/\mathscr{A})$. □

Another theorem tells us that we can imply Assumption 3.2 also from κ-categoricity for suitable κ. The result is also due to Shelah and this proof is from [1].

Definition 3.22 *We say that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* κ*-categorical, if whenever* $\mathscr{A}, \mathscr{B} \in \mathbb{K}$ *and* $|\mathscr{A}| = |\mathscr{B}| = \kappa$*, then* $\mathscr{A}$ *and* $\mathscr{B}$ *are isomorphic.*

For convenience we define λ-*dense* to be the concept that is usually called λ-*dense without endpoints.*

Definition 3.23 *Let* $(I, <)$ *be a linear ordering and* $C, D \subset I$*. When* $c < d$ *for all* $c \in C, d \in D$*, we denote* $C < D$*. We say that* $(I, <)$ *is* λ*-dense, if for all* $C, D \subset I$*,* $|C|, |D| < \lambda$ *and* $C < D$*, there is* $i \in I$ *such that* $C < \{i\} < D$*, and for all* $C \subset I$*,* $|C| < \lambda$*, there are* $i, j \in I$ *such that* $\{i\} < C < \{j\}$*.*

We say that $(I, <)$ *is dense, if it is* $\aleph_0$*-dense.*

Theorem 3.24 (Existence of free extension) *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* κ*-categorical for* κ *such that* $\kappa = \kappa^{\aleph_0} \geq \lambda$*. Then if* $\mathscr{A}$ *is an* ω*-saturated model,* $\mathscr{A} \subset B$*,* $|B| \leq \lambda$ *and* $t^w(\bar{a}/\mathscr{A})$ *does not split over finite* $E \subset \mathscr{A}$*, there is* $\bar{b}$ *such that* $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$ *and* $t^w(\bar{b}/B)$ *does not split over* E*.*

Proof: Let $\mathscr{A}_0 \preccurlyeq_{\mathbb{K}} \mathscr{A}$ be countable and ω-saturated such that $E \subset \mathscr{A}$. If we find such $\bar{b}$ that $t^w(\bar{b}/\mathscr{A}_0) = t^w(\bar{b}/\mathscr{A}_0)$ and that $t^w(\bar{b}/B)$ does not split over E, we get that also $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$. That is because for every finite $A \subset \mathscr{A}$, we have that $t^w(\bar{b}/\mathscr{A}_0) = t^w(\bar{a}/\mathscr{A}_0)$, $\bar{b} \downarrow^s_{\mathscr{A}_0} \mathscr{A}_0 \cup A$ and $\bar{a} \downarrow^s_{\mathscr{A}_0} \mathscr{A}_0 \cup A$. Then $t^g(\bar{b}/A) = t^g(\bar{a}/A)$ follows from uniqueness. Thus we may assume that $\mathscr{A}$ is countable.

Our second remark is that we can construct a ω_1-saturated[4] model of size κ, and then from κ-categoricity it follows that every model of size κ is ω_1-saturated. Because $\kappa^{\aleph_0} = \kappa$, there are only κ-many countable subsets of a model of size κ, and by ω-stability, only countably many different weak types over each countable subset. Then we can construct an increasing chain of models $\mathscr{A}_i$, $i < \omega_1$, where $|\mathscr{A}_i| = \kappa$ for all $i < \omega_1$ and every type over a countable subset of $\mathscr{A}_i$ is satisfied in $\mathscr{A}_{i+1}$. Then $\bigcup_{i<\omega_1} \mathscr{A}_i$ is ω_1-saturated and of size κ.

Lemmas 3.11 and 3.12 did not use Theorem 3.19. From 3.11 we get that there is a countable order-indiscernible sequence in $\mathfrak{M}$, and from 3.12 also an order-indiscernible $(I,<)$, where $I \subset \mathfrak{M}$, $|I| = \kappa$ and $(I,<)$ is a dense linear order. Let $SH(J)$ denote the closure of $J \subset I$ with τ^*. The set $B \cup \{\bar{a}\}$ is included in $\mathscr{B}$ for some model $\mathscr{B}$ of size κ. From κ-categoricity we get that $\mathscr{B}$ and $SH(I)$ are isomorphic. Thus there is an automorphism $f \in \mathrm{Aut}(\mathfrak{M})$ mapping $\mathscr{B}$ to $SH(I)$. If we find $\bar{b}$ as in the claim for $f(\mathscr{A}), f(B)$ and $f(\bar{a})$, we can take $f^{-1}(\bar{b})$ for the claim. Thus we may assume that $\mathscr{B} \cup \{\bar{a}\} \subset SH(I)$. We have that $B \subset SH(K)$ for some $K \subset I$ such that $|K| = \lambda$. We assumed that $\mathscr{A}$ is countable, and thus $\mathscr{A} \preccurlyeq_{\mathbb{K}} SH(J_0)$ for some countable $J_0 \subset I$. We can take J_0 such that $(J_0,<) \cong (\mathbb{Q},<)$ because I is dense. Then again $J_0 \preccurlyeq_{\mathbb{K}} \mathscr{A}_1$ for some countable ω-saturated model $\mathscr{A}_1 \preccurlyeq_{\mathbb{K}} SH(I)$. This way we can get a increasing chain of models $\mathscr{A}_n$ and $SH(J_n)$ such that $|A_n| = |SH(J_n)| = \aleph_0$, $A_n \preccurlyeq_{\mathbb{K}} SH(J_n) \preccurlyeq_{\mathbb{K}} SH(I)$, $(J_n,<) \cong (\mathbb{Q},<)$ and A_n is ω-saturated for all $n < \omega$. Finally $J = \bigcup_{n<\omega}(J_n,<) \cong (\mathbb{Q},<)$, because it is a countable dense linear order, and $SH(J)$ is a countable ω-saturated model. By restricted existence of free extension there is $\bar{a}' \in \mathfrak{M}$ such that $t^w(\bar{a}'/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$ and $t^w(\bar{a}'/SH(J))$ does not split over E. Because $SH(I)$ is ω_1-saturated, there is such $\bar{a}'$ in $SH(I)$.

We use again Lemma 3.12 to find an order-indiscernible $(I',<)$ in $\mathfrak{M}^*$ such that $(I',<)$ is a λ^+-dense linear order, $(I,<) \cong (I'',<)$ for some $I'' \subset I'$ and for every finite $n < \omega$ and $i_0 < ... < i_n \in I$, $j_0 < ... < j_n \in I'$, we have that $t^*(i_0,...,i_n/\emptyset) = t^*(j_0,...,j_n/\emptyset)$. Then we have an automorphism of $\mathfrak{M}^*$ mapping I to I'', and thus have that I is a suborder of a order-indiscernible λ^+-dense linear order in $\mathfrak{M}^*$. We call this order I^+.

Let $i_0 < ... < i_{n-1} \in I$ and functions $F^{n_0}_{k_0}, ..., F^{n_p}_{k_p} \in \tau^*$ be such that

$$\bar{a}' = ((F^{n_0}_{k_0})^{\mathfrak{M}}(i^0_0, ..., i^0_{n_0-1}), ..., (F^{n_p}_{k_p})^{\mathfrak{M}}(i^p_0, ..., i^p_{n_p-1})) \tag{3.5}$$

[4] A model $\mathscr{A}$ is ω_1-saturated, if for every $\bar{a}$ and every countable $B \subset \mathscr{A}$ there is $\bar{b} \in \mathscr{A}$ such that $t^w(\bar{b}/B) = t^w(\bar{a}/B)$.

and $\{i_0, ..., i_{n-1}\} = \{i^0_0, ..., i^0_{n_0-1}, ..., i^p_0, ..., i^p_{n_p-1}\}$. Then we find $j_0 < ... < j_{n-1} \in I^+$ such that

1. if $i_k \in J$, then $j_k = i_k$,
2. $i_k < j$ if and only if $j_k < j$ for all $j \in J$,
3. if $i_k \notin J$, then $j_k \notin J \cup K$,
4. if there is $k \in K \setminus J$ such that $j_k < k < j_{k+1}$, then there are infinitely many $j \in J$ such that $j_k < j < j_{k+1}$,
5. if there is $k \in K$ between some j_k and $j \in J$, then there are infinitely many $i \in J$ in that same interval,
6. if there are $k \in K$ such that $k < i_0$, then there is infinitely many such $j \in J$ and similarly for $k > j_{n-1}$.

First we look at such $k, ..., k+p$ that $[i_k, i_{k+p}] \cap J = \emptyset$ and p has been chosen maximally. If there are no $j \in J$ such that $j < i_k$, we get from the λ^+-density of I^+ such $j_k < ... < j_{k+p}$ that $j_{k+p} < j$ for all $j \in K \cup J$. Symmetrically if there are no $j \in J$ such that $j > i_{k+p}$. Then clearly condition 6 holds. Next assume that there are elements of J on both sides of the interval $[i_k, i_{k+p}]$. In that case condition 6 holds because J is dense. Define $j_{inf} = \inf\{j \in J : j > i_{k+p}\}$ and $j_{sup} = \sup\{j \in J : j < i_k\}$. Both i_{inf} and i_{sup} can't be in J, because J is dense. Assume that i_{sup} is not in J. Let $C = \{i \in J \cup K : i < j_{inf}\}$ and $D = \{i \in J \cup K : i \geq j_{inf}\}$. By λ^+-density of I^+, there are $j_k < ... < j_{k+p} \in I^+$ such that $C < \{j_k, ..., j_{k+p}\} < D$. Now there are no elements of K between j_{k+p} and i_{inf} or between j_{k_1} and j_{k_2}, when $k_1, k_2 \in \{k, ..., k+p\}$. Also there are always infinitely many elements of J between j_{inf} and some $i \in J$ such that $i > j_{inf}$ and also between j_{sup} and some $i \in J$ such that $i < j_{sup}$. Thus we see that 5 holds for these $j_k, ..., j_{k+p}$. The case when i_{inf} is not in J goes similarly.

Then we look at such $i_k = j_k$ that $i_k \in J$. Conditions 5 and 6 follow from the density of J. Also if j_{k-1} or j_{k+1} are in J, 4 follows from the density of J. If not, 4 follows from the condition 5 for j_{k-1} or j_{k+1}. We see that 4 holds for all $j_0, ..., j_{n-1}$. That is because if there is none or only one $j \in J$ such that $j_k \leq j \leq j_{k+1}$, it follows from above construction that there are no element of K in that interval. If there are at least two such j, then there are also infinitely many.

Finally let $\bar{b}$ be generated from $j_0, ..., j_{n-1}$ as $\bar{a}'$ was from $i_0, ..., i_{n-1}$, that is

$$\bar{b} = ((F^{n_0}_{k_0})^{\mathfrak{M}}(j^0_0, ..., j^0_{n_0-1}), ..., (F^{n_p}_{k_p})^{\mathfrak{M}}(j^p_0, ..., j^p_{n_p-1})),$$

where $j^r_{q_s} = j_k$ if and only if $i^r_{q_s} = i_k$ in 3.5. We claim that $t^*(\bar{a}'/SH(J_0)) = t^*(\bar{b}/SH(J_0))$ for every finite $J_0 \subset J$. This is because we have an order-preserving map f such that $f(i_k) = j_k$ for all $k \in \{0, ..., n-1\}$ and $f(j) = j$

for all $j \in J_0$. Because $(I,<)$ is order-indiscernible, this f extends to $F \in \mathrm{Aut}(\mathfrak{M}^*)$. Then $F(\bar{a}') = \bar{b}$ and $F\restriction_{SH(J_0)} = \mathrm{Id}_{SH(J_0)}$. Because $\mathscr{A} = SH(J)$, we have that $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}'/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$.

Then we claim that $t^w(\bar{b}/B)$ does not split over E. Assume that there would be some $\bar{c}, \bar{d} \in B \subset SH(K)$ witnessing the contrary. Let $r_0 < ... < r_m \in J$ be such that $E \cup (\{\bar{c},\bar{d}\} \cap SH(J)) \subset SH(\{r_0, ..., r_m\})$. Then let $p_0, ..., p_{m'} \in K \setminus J$ be such that $(\{\bar{c},\bar{d}\} \setminus SH(J)) \subset SH(\{p_0, ..., p_{m'}\})$. Let f be order-preserving such that $f(j_k) = i_k$ for $k \in \{0, ..., n-1\}$ and $f(r_k) = r_k$ for $k \in \{0, ..., m\}$. Look at such $p_{j_0} < ... < p_{j_k}$ that $p_{j_k} < \{j_0, ..., j_{n-1}, r_0, ..., r_m\}$. Because $p_{j_k} \in K$ we get from condition 6 elements $f(p_{j_0}) < ... < f(p_{j_k}) \in J$ such that $f(p_{j_k}) < p_{j_k}$. Then, because of 2, also $f(p_{j_k}) < \{i_0, ..., i_{n-1}, r_0, ..., r_m\}$. Other cases similarly. For $p_{j_0} < ... < p_{j_k}$ between some two elements in $\{j_0, ..., j_{n-1}, r_0, ..., r_m\}$, it depends whether they belong to J or not, if we use the $\aleph_0$-density of $(J,<)$ or properties 4 and 5 to find suitable $f(p_{j_0}) <, ..., < f(p_{j_k}) \in J$. Finally we find an order-preserving $f : \{j_0, ..., j_{n-1}, r_0, ..., r_m, p_0, ..., p_{m'}\} \to J$, which extends to $F \in \mathrm{Aut}(\mathfrak{M}^*)$. Then $F(\bar{c}), F(\bar{d}) \in SH(J)$ and $F\restriction_{E\cup\{\bar{a}'\}} = \mathrm{Id}_{E\cup\{\bar{a}'\}}$, thus $F(\bar{c}), F(\bar{d})$ witness that $t^w(\bar{a}'/SH(J))$ splits over E, a contradiction. □

We note that if we assume κ-kategoricity for $\kappa \geq 2^{\aleph_0}$, we do not need to assume ω-stability for $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$.

Theorem 3.25 *Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be a local abstract elementary class, which is* κ*-categorical for* $\kappa \geq 2^{\aleph_0}$. *Then it is* ω*-stable.*

Proof: Let A be countable. As we saw in Remark 3.8, there are at most $2^{\aleph_0}$-many $*$-types over A, and hence at most such many weak types also. Then let $\mathscr{B}$ be a model such that $A \subset \mathscr{B}$, $|\mathscr{B}| = \kappa$ and every weak type over A is represented in $\mathscr{B}$. Then let $(\kappa, <) \subset \mathfrak{M}^*$ be an order-indiscernible sequence of order-type κ, and denote again by $SH(\kappa)$ the model we get closing $(\kappa,<)$ with the functions of τ^*. From categoricity we get $f \in \mathrm{Aut}(\mathfrak{M})$ mapping $\mathscr{B}$ to $SH(\kappa)$. Then $f(A) \subset SH(J)$ for some countable $J \subset \kappa$. If $\bar{a}$ and $\bar{b}$ in $\mathscr{B}$ have different weak type over A, then $f(\bar{a})$ and $f(\bar{b})$ in $SH(\kappa)$ have different weak type over $SH(J)$. It is enough to show that there are at most countably many tuples in $SH(\kappa)$ with pairwise different $*$-type over $SH(J)$.

Every tuple $\bar{b} \in SH(\kappa)$ is generated by finitely many functions of τ^* from a finite suborder $\bar{i}$ of κ. Tuples $\bar{a}$ and $\bar{b}$ have same $*$-type over $SH(\alpha)$ if and only if there are $\bar{i}$ and $\bar{j}$ in κ such that $\bar{a}$ can be generated from $\bar{i}$ similarly and with the same functions that $\bar{b}$ can be generated from $\bar{j}$ and a partial order-preserving $f : \kappa \to \kappa$ such that $f(\bar{i}) = \bar{j}$ and $f\restriction_J = \mathrm{Id}_J$.

Let $(\bar{a}_i)_{i<\omega_1}$ be a sequence of tuples in $SH(\kappa)$. Choose a finite sequence of functions in τ^* and suborders $\bar{j}_i$ in κ for each $\bar{a}_i$. There are only countably many different finite sequences of functions in τ^*, thus by the pigeonhole principle there is a subsequence $(\bar{a}_{n_i})_{i<\omega_1}$ such that every $\bar{a}_{n_i}$ can be generated from some $\bar{j}_{n_i} \in \kappa$ with the same functions. Furthermore we can choose $\bar{a}_{n_i}$ such that they are generated similarly, because there are only finitely many ways to order $\bar{j}_{n_i}$. Also there are only countably many ways to order a finite set compared to the countable well-order J. Thus we may find indexes n_α and n_β such that when we denote $\bar{j}_{n_\alpha} = (k_0, ..., k_m)$ and $\bar{j}_{n_\beta} = (p_0, ..., p_m)$, we have that $k_n = j$ if and only if $p_n = j$ and $k_n < j$ if and only if $p_n < j$ for all $0 \leq n \leq m$ and $j \in J$. Then we have a partial mapping as above, and hence $t^*(\bar{a}_{n_\alpha}/SH(J)) = t^*(\bar{a}_{n_\beta}/SH(J))$ for some $n_\alpha, n_\beta < \omega_1$, and tuples$(\bar{a}_i)_{i<\omega_1}$ don't have pairwise disjoint $*$-type over $SH(J)$. □

4 U-rank

In this section we assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a local abstract elementary class with ω-stability and existence of free extension. Then we may use all properties derived from these in the previous chapters, including symmetry. Again we work inside a monster model.

Definition 4.1 *Let $\mathscr{A}$ be countable and ω-saturated model. Define U-rank of $\bar{a}$ over $\mathscr{A}$, $U(\bar{a}/\mathscr{A})$, by induction:*

1. *Always $U(\bar{a}/\mathscr{A}) \geq 0$.*
2. *$U(\bar{a}/\mathscr{A}) \geq \beta + 1$ iff there is countable ω-saturated model $\mathscr{B}$ such that $\mathscr{A} \subset \mathscr{B}$, $U(\bar{a}/\mathscr{B}) \geq \beta$ and $\bar{a} \not\downarrow^s_{\mathscr{A}} \mathscr{B}$*

For a countable ω-saturated model $\mathscr{A}$, define

$$U(\bar{a}/\mathscr{A}) = \min\{\alpha : U(\bar{a}/\mathscr{A}) \ngeq \alpha + 1\}$$

if such an ordinal exists. Then define U-rank for arbitrary ω-saturated model $\mathscr{A}$ as

$$U(\bar{a}/\mathscr{A}) = \min\{U(\bar{a}/\mathscr{A}') : \mathscr{A}' \subset \mathscr{A} \text{ countable } \omega\text{-saturated model.}\}$$

For a countable ω-saturated model $\mathscr{A}$ we say that $U(\bar{a}/\mathscr{A})$ is defined if there exists an ordinal α such that $U(\bar{a}/\mathscr{A}) \ngeq \alpha + 1$. Also the above minimum is defined for arbitrary $\mathscr{A}$ if it is defined for some countable ω-saturated model $\mathscr{A}' \subset \mathscr{A}$. The next lemma shows that $U(\bar{a}/\mathscr{A})$ is actually defined for all ω-saturated $\mathscr{A}$.

Lemma 4.2 *$U(\bar{a}/\mathscr{A})$ is defined for all $\bar{a}$ and all ω-saturated models $\mathscr{A}$.*

Proof: For a countable ω-saturated model $\mathscr{A}$, denote

$$\alpha_{\mathscr{A}} = \sup\{U(\bar{a}/\mathscr{A}) : \bar{a} \in \mathfrak{M} \text{ and } U(\bar{a}/\mathscr{A}) \text{ is defined. }\} + 1.$$

Furthermore, let $\mathbf{A} = \{\mathscr{A} \subset \mathfrak{M} : \mathscr{A}\ \omega\text{-saturated, countable model }\}$ and $\alpha = \sup\{\alpha_{\mathscr{A}} : \mathscr{A} \in \mathbf{A}\}$. Assume the contrary, that there would be some $\mathscr{A}_0 \in \mathbf{A}$ and $\bar{a}$ such that $U(\bar{a}/\mathscr{A}_0)$ is not defined, i.e. $U(\bar{a}/\mathscr{A}_0) > \beta$ for all ordinals β. Then also $U(\bar{a}/\mathscr{A}_0) > \alpha$. Assume we have defined $\mathscr{A}_i \in \mathbf{A}$ for $i \leq n$ such that

1. when $i \leq j$, $\mathscr{A}_i \subset \mathscr{A}_j$,
2. when $i < n$, $\bar{a} \not\downarrow^{s}_{\mathscr{A}_i} \mathscr{A}_{i+1}$ and
3. $U(\bar{a}/\mathscr{A}_i) > \alpha$.

Now $U(\bar{a}/\mathscr{A}_n) \geq \alpha + 1$ and from the definition of U-rank we then get some $\mathscr{A}_{n+1} \in \mathbf{A}$ such that $\mathscr{A}_n \subset \mathscr{A}_{n+1}$, $\bar{a} \not\downarrow^{s}_{\mathscr{A}_n} \mathscr{A}_{n+1}$ and $U(\bar{a}/\mathscr{A}_{n+1}) \geq \alpha \geq \alpha_{\mathscr{A}_{n+1}}$. Thus from the definition of $\alpha_{\mathscr{A}_{n+1}}$ we get that $U(\bar{a}/\mathscr{A}_{n+1})$ is not defined, and particularly, $U(\bar{a}/\mathscr{A}_{n+1}) > \alpha$.

Finally $\mathscr{A} = \bigcup_{i<\omega} \mathscr{A}_i$ is a countable model. We would like to get a contradiction with Lemma 2.12, but the lemma forbids a chain of finite sets, and our sets $\mathscr{A}_i$ are countable. Next we find finite sets B_i such that $\bigcup_{i<\omega} B_i = \mathscr{A}$ and $t^w(\bar{a}/B_{i+1})$ splits over B_i for all $i < \omega$. To assure that $\mathscr{A} \subset \bigcup_{i<\omega} B_i$, we write $\mathscr{A} = \{a_i : i < \omega\}$ and make a_i be an element of B_i for each $i < \omega$. We may assume that the a_i's are chosen so that $a_i \in \mathscr{A}_i$ for $i < \omega$.

Let $a_0 \in B_0 \subset \mathscr{A}_0$. Then assume we have defined an increasing chain of finite sets B_i for $i \leq n$ such that $a_i \in B_i \subset \mathscr{A}_i$ for $i \leq n$ and $t^w(\bar{a}/B_{i+1})$ splits over B_i for $i < n$. Then $t^w(\bar{a}/\mathscr{A}_{n+1})$ splits over $B_n \subset \mathscr{A}_n$, and there are some $c_n, d_n \in \mathscr{A}_{n+1}$ witnessing that. We can take $B_{n+1} = B_n \cup \{a_{n+1}, c_n, d_n\} \subset \mathscr{A}_{n+1}$.

These $\bar{a}$ and $(B_i)_{i<\omega}$ contradict Lemma 2.12. □

These two remarks follow easily from the definition:

Remark 4.3 *If $U(\bar{a}/\mathscr{A}) = \alpha$ and g is an automorphism of $\mathfrak{M}$ then $U(g(a)/g(\mathscr{A})) = \alpha$.*

Remark 4.4 *If $\mathscr{A}$ and $\mathscr{B}$ are ω-saturated models such that $\mathscr{A} \subset \mathscr{B}$, then $U(\bar{a}/\mathscr{B}) \leq U(\bar{a}/\mathscr{A})$.*

Definition 4.5 *We say that $\bar{a}$ and a set A are finitely equivalent to $\bar{a}'$ and A', write*

$$(\bar{a}, A) \equiv_{\emptyset} (\bar{a}', A')$$

if there is a bijective mapping $f : \bar{a} \cup A \to \bar{a}' \cup A'$ such that $f(\bar{a}) = \bar{a}'$ and for all $\bar{b} \in A$ $t^g(\bar{a}^\frown\bar{b}/\emptyset) = t^g(\bar{a}'^\frown f(\bar{b})/\emptyset)$.

We see that if $t^w(\bar{a}/A) = t^w(\bar{a'}/A)$, then $(\bar{a}, A) \equiv_{\emptyset} (\bar{a}', A)$.

Remark 4.6 *If $\mathscr{A}$ and $\mathscr{A}'$ are ω-saturated models such that $(\bar{a}, \mathscr{A}) \equiv_{\emptyset} (\bar{a}', \mathscr{A}')$, then $U(\bar{a}/\mathscr{A}) = U(\bar{a}'/\mathscr{A}')$.*

Proof: By the definition of U-rank, it is enough to prove the claim for all countable $\mathscr{A}$ and $\mathscr{A}'$. Hence we assume that $\mathscr{A}$ and $\mathscr{A}$ are countable.

Let $f : \bar{a} \cup \mathscr{A} \to \bar{a}' \cup \mathscr{A}'$ be the mapping from the definition 4.5. Now $f \restriction_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}'$ extends to an automorphism g.

When $\bar{c} \in \mathscr{A}'$ finite, we have that $g^{-1}(\bar{c}) = f^{-1}(\bar{c}) \in \mathscr{A}$ and $t^g(g(\bar{a})^\frown \bar{c}/\emptyset) = t^g(\bar{a}^\frown f^{-1}(\bar{c})/\emptyset) = t^g(\bar{a}'^\frown \bar{c}/\emptyset)$. Thus $t^w(g(\bar{a})/\mathscr{A}') = t^w(\bar{a}'/\mathscr{A}')$ and we get from Theorem 2.23 an automorphism h such that $h(g(\bar{a})) = \bar{a}'$ and $h \restriction_{\mathscr{A}'} = \mathrm{Id}_{\mathscr{A}'}$.

Now $h \circ g$ is an automorphism, $h \circ g(\bar{a}) = \bar{a}'$ and $h \circ g(\mathscr{A}) = \mathscr{A}'$. The claim follows from Remark 4.3. □

Lemma 4.7 *Assume that $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$, $\mathscr{A} \subset \mathscr{B}$ and $\mathscr{A}, \mathscr{B}$ are countable, ω-saturated models. Then if $U(\bar{a}/\mathscr{A}) \geq \alpha$, also $U(\bar{a}/\mathscr{B}) \geq \alpha$.*

Proof: The proof is by induction on α, and we prove the implication for all $\mathscr{A}$, $\mathscr{B}$ and $\bar{a}$ simultaneously. If α is 0 or a limit ordinal, the induction step is clear. Assume that $\alpha = \beta + 1$ and that $\mathscr{C}$ is an ω-saturated countable model such that $\mathscr{A} \subset \mathscr{C}$, $\bar{a} \not\downarrow^s_{\mathscr{A}} \mathscr{C}$, and $U(\bar{a}/\mathscr{C}) \geq \beta$.

We use Lemma 2.21 to get a tuple $\bar{a}'$ and countable set $\mathscr{C}$’ such that $t^w(\bar{a}^\frown \mathscr{C}/\mathscr{A}) = t^w(\bar{a}'^\frown \mathscr{C}'/\mathscr{A})$ and $\bar{a}'^\frown \mathscr{C}' \downarrow^s_{\mathscr{A}} \mathscr{B}$. Then also $(\bar{a}', \mathscr{C}') \equiv_{\emptyset} (\bar{a}', \mathscr{C})$. Because we may gain an automorphism mapping $\mathscr{C}$ to $\mathscr{C}'$, we see that also $\mathscr{C}'$ is an ω-saturated model. Then from Remark 4.6 we get that $U(\bar{a}'/\mathscr{C}') \geq \beta$. Also $\mathscr{A} \subset \mathscr{C}'$, $t^w(\bar{a}'/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$ and we can also easily see that $\bar{a}' \not\downarrow^s_{\mathscr{A}} \mathscr{C}'$.

Let $\mathscr{D}$ be a countable ω-saturated model such that $\mathscr{C}' \cup \mathscr{B} \subset \mathscr{D}$. From the existence of free extension we get $\bar{a}^*$ such that $t^w(\bar{a}^*/\mathscr{C}') = t^w(\bar{a}'/\mathscr{C}')$ and $\bar{a}^* \downarrow^s_{\mathscr{C}'} \mathscr{D}$. Then also $\mathscr{C}' \subset \mathscr{D}$ and $U(\bar{a}^*/\mathscr{C}') = U(\bar{a}'/\mathscr{C}') \geq \beta$, and from induction we get that

$$U(\bar{a}^*/\mathscr{D}) \geq \beta. \tag{4.6}$$

Next we would like to show that $t^w(\bar{a}^*/\mathscr{B}) = t^w(\bar{a}/\mathscr{B})$. In order to do that, we take arbitrary finite $\bar{b} \in \mathscr{B}$ and claim that

$$\bar{a}^* \downarrow^s_{\mathscr{A}} \bar{b}. \tag{4.7}$$

Let $\bar{b}'$ be a free extension such that $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{b}'/\mathscr{A})$ and $\bar{b}' \downarrow^s_{\mathscr{A}} \mathscr{C}'$. Let $\bar{c} \in \mathscr{C}'$ be finite. Because $\mathscr{C}' \downarrow^s_{\mathscr{A}} \mathscr{B}$, we get from symmetry that $\bar{b} \downarrow^s_{\mathscr{A}} \bar{c}$. By monotonicity $\bar{b}' \downarrow^s_{\mathscr{A}} \bar{c}$ and we get from the uniqueness of free extension that $t^w(\bar{b}/\mathscr{A} \cup \{\bar{c}\}) = t^w(\bar{b}'/\mathscr{A} \cup \{\bar{c}\})$. Because this holds for all finite $\bar{c} \in \mathscr{C}'$, we get that $t^w(\bar{b}/\mathscr{C}') = t^w(\bar{b}'/\mathscr{C}')$. Then also $\bar{b} \downarrow^s_{\mathscr{A}} \mathscr{C}'$.

Because $\bar{a}^* \downarrow^s_{\mathscr{C}'} \mathscr{D}$, we get that $\bar{a}^* \downarrow^s_{\mathscr{C}'} \bar{b}$ and again from symmetry that $\bar{b} \downarrow^s_{\mathscr{C}'} \bar{a}^*$. Now we have that $\mathscr{A} \subset \mathscr{C}' \subset \mathscr{C}' \cup \{\bar{a}^*\}$, $\mathscr{C}'$ ω-saturated, $\bar{b} \downarrow^s_{\mathscr{C}'} \mathscr{C}' \cup \{\bar{a}^*\}$ and $\bar{b} \downarrow^s_{\mathscr{A}} \mathscr{C}'$. We may use transitivity to get $\bar{b} \downarrow^s_{\mathscr{A}} \mathscr{C}' \cup \{\bar{a}^*\}$. Claim (4.7) follows from symmetry.

Now we take a free extension $\bar{d}$ such that $t^w(\bar{d}/\mathscr{A}) = t^w(\bar{a}^*/\mathscr{A})$ and $\bar{d} \downarrow^s_{\mathscr{A}} \mathscr{B}$. Then from (4.7) we get that for all finite $\bar{b} \in \mathscr{B}$ both $\bar{d} \downarrow^s_{\mathscr{A}} \bar{b}$ and $\bar{a}^* \downarrow^s_{\mathscr{A}} \bar{b}$. Again we get by uniqueness that $t^w(\bar{a}^*/\mathscr{A} \cup \bar{b}) = t^(\bar{d}/\mathscr{A} \cup \bar{b})$ for all finite $\bar{b} \in \mathscr{B}$, and thus $t^w(\bar{a}^*/\mathscr{B}) = t^w(\bar{d}/\mathscr{B})$. Hence also $\bar{a}^* \downarrow^s_{\mathscr{A}} \mathscr{B}$.

Then because $\bar{a}^* \downarrow^s_{\mathscr{A}} \mathscr{B}$, $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$ and $t^w(\bar{a}/\mathscr{A}) = t^w(\bar{a}^*/\mathscr{A})$, we again get from uniqueness that

$$t^w(\bar{a}/\mathscr{B}) = t^w(\bar{a}^*/\mathscr{B}). \tag{4.8}$$

Because we have that $\mathscr{B} \subset \mathscr{D}$, $\mathscr{D}$ ω-saturated and we have shown (4.6), we would like to show that also

$$\bar{a}^* \not\downarrow^s_{\mathscr{B}} \mathscr{D}. \tag{4.9}$$

Assume the contrary, that $\bar{a}^* \downarrow^s_{\mathscr{B}} \mathscr{D}$. Then we get from (4.8) and $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$ that $\bar{a}^* \downarrow^s_{\mathscr{A}} \mathscr{B}$ and furthermore from transitivity that $\bar{a}^* \downarrow^s_{\mathscr{A}} \mathscr{D}$. But then because $\mathscr{C}' \subset \mathscr{D}$, also $\bar{a}^* \downarrow^s_{\mathscr{A}} \mathscr{C}'$. This is a contradiction, because we chose $\bar{a}^*$ so that $t^w(\bar{a}^*/\mathscr{C}') = t^w(\bar{a}'/\mathscr{C}')$ and we know that $\bar{a}' \not\downarrow^s_{\mathscr{A}} \mathscr{C}'$.

We have now that

$$U(\bar{a}^*/\mathscr{B}) \geq \alpha. \tag{4.10}$$

Then finally from (4.10), (4.8) and Remark 4.6 we get that $U(\bar{a}/\mathscr{B}) \geq \alpha$. □

Theorem 4.8 *For ω-saturated models $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{A} \subset \mathscr{B}$, $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$ if and only if $U(\bar{a}/\mathscr{A}) = U(\bar{a}/\mathscr{B})$.*

Proof: We prove the claim first for countable $\mathscr{A}$ and $\mathscr{B}$. If $\bar{a} \not\downarrow^s_{\mathscr{A}} \mathscr{B}$, we can take $\mathscr{B}$ in Definition 4.1 to show that $U(\bar{a}/\mathscr{A}) \geq U(\bar{a}/\mathscr{B}) + 1$. Thus from $U(\bar{a}/\mathscr{A}) = U(\bar{b}/\mathscr{B})$ it follows that $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$. Also if we have that $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$, we get from Lemma 4.7 that $U(\bar{a}/\mathscr{A}) \leq U(\bar{a}/\mathscr{B})$, and then by 4.4 $U(\bar{a}/\mathscr{A}) = U(\bar{a}/\mathscr{B})$.

Then let $\mathscr{A}$ and $\mathscr{B}$ be of arbitrary size. Assume that $U(\bar{a}/\mathscr{A}) = U(\bar{a}/\mathscr{B})$. Let $\mathscr{B}' \subset \mathscr{B}$ be a countable ω-saturated model such that $U(\bar{a}/\mathscr{B}')$ is minimal. Then there must be some countable ω-saturated $\mathscr{A}' \subset \mathscr{A}$ such that $U(\bar{a}/\mathscr{A}') = U(\bar{a}/\mathscr{B}')$. Now if $\bar{a} \not\downarrow^s_{\mathscr{A}} \mathscr{B}$, also $\bar{a} \not\downarrow^s_{\mathscr{A}'} \mathscr{B}$ and as in 3.3, including all the necessary witnesses we can find a countable ω-saturated model $\mathscr{B}'' \subset \mathscr{B}$ such that $\mathscr{A}' \cup \mathscr{B}' \subset \mathscr{B}''$ and $\bar{a} \not\downarrow^s_{\mathscr{A}'} \mathscr{B}''$. Now $U(\bar{a}/\mathscr{B}'') \neq U(\bar{a}/\mathscr{A}') = U(\bar{a}/\mathscr{B}')$ and because $\mathscr{B}' \subset \mathscr{B}''$, $U(\bar{a}/\mathscr{B}'') \leq U(\bar{a}/\mathscr{B}')$. This contradicts the minimality of $U(\bar{a}/\mathscr{B}')$. Thus from $U(\bar{a}/\mathscr{A}) = U(\bar{a}/\mathscr{B})$ we get that $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$. Then assume that $\bar{a} \downarrow^s_{\mathscr{A}} \mathscr{B}$. Let $\mathscr{A}'$ be a countable ω saturated model such that $\bar{a} \downarrow^s_{\mathscr{A}'} \mathscr{B}$ and $\mathscr{B}'$ again countable such that $U(\bar{a}/\mathscr{B}) = U(\bar{a}/\mathscr{B}')$. Then let $\mathscr{B}''$ be a countable ω-saturated model such that $\mathscr{A}' \cup \mathscr{B}' \subset \mathscr{B}'' \subset \mathscr{B}$. Now because $\bar{a} \downarrow^s_{\mathscr{A}'} \mathscr{B}''$, we have that $U(\bar{a}/\mathscr{A}') = U(\bar{a}/\mathscr{B}'')$. Then because $\mathscr{B}' \subset \mathscr{B}''$, we have that $U(\bar{a}/\mathscr{B}'') \leq U(\bar{a}/\mathscr{B}')$, and thus $U(\bar{a}/\mathscr{B}'') = U(\bar{a}/\mathscr{B})$. We get that $U(\bar{a}/\mathscr{A}) \leq U(\bar{a}/\mathscr{A}') = U(\bar{a}/\mathscr{B})$, and because $\mathscr{A} \subset \mathscr{B}$, $U(\bar{a}/\mathscr{A}) = U(\bar{a}/\mathscr{B})$. □

4.1 Other results

Based on the results of this paper in [5] the following theorems are proved.

Definition 4.9

1. *Suppose $A \subset \mathscr{A}$. We say that $\mathscr{A}$ is minimal over A if there is no $\mathscr{B} \preccurlyeq_{\mathbb{K}} \mathscr{A}$ such that $A \subset \mathscr{B}$ and $\mathscr{B} \neq \mathscr{A}$.*

2. *Suppose $\mathscr{A}$ is ω-saturated. We say that $(\bar{a}_i)_{i<\alpha}$ is a Morley sequence (over $\mathscr{A}$) if for all $i < j < \alpha$, $t^w(\bar{a}_i/\mathscr{A}) = t^w(\bar{a}_j/\mathscr{A})$ and for all $i < \alpha$, $\bar{a}_i \downarrow^s_{\mathscr{A}} \bigcup_{j<i} \bar{a}_j$.*

3. *We write bcl(A) for the set of those tuples $\bar{a} \in \mathfrak{M}$ such that the number of realizations of $t^w(\bar{a}/A)$ in $\mathfrak{M}$ is less or equal to $|A| + \omega$. Then we also say that $t^w(\bar{a}/A)$ is bounded.*

4. *Suppose that $\mathscr{A}$ is ω-saturated. We say that $t^w(\bar{a}/\mathscr{A})$ is minimal if it is not bounded but for all $A \supseteq \mathscr{A}$ and $\bar{b}$ such that $t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})$ the following holds: if $\bar{b} \downarrow^s_{\mathscr{A}} A$, then $t^w(\bar{b}/A)$ is bounded.*

5. *Suppose $A \subset C$. We say that C is atomic over A if for all $\bar{a} \in C$ there is finite $B \subset A$ such that for all $\bar{b}$, if $t^w(\bar{b}/B) = t^w(\bar{a}/B)$, then $t^w(\bar{b}/A) = t^w(\bar{a}/A)$.*

6. *Suppose $A \subset C$. We say that C is prime over A if for all ω-saturated $\mathscr{B}$ the following holds: If $f : A \to \mathscr{B}$ is (weak) type-preserving, then there is $F \in \mathrm{Aut}(\mathfrak{M})$ such that $f \subset F$ and $F(C) \subset \mathscr{B}$.*

Theorem 4.10 *Assume $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is tame local abstract elementary class and κ-categorical for all uncountable κ. Suppose $\mathscr{A}$ is uncountable, $\mathscr{B} \subset \mathscr{A}$ a countable ω-saturated model, $t^w(\bar{a}/\mathscr{B})$ is a minimal type and $(\bar{a}_i)_{i<\alpha} \subset \mathscr{A}$ is a maximal Morley sequence over $\mathscr{B}$ such that $t^w(\bar{a}/\mathscr{B}) = t^w(\bar{a}_i/\mathscr{B})$ for all $i < \alpha$. Then $\mathscr{A}$ is minimal, atomic and prime over $\mathscr{B} \cup \bigcup_{i<\alpha} \bar{a}_i$.*

Theorem 4.11 *Assume* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is tame local abstract elementary class and* κ*-categorical for all uncountable* κ. *Suppose* $\mathscr{A}$ *is countable* ω*-saturated and* $t^w(\bar{a}/\mathscr{A})$ *is a minimal type. Let* $P = \{\bar{b} \in \mathfrak{M} : t^w(\bar{b}/\mathscr{A}) = t^w(\bar{a}/\mathscr{A})\}$ *and* cl *an operation on* P *such that for all* $X \subset P$, $cl(X) = bcl(\mathscr{A} \cup X) \cap P$. *Then* (P, cl) *is a pregeometry.*

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