Stochastic equilibrium in financial markets

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Abstract

In this paper we will study the price-forming of securities in purely financial markets when the agents have quadratic utility functions for final wealth. We will emphasize a model where the utility parameters are sampled and agents' acts are somewhat random even in a homogeneous environment. In the scale of the whole economy some behavior is still expected and we study the deviations from this behavior.

1 Security demand and equilibrium

Consider a set of agents $i = 1, \ldots, n$ acting on a two-period financial markets with *securities* $j = 1, \ldots, \ell$ bearing risk and a safe security $j = \ell + 1$ with a fixed payoff. At the next period there are *states* $s = 1, \ldots, S$ one of which will reveal. The securities have state-dependent payoffs tomorrow in money, $\psi^j(s)$.

$$\mathbf{\Psi} = \begin{pmatrix} \psi^1(1) & \psi^2(1) & \ldots & \psi^{\ell+1}(1) \\ \psi^1(2) & \psi^2(2) & \ldots & \psi^{\ell+1}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi^1(S) & \psi^2(S) & \ldots & \psi^{\ell+1}(S) \end{pmatrix}.$$

Especially for the $\ell + 1^{st}$ commodity $\psi^{\ell+1}(s) \equiv 1 \ \forall s = 1, \ldots, S$ for which the *price* $p^{\ell+1} = 1$. Hence it can be considered as the numeraire. For the different states agents assign probabilities $q_i(s)$, $i = 1, \ldots, n$. Furthermore, agents have *initial endowments* in assets $e_i^1, \ldots, e_i^{\ell+1}$ and a *utility function* $u_i : \mathbb{R} \to \mathbb{R}$ for final wealth with a special quadratic form:

$$u_i(x) = x - \frac{x^2}{2a_i} \tag{1.1}$$

The parameter a_i^{-1} has a risk-aversion interpretation – the bigger it is, the further we are from risk-neutrality. Argument x refers to the terminal wealth of a *feasible* and *optimal* consumption allocation, or *portfolio*, as used more often.

We assume that the portfolio-holders or agents have unique beliefs $\mathbf{q}$ of future and agreement on $\mathbf{\Psi}$. We define the portfolios and future beliefs by

$$\mathbf{x}_i = \begin{pmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^{\ell+1} \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} q(1) \\ q(2) \\ \vdots \\ q(S) \end{pmatrix}.$$

We will first discuss the selection of an optimal and feasible portfolio. For this, choose one agent and supress the agent index i everywhere.

1.1 Individual security demand

Recall that the instantaneous utility of a terminal wealth was $u : \mathbb{R} \to \mathbb{R}$. If we see this from today, the utility will be $\mathbf{u} : \mathbb{R}^S \to \mathbb{R}^S$, as there are S states tomorrow. The utility of a whole portfolio $\mathbf{x}$ will then be $\mathbf{u} : \mathbb{R}^{(\ell+1)\times S} \to \mathbb{R}^S$

$$\mathbf{u}(\mathbf{\Psi x}) = (u(\mathbf{\Psi}(1)\mathbf{x}), \ldots, u(\mathbf{\Psi}(S)\mathbf{x}))^\top. \tag{1.2}$$

To define the optimal portfolio, an agent wants to maximize a utility function $U : \mathbb{R}^{\ell+1} \to \mathbb{R}$. A natural choice is the *expected utility* [1]

$$U(\mathbf{x}) = \mathbf{q}^\top \mathbf{u}(\mathbf{\Psi x}). \tag{1.3}$$

Besides optimal, the portfolio must also be feasible and thus we have a convex programming problem

$$\max\{U(\mathbf{x}) = \mathbf{q}^\top \mathbf{u}(\mathbf{\Psi x}) \,|\, \mathbf{p}^\top \mathbf{x} = \mathbf{p}^\top \mathbf{e}\}. \tag{1.4}$$

The Lagrangean is

$$L(\mathbf{x}; \lambda) = \mathbf{q}^\top \mathbf{u}(\mathbf{\Psi x}) - \lambda \mathbf{p}^\top (\mathbf{x} - \mathbf{e})$$

and the first-order condition is

$$\begin{aligned} \nabla L(\mathbf{x}; \lambda) &= \nabla(\mathbf{\Psi x})\mathbf{u}'(\mathbf{\Psi x})\mathbf{q} - \lambda \nabla(\mathbf{x} - \mathbf{e})\mathbf{p} \\ &= \mathbf{\Psi}^\top \mathbf{u}'(\mathbf{\Psi x})\mathbf{q} - \lambda \mathbf{p} = \mathbf{0}, \end{aligned}$$

where $\mathbf{u}'(\mathbf{\Psi x}) = \mathrm{diag}[u'(\mathbf{\Psi}(s)\mathbf{x})] \in \mathbb{R}^{S\times S}$. This produces the system,

[1] $U = f \circ \mathbf{g} : \mathbb{R}^{\ell+1} \to \mathbb{R}^S \to \mathbb{R}$, where $\mathbf{g}(\mathbf{x}) \doteq \mathbf{\Psi x}$ and $f(\mathbf{y}) \doteq \mathbf{q}^\top \mathbf{u}(\mathbf{y})$.

$$(*) \quad \begin{cases} \sum_{s=1}^{S} q(s)u'[\sum_{j=1}^{\ell+1}\psi^j(s)x^j]\psi^1(s) & = \lambda p^1 \\ \sum_{s=1}^{S} q(s)u'[\sum_{j=1}^{\ell+1}\psi^j(s)x^j]\psi^2(s) & = \lambda p^2 \\ \vdots & \vdots \\ \sum_{s=1}^{S} q(s)u'[\sum_{j=1}^{\ell+1}\psi^j(s)x^j]\psi^{\ell+1}(s) & = \lambda p^{\ell+1}. \end{cases}$$

Now put $u'(x) = 1 - \frac{x}{a}$. The system of equations $(*)$ can be written shortly as

$$\boldsymbol{\mu}_\psi - \frac{1}{a}\boldsymbol{\Sigma}_\psi \mathbf{x} = \lambda \mathbf{p},$$

where $\boldsymbol{\mu}_\psi = \boldsymbol{\Psi}^\top \mathbf{q}$ and

$$[\boldsymbol{\Sigma}_\psi]^{j,k} = \sum_{s=1}^{S} \psi^j(s)\psi^k(s)q(s).$$

For $\psi^{\ell+1}(\cdot) = \mathbf{1} \in \mathbb{R}^S$ and $p^{\ell+1} = 1$, hence $\lambda = 1 - \frac{1}{a}\,\boldsymbol{\mu}_\psi^\top \mathbf{x}$ and

$$\boldsymbol{\mu}_\psi - \frac{1}{a}\boldsymbol{\Sigma}_\psi \mathbf{x} = \mathbf{p} - \frac{1}{a}\,\boldsymbol{\mu}_\psi^\top \mathbf{x}\mathbf{p} = \mathbf{p} - \frac{1}{a}\,\mathbf{p}\boldsymbol{\mu}_\psi^\top \mathbf{x}.$$

The demand i.e. optimal and feasible portfolio is then

$$\mathbf{x}(\mathbf{p}) = a[\mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi]^{-1}(\mathbf{p} - \boldsymbol{\mu}_\psi), \tag{1.5}$$

where $\mathbf{p} \otimes \boldsymbol{\mu}_\psi$ denotes the tensor (Kronecker-) product $\mathbf{p}\boldsymbol{\mu}_\psi^\top \in \mathbb{R}^{(\ell+1)\times(\ell+1)}$.

1.2 Equilibrium

We now add the subindex i in a, $\mathbf{x}(\mathbf{p})$ and $\mathbf{e}$ to indicate the agent. Denote the *individual excess demand* by

$$\boldsymbol{\zeta}_i(\mathbf{p}) = \mathbf{x}_i(\mathbf{p}) - \mathbf{e}_i = a_i[\mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi]^{-1}(\mathbf{p} - \boldsymbol{\mu}_\psi) - \mathbf{e}_i,$$

a vector in $\mathbb{R}^\ell$, like $\mathbf{e}_i$ and $\mathbf{p} - \boldsymbol{\mu}_\psi$, while $[\mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi]^{-1}$ is in $\ell \times \ell$. For an economy with n agents we use the following notation:

$$\bar{\boldsymbol{\zeta}}(\mathbf{p}) = \bar{a} \otimes [[\mathbf{p} \otimes \mu_\psi - \boldsymbol{\Sigma}_\psi]^{-1}(\mathbf{p} - \mu_\psi)] - \bar{\mathbf{e}}, \tag{1.6}$$

where $\bar{\mathbf{e}}$, $\bar{\boldsymbol{\zeta}}(\mathbf{p}) \in \mathbb{R}^{n\times\ell}$, $\bar{a} \in \mathbb{R}^n$ and hence the rest is in $\mathbb{R}^{n\times\ell}$ as ought to be. The *total excess demand* $\bar{\mathbf{Z}}(\mathbf{p})$ is the sum of the n individual excess demands

$$\bar{\mathbf{Z}}(\mathbf{p}) = \bar{\boldsymbol{\zeta}}(\mathbf{p})^\top \mathbf{1}.$$

We get the market clearing condition of *equilibrium*:

$$[\bar{a} \otimes [(\mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi)^{-1}(\mathbf{p} - \boldsymbol{\mu}_\psi)]]^\top \mathbf{1} - \bar{\mathbf{e}}^\top \mathbf{1} = \mathbf{0}.$$

Let us look at the *equilibrium prices* of the securities. They satisfy

$$(\mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi)^{-1}(\mathbf{p} - \boldsymbol{\mu}_\psi)\bar{a}^\top \mathbf{1} - \bar{\mathbf{e}}^\top \mathbf{1} = \mathbf{0}.$$

Denote

$$\begin{aligned} S(p) &\doteq (\mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi)^{-1}(\mathbf{p} - \boldsymbol{\mu}_\psi) = \frac{\bar{\mathbf{e}}^\top \mathbf{1}}{\bar{a}^\top \mathbf{1}} \\ \Leftrightarrow \mathbf{p} - \boldsymbol{\mu}_\psi &= \frac{1}{\bar{a}^\top \mathbf{1}}(\mathbf{p} \otimes \boldsymbol{\mu}_\psi \bar{\mathbf{e}}^\top \mathbf{1} - \boldsymbol{\Sigma}_\psi \bar{\mathbf{e}}^\top \mathbf{1}) \\ \Leftrightarrow \mathbf{p} - \frac{1}{\bar{a}^\top \mathbf{1}} \mathbf{p} \boldsymbol{\mu}_\psi^\top \bar{\mathbf{e}}^\top \mathbf{1} &= \boldsymbol{\mu}_\psi - \frac{1}{\bar{a}^\top \mathbf{1}} \boldsymbol{\Sigma}_\psi \bar{\mathbf{e}}^\top \mathbf{1}. \end{aligned}$$

We get the formula for the equilibrium prices,

$$\mathbf{p} = \frac{\bar{a}^\top \mathbf{1} \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi \bar{\mathbf{e}}^\top \mathbf{1}}{\bar{a}^\top \mathbf{1} - \boldsymbol{\mu}_\psi^\top \bar{\mathbf{e}}^\top \mathbf{1}} \doteq \bar{\mathbf{p}}_n. \tag{1.7}$$

Remark 1.1. Write $a \doteq \theta^1$, $e^1 \doteq \theta^2$, ..., $e^\ell \doteq \theta^{\ell+1}$ and the total excess demand $\bar{\mathbf{Z}}(\theta; \mathbf{p})$ can be defined more precisely

$$\bar{\mathbf{Z}}(\theta; \mathbf{p}) = \boldsymbol{\zeta}(\theta_1; \mathbf{p}) + ... + \boldsymbol{\zeta}(\theta_n; \mathbf{p}) \doteq \mathrm{A}(p)\bar{\mathbf{S}}(\theta), \tag{1.8}$$

where $\mathrm{A}(p)$ is a $\mathbb{R}^{\ell \times (\ell+1)}$-matrix,

$$\mathrm{A}(p) = \begin{pmatrix} S(p) & -1 & 0 & \dots & 0 \\ S(p) & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S(p) & 0 & 0 & \dots & -1 \end{pmatrix}$$

and $\bar{\mathbf{S}}(\theta)$ is the sum of the individual characteristics. Using vector $\boldsymbol{\zeta}(\mathbf{p})$, $\bar{\mathbf{Z}}(\theta, \mathbf{p}) = \boldsymbol{\zeta}(p)^\top \mathbf{1} = \mathrm{A}(p)\Theta^\top \mathbf{1}$ so that $\boldsymbol{\zeta}(p) = \Theta \mathrm{A}(p)^\top$, $\Theta \in \mathbb{R}^{n \times (\ell+1)}$.

Remark 1.2 (Capital asset prices). Put $\mathbf{m} \doteq (\boldsymbol{\mu}_\psi - \mathbf{p})$, $W \doteq \mathbf{p}^\top \mathbf{e}$ and $\mathbf{C}_\psi = [\mathbf{C}_\psi^{j,k}] \doteq [\mathrm{cov}(\psi^j, \psi^k)] = \boldsymbol{\Sigma}_\psi - \boldsymbol{\mu}_\psi \otimes \boldsymbol{\mu}_\psi$. We can write (1.5) as

$$\mathbf{x}(\mathbf{p}) = (a - W)[\mathbf{C}_\psi + \mathbf{m} \otimes \mathbf{m}]^{-1}\mathbf{m}. \tag{1.9}$$

Write

$$[\mathbf{C}_\psi + \mathbf{m} \otimes \mathbf{m}]\mathbf{x} = (a - W)\mathbf{m} \Leftrightarrow$$
$$[\boldsymbol{\Sigma}_\psi - \mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\mu}_\psi \otimes \mathbf{p} + \mathbf{p} \otimes \mathbf{p}]\mathbf{x} = a\mathbf{m} - (\mathbf{m} \otimes \mathbf{p})\mathbf{e}.$$

The last two terms cancel from both sides by the equilibrium condition $\mathbf{x} = \mathbf{e}$, which results in (1.5). The security demand of (1.9) is proportional to $\mathbf{C}_\psi^{-1}\mathbf{m}$, the solution of the *mean-variance* formulation of the CAPM. See [4].

2 Random economy

Take, not only s, but also a and $\mathbf{e}$ as random variables with a joint-distribution $f(a, \mathbf{e})$. We define

$$\boldsymbol{\mu}(\mathbf{p}) = \mathbb{E}\boldsymbol{\zeta}_i(\mathbf{p}) = \int\int \boldsymbol{\zeta}_i(a, \mathbf{e}; \mathbf{p}) f(a, \mathbf{e}) d\mathbf{e} da.$$

Each $\boldsymbol{\zeta}_i(\mathbf{p})$ is a realization via $(a, \mathbf{e})$. When $\boldsymbol{\mu}(\mathbf{p}^*) = \mathbf{0}$ we call $\mathbf{p}^*$ an *expected equilibrium price*. Let us solve the expected equilibrium prices:

$$\begin{aligned} \boldsymbol{\mu}(\mathbf{p}) = \mathbb{E}a[\mathbf{p} \otimes \boldsymbol{\mu}_\psi - \boldsymbol{\Sigma}_\psi]^{-1}(\mathbf{p} - \boldsymbol{\mu}_\psi) - \mathbb{E}\mathbf{e} = \mathbf{0} \Leftrightarrow \\ \frac{1}{\mathbb{E}a}\mathbf{p}\boldsymbol{\mu}_\psi^\top \mathbb{E}\mathbf{e} - \mathbf{p} = \frac{1}{\mathbb{E}a}\boldsymbol{\Sigma}_\psi \mathbb{E}\mathbf{e} - \boldsymbol{\mu}_\psi \Leftrightarrow \\ \mathbf{p} = \frac{\boldsymbol{\Sigma}_\psi \mathbb{E}\mathbf{e} - \mathbb{E}a\boldsymbol{\mu}_\psi}{\boldsymbol{\mu}_\psi^\top \mathbb{E}\mathbf{e} - \mathbb{E}a} \doteq \mathbf{p}^*. \end{aligned} \tag{2.1}$$

Recall that (1.7) equals

$$\bar{\mathbf{p}}_n = \frac{\boldsymbol{\Sigma}_\psi \frac{1}{n}\bar{\mathbf{e}}^\top \mathbf{1} - \frac{1}{n}\bar{a}^\top \mathbf{1}\boldsymbol{\mu}_\psi}{\boldsymbol{\mu}_\psi^\top \frac{1}{n}\bar{\mathbf{e}}^\top \mathbf{1} - \frac{1}{n}\bar{a}^\top \mathbf{1}}.$$

Now we see that $w.p.\,1$ as $n \to \infty$, $\bar{\mathbf{p}}_n \to \mathbf{p}^*$. This is the law of large numbers.

2.1 The Gärtner-Ellis theorem

The total characteristic is denoted

$$\bar{\mathbf{S}}(\theta) = \theta_1 + ... + \theta_n \doteq (a_1, \mathbf{e}_1^\top)^\top + ... + (a_n, \mathbf{e}_n^\top)^\top,$$

which has the (limiting) free energy function

$$c_\theta(\mathbf{u}) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\{\exp[\mathbf{u}^\top \bar{\mathbf{S}}(\theta)]\}.$$

The convex conjugate (or the Legendre– Fenchel transform) of it is

$$I_\theta(x) \doteq \sup_{\mathbf{u}}[\mathbf{u}^\top \mathbf{x} - c_\theta(\mathbf{u})]. \tag{2.2}$$

According to the Gärtner-Ellis theorem, for an open set G and a closed set F, the LDP holds for $n^{-1}\bar{\mathbf{S}}$:

$$\begin{aligned} \lim_{n\to\infty} \sup \frac{1}{n} \log \mathbb{P}\{n^{-1}\bar{\mathbf{S}}(\theta) \in F\} \quad &\leq \quad -\inf_{x\in F} I_\theta(x) \\ \lim_{n\to\infty} \inf \frac{1}{n} \log \mathbb{P}\{n^{-1}\bar{\mathbf{S}}(\theta) \in G\} \quad &\geq \quad -\inf_{x\in G} I_\theta(x). \end{aligned}$$

For instance, with θ_i iid, $\mathbb{P}(n^{-1}\bar{\mathbf{S}}(\theta) \approx x) \approx \mathrm{e}^{-nI_\theta(x)}$, where $x \not\approx \mathbb{E}\theta_1$. In this special case the Gärtner-Ellis theorem is called the Cramér's theorem.

2.2 Deviations from the expected behavior

We are interested in the asymptotics of $\mathbb{P}(n^{-1}\bar{\mathbf{Z}}(\theta;\mathbf{p}) \approx \mathbf{0})$ while $\mu(\mathbf{p}) \neq \mathbf{0}$. Equally one may think of the event $\mathbf{p} \neq \mathbf{p}^*$ while the prices $\mathbf{p}$ seem to be in equilibrium i.e. with zero total excess demand $\bar{\mathbf{Z}}(\theta;\mathbf{p})$, which was defined as $\bar{\mathbf{Z}}(\theta;\mathbf{p}) = \zeta_1(\theta;\mathbf{p}) + ... + \zeta_n(\theta;\mathbf{p}) \doteq \mathrm{A}(\mathbf{p})\bar{\mathbf{S}}(\theta)$, where $\mathrm{A}(\mathbf{p})$ was defined in remark (1.1).

With this linear form, we see that the function $\bar{\mathbf{Z}}$ is continuous and satisfies the requirements of the contraction principle, see e.g. [3], Theorem 4.2.1. By the contraction principle, the LDP holds for $n^{-1}\bar{\mathbf{Z}}(\theta;\mathbf{p})$ with an excess demand-rate

$$I(\mathbf{z};\mathbf{p}) = \inf_{\mathbf{y}:\mathrm{A}(\mathbf{p})\mathbf{y}=\mathbf{x}} I_\theta(\mathbf{y}). \tag{2.3}$$

For the random equilibrium prices take $\mathbf{z} = \mathbf{0}$ representing the equation $\bar{\mathbf{Z}}(\theta;\mathbf{p}) = \mathbf{0}$. Our equilibrium-rate is then

$$I(\mathbf{0};\mathbf{p}) = \sup_{\mathbf{u}\in\mathbb{R}^{\ell+1}} [\mathbf{0} - \mathbf{c}(\mathbf{u};\mathbf{p})] = -\inf_{\mathbf{u}\in\mathbb{R}^{\ell+1}} \mathbf{c}(\mathbf{u};\mathbf{p}).$$

Note that $\mathbf{c}(\mathbf{u};\mathbf{p})$ it is not $c_\theta(\mathbf{u})$ but a different function. However $\mathbf{u}^\top\bar{\mathbf{Z}}(\theta;\mathbf{p}) = (\mathrm{A}(\mathbf{p})^\top\mathbf{u})^\top\bar{\mathbf{S}}(\theta)$ which implies

$$c(\mathbf{u};\mathbf{p}) = c_\theta(\mathrm{A}(\mathbf{p})^\top\mathbf{u}) \quad \text{and}$$

$$I(\mathbf{p}) = -\inf_{\mathbf{u}\in\mathbb{R}^{\ell+1}} c_\theta(\mathrm{A}(\mathbf{p})^\top\mathbf{u}) = -c_\theta(\mathrm{A}(\mathbf{p})^\top\mathbf{u}(\mathbf{p})),$$

where $\mathbf{u}(\mathbf{p})$ is a unique minimum as the function $c_\theta(\cdot)$ is convex. In this point

$$\nabla_{\mathbf{u}} c_\theta(\mathrm{A}(\mathbf{p})^\top\mathbf{u}) = \mathbf{0}.$$

Using the convex duality: $\nabla_{\mathbf{x}} I_\theta(\mathbf{x}) = \mathbf{u}(\mathbf{p})$, s.t. $\nabla_{\mathbf{u}} c_\theta(\mathrm{A}(\mathbf{p})^\top\mathbf{u}) = \mathbf{x}$, we get

$$I(\mathbf{p}) = -c_\theta(\mathrm{A}(\mathbf{p})^\top\nabla_{\mathbf{x}} I_\theta(\mathbf{x})).$$

Especially for the equilibrium prices $\mathbf{x} = \mathbf{0}$ and the rate will be

$$I(\mathbf{p}) = -c_\theta(\mathrm{A}(\mathbf{p})^\top\nabla_{\mathbf{x}} I_\theta(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}). \tag{2.4}$$

To make things more clear we will next present an example where the characteristic parameters are independently sampled from the multinormal distribution.

Example 2.1. Preferences $\boldsymbol{\theta}_i$ i.i.d. $\sim$ mn$(\bar{\boldsymbol{\theta}}, \mathbf{Q})$ with mean $\bar{\boldsymbol{\theta}} = \mathbb{E}\boldsymbol{\theta}_1$ and covariance matrix $\mathbf{Q} = \mathbb{E}[(\boldsymbol{\theta}_1 - \bar{\boldsymbol{\theta}})(\boldsymbol{\theta}_1 - \bar{\boldsymbol{\theta}})^\top]$. Assume $\mathbf{Q}$ invertible.

Now $\mathbf{S}_n(\boldsymbol{\theta}) = \sum_{i=1}^n \boldsymbol{\theta}_i \sim$ mn$(n\bar{\boldsymbol{\theta}}, n\mathbf{Q})$ i.e. the density is

$$f(\boldsymbol{\theta}) = [(2\pi)|\mathbf{Q}|]^{-1/2} \exp[-\frac{1}{2}(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}})^\top \mathbf{Q}^{-1}(\boldsymbol{\theta} - \bar{\boldsymbol{\theta}})].$$

The Laplace transform of θ is well-known,

$$\mathbb{E}[\mathrm{e}^{\mathbf{u}^\top \boldsymbol{\theta}}] = \mathrm{e}^{\mathbf{u}^\top \bar{\boldsymbol{\theta}} + \frac{1}{2}\mathbf{u}^\top \mathbf{Q}\mathbf{u}}$$

and correspondingly for $\mathbf{S}_n(\boldsymbol{\theta})$

$$\mathbb{E}[\mathrm{e}^{\mathbf{u}^\top \mathbf{S}_n}] = \mathrm{e}^{nu^\top \bar{\boldsymbol{\theta}} + \frac{n}{2}\mathbf{u}^\top \mathbf{Q}\mathbf{u}}. \tag{2.5}$$

Log of this is $c_{\boldsymbol{\theta}}(\mathbf{u})$ and the convex conjugate of it is $I_{\boldsymbol{\theta}}(\mathbf{x}) = \sup_{\mathbf{u}\in\mathbb{R}^{\ell+1}}[\mathbf{u}^\top\mathbf{x} - c_\theta(\mathbf{u})]$

$$= \sup_{\mathbf{u}\in\mathbb{R}^{\ell+1}} [\mathbf{u}^\top \mathbf{x} - n\mathbf{u}^\top \bar{\boldsymbol{\theta}} - \frac{n}{2}\mathbf{u}^\top \mathbf{Q}\mathbf{u}]. \tag{2.6}$$

$\nabla_{\mathbf{u}} I_{\boldsymbol{\theta}}(\mathbf{x}) = 0 \Rightarrow$ optimum $\hat{\mathbf{u}} = \mathbf{Q}^{-1}(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}})$. Substitute to (2.6).

$$\begin{aligned}
I_{\boldsymbol{\theta}}(\mathbf{x}) &= \Big[\mathbf{Q}^{-1}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)\Big]^\top \mathbf{x} - n\Big[\mathbf{Q}^{-1}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)\Big]^\top \bar{\boldsymbol{\theta}} - \\
&\quad -\frac{n}{2}\Big[\mathbf{Q}^{-1}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)\Big]^\top \mathbf{Q}\Big[\mathbf{Q}^{-1}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)\Big] \\
&= \Big[\mathbf{Q}^{-1}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)\Big]^\top (\mathbf{x} - n\bar{\boldsymbol{\theta}}) \\
&\quad - \frac{1}{2}\Big[\mathbf{Q}^{-1}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)\Big]^\top (\mathbf{x} - n\bar{\boldsymbol{\theta}}) \\
&= \frac{n}{2}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)^\top \mathbf{Q}^{-1}\big(\frac{\mathbf{x}}{n} - \bar{\boldsymbol{\theta}}\big)
\end{aligned} \tag{2.7}$$

The LDP holds with rate $I_{\boldsymbol{\theta}}(\mathbf{x})$. Put $\boldsymbol{\zeta}(\boldsymbol{\theta}) = aS(p) - \mathbf{e}$ where $\boldsymbol{\theta}_i$, $i = 1, \ldots, n$. In matrix form $\mathbf{Z}_n(\boldsymbol{\theta}; \mathbf{p}) = \mathrm{A}(\mathbf{p})\mathbf{S}_n(\boldsymbol{\theta})$, which is a continuous transformation. Thus due to the contraction principle we have that for $\mathbf{Z}_n(\boldsymbol{\theta}; \mathbf{p}) = \mathbf{S}_n(\boldsymbol{\zeta}(\boldsymbol{\theta})) = \mathrm{A}(\mathbf{p})\mathbf{S}_n(\boldsymbol{\theta})$ and the LDP holds for $n^{-1}\mathbf{Z}_n(\boldsymbol{\theta}; \mathbf{p})$ with rate $I(\mathbf{z}; \mathbf{p}) = \inf_{\mathbf{y}:\mathrm{A}(\mathbf{p})\mathbf{y}=\mathbf{z}} I_{\boldsymbol{\theta}}(\mathbf{y})$.

The rate at which the probability of seeing a random equilibrium price at a large economy, with pricesystem $\mathbf{p}$ s.t. $\mathbf{p} \neq \mathbf{p}^*$ was of the form $I(\mathbf{p}) = -\inf_{\mathbf{u}\in\mathbb{R}^{\ell+1}} c_{\boldsymbol{\theta}}(\mathrm{A}(\mathbf{p})^\top \mathbf{u})$, equivalent to that of

$$\begin{aligned}
I(\mathbf{p}) &= -c_{\boldsymbol{\theta}}(\mathrm{A}(\mathbf{p})^\top \mathbf{u}(\mathbf{p})) \\
&= -c_{\boldsymbol{\theta}}(\mathrm{A}(\mathbf{p})^\top \nabla_{\mathbf{x}} I_{\boldsymbol{\theta}}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}) \\
&= -c_{\boldsymbol{\theta}}(-\mathrm{A}(\mathbf{p})^\top \mathbf{Q}^{-1}\bar{\boldsymbol{\theta}}) \\
&= n[\mathrm{A}(\mathbf{p})^\top \mathbf{Q}^{-1}\bar{\boldsymbol{\theta}}]^\top \bar{\boldsymbol{\theta}} \\
&\quad - \frac{n}{2}[\mathrm{A}(\mathbf{p})^\top \mathbf{Q}^{-1}\bar{\boldsymbol{\theta}}]^\top \mathbf{Q}[\mathrm{A}(\mathbf{p})^\top \mathbf{Q}^{-1}\bar{\boldsymbol{\theta}}].
\end{aligned}$$

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