THE APOLLONIAN METRIC AND BILIPSCHITZ MAPPINGS

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This thesis consists of two manuscripts, [9] and [10], intended for independent publication and of the extended abstract that you are now reading. The topic of the dissertation is the Apollonian metric. The results can be classified on one hand as being estimates of the Apollonian metric by simpler metrics along the line of investigation started by Alan Beardon, [3], and on the other hand as generalizations of results by Frederick Gehring and Kari Hag, [7], describing Apollonian bilipschitz mappings.

This extended abstract consists of a short historical overview of the developments of the Apollonian metric, a more detailed presentation of some recent results pertaining to the present investigation and the statement of some of our most interesting results. The notation and terminology used conforms largely to that of [2] and [17], see Section 2 of [9] for details.

The Apollonian metric was first introduced by Dan Barbilian in 1935 in the paper [1]. It has since been considered by P. Kelly [13], L. M. Blumenthal [4] and W.-G. Boskoff [5] under the name of the Barbilian metric. These investigations are based on a qualitative approach. Thus Kelly concludes for instance that the metric is not of interest in arbitrary domains, since there are, in general, no geodesics in this case. Similarly, Boskoff is more interested in generalizations of the metric as we will define it (see (1)), developing an axiomatic approach to the Barbilian metric.

The approach that we adopt is of a more quantitative nature. Articles along this line of investigation have appeared recently, inspired by the article [3] of Alan Beardon's. In that article, Beardon coins the name and defines the Apollonian metric, α_G, unaware of the previous investigations mentioned above, by

$$\alpha_G(x,y) := \sup_{a,b\in\partial G} \log \frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|}. \tag{1}$$

where $x, y \in G \subsetneq \mathbb{R}^n$, and G^c (the complement of G) is not contained in a hyperplane or sphere. If G^c is contained in a hyperplane or sphere then α_G is still well defined, but it is only a pseudo-metric (i.e. there exist distinct $x, y \in G$ such that $\alpha_G(x, y) = 0$). Beardon's quantitative approach to the Apollonian metric has also been adopted in [14], [15], [7] and [12]. Beardon proved various inequalities relating the Apollonian metric to other well-known metrics, such as the quasihyperbolic metric (see [8]), the Klein-Hilbert metric (see [11]), the hyperbolic metric (see [7]) and the j_G metric, defined for $x, y \in G \subsetneq \mathbb{R}^n$ by

$$j_G(x,y) := \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right).$$

An estimate of α_G by a metric cannot give a "good" lower bound if it is valid in all domains $G \subsetneq \mathbb{R}^n$, since α_G is a pseudo-metric in some domains. For instance, Beardon's estimate [3,

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Theorem 6.1]

$$\alpha_G/2 \le h_G \le 4\sinh(\alpha_G/2),$$

where h_G denotes the hyperbolic metric, is valid only in simply connected convex planar domains. (An inequality of the type $d_G \le c d'_G$ means that $d_G(x,y) \le c d'_G(x,y)$ for all $x, y \in G$.) On the other hand, Pasi Seittenranta's result [15, Theorem 3.11],

$$\delta_G - \log 3 \le \alpha_G \le \delta_G,$$

where δ_G denotes Seittenranta's cross ratio metric (see [15, (1.1)]), is valid in every domain $G \subsetneq \mathbb{R}^n$ but does not provide a linear (as opposed to affine) lower bound. Notice that this means that Seittenranta's lower bound tells us nothing when $\delta_G \le \log 3$.

On a different track, Gehring and Hag [7, Theorem 3.1] proved that for a simply connected planar domain G the inequality

$$\frac{h_G}{K} \le \alpha_G \le 2h_G \tag{2}$$

holds if and only if G is a quasidisk. Apart from the fact that this inequality provides a new characterization of planar quasidisks it is interesting in the present context since it is a comparison result valid for quite large a class of domains. Note, however, that the restriction to planar domains is essential – in $\overline{\mathbb{R}^n}$ (with $n \ge 3$) the hyperbolic metric is not even defined for domains other than balls.

It is well known (e.g. [6]) that $j_G \le c h_G$ in quasidisks. Hence it follows from (2) that $j_G \le c\alpha_G$ in quasidisks. In [9] we will show that this result is also valid in quasiballs in $\mathbb{R}^n$, $n \ge 3$.

Corollary 1.3, [9]**.** *If $G \subsetneq \mathbb{R}^n$ is a K–quasiball then there exists a constant L depending only on K and n such that $j_G/L \le \alpha_G \le 2j_G$.*

In fact, we are able to give a geometrical characterization, in terms of an interior double ball condition, of those domains that satisfy the comparison condition $j_G/L \le \alpha_G \le 2j_G$, see [9, Theorem 5.9].

Using this corollary we will prove a partial generalization of the following theorem.

Theorem 3.11, [7]**.** *Let $G \subsetneq \mathbb{R}^2$ be a quasidisk and $f\colon G \to G'$ be an Apollonian bilipschitz mapping.*

1. *If G' is a quasidisk then f is quasiconformal in G and $f = g|_G$, where $g\colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is quasiconformal.*
2. *If f is quasiconformal in G then G' is a quasidisk and $f = g|_G$, where $g\colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is quasiconformal.*

Note that in [7] it is required that f be an isometry instead of a bilipschitz mapping, but their proof is valid *mutatis mutandis* for bilipschitz mappings as well. Also, the wording of the theorem has been changed so as to make its connection with the following results clearer.

We will show that the first statement of the previous theorem is true also for $n \ge 3$.

Theorem 1.6, [9]**.** *Let $G \subsetneq \mathbb{R}^n$ be a quasiball and $f\colon G \to \mathbb{R}^n$ be an Apollonian bilipschitz mapping. If $f(G)$ is a quasiball then $f = g|_G$, where $g\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is quasiconformal.*

In [9] we also ask under what conditions a Euclidean bilipschitz mapping is Apollonian bilipschitz as well. Let us define two quantities that give information about this: for $L \ge 1$

define

$$\alpha_L(G) := \sup_f \sup_{x,y\in G} \max\left(\frac{\alpha_{f(G)}(f(x),f(y))}{\alpha_G(x,y)}, \frac{\alpha_G(x,y)}{\alpha_{f(G)}(f(x),f(y))}\right),$$

where the first supremum is over all Euclidean L–bilipschitz mappings f mapping G into $\mathbb{R}^n$ (with the understanding that terms with zero denominators are ignored). Define also

$$\alpha'_L(G) := \sup_g \sup_{x,y\in G} \{\alpha_G(g(x),g(y))/\alpha_G(x,y)\},$$

where the first supremum is over all Euclidean L–bilipschitz mappings g mapping G onto G (again ignoring zero-denominator-terms).

Corollary 1.7, [9]**.** *If $G \subsetneq \mathbb{R}^n$ is a K–quasiball then there exists a function $\phi\colon [1,\infty) \to [1,\infty)$ depending only on K and n such that $\alpha_L(G) < \phi(L)$ for every $1 \le L < \infty$.*

The following result provides a connection between domains in which the Apollonian metric can be estimated by the j_G metric and domains for which every bilipschitz mapping is Apollonian bilipschitz.

Theorem 1.8, [9]**.** *Let $G \subsetneq \mathbb{R}^n$ be a domain such that $\alpha_L(G) < \infty$ for some $L > 1$. Then $j_G \le K\alpha_G$, where the constant K depends only on L and $\alpha_L(G)$.*

Conversely, let G be a domain such that $j_G \le K\alpha_G$ for some K. Then there exists a constant $L_0 > 1$ such that $\alpha_L(G) < \infty$ for every $1 \le L < L_0$.

The following result shows that the weaker bilipschitz condition $\alpha'_L(G) < \infty$ is equivalent to the comparison property.

Corollary 1.9, [9]**.** *Let $G \subsetneq \mathbb{R}^n$ be a domain. The following two conditions are equivalent:*

(1) *There exists a constant K such that $j_G \le K\alpha_G$.*
(2) *For every $L \ge 1$ we have $\alpha'_L(G) < \infty$.*

In the second manuscript, [10], we consider in greater depth how Theorem 3.11 from [7], cited above, can be generalized. However, also in this case the problem itself goes back to A. Beardon, who speculated in [3] that the isometries of the Apollonian metric are only the Möbius mappings, at least for many domains, and proved that this is so in a special case, see [3, Theorem 1.3]. Z. Ibragimov has also recently proved some new results in this direction, see [12]. Apart from Theorem 3.11, Gehring and Hag proved two other results that also pertain to this problem.

Theorem 3.16, [7]**.** *Let $G \subseteq \mathbb{R}^2$ be a disk and let $f\colon G \to \mathbb{R}^2$ be an Apollonian isometry. The following conditions are equivalent:*

(1) *$f(G)$ is a disk.*
(2) *f is a Möbius mapping of G.*

Moreover, if either of the two conditions holds then $f = g|_G$, where $g\colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is a Möbius mapping.

As a last result from the paper of Gehring and Hag we quote the following theorem, which is a stronger version of the previous one:

Theorem 3.29, [7]**.** *If $G \subseteq \mathbb{R}^2$ is a disk and $f\colon G \to \mathbb{R}^2$ is an Apollonian isometry then*

(1) *$f(G)$ is a disk and*
(2) *$f = g|_G$, where $g\colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is a Möbius mapping.*

Note that this result solves Beardon's problem for the disk. In [10] we complement these results by three new ones, two in space and one in $\mathbb{R}^2$. Our first result is the strong version of Theorem 3.16, [7] and is valid only in the plane.

Theorem 1.7, [10]**.** *If $G \subseteq \mathbb{R}^2$ is a quasidisk and $f\colon G \to \mathbb{R}^2$ is an Apollonian bilipschitz mapping then*

(1) *$f(G)$ is a quasidisk and*
(2) *$f = g|_G$, where $g\colon \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is quasiconformal.*

The next result is an extension of Theorem 3.16, [7], which is, however, not stated in terms of quasiballs but in terms of A-uniform domains. A-uniform domains are introduced in Definition 6.5 of [10] and are defined as those domains that satisfy the relation $k_G \le K\alpha_G$ for some fixed $K \ge 1$, where k_G denotes the quasihyperbolic metric from [8]. We show that in general quasiballs are A-uniform domains (Corollary 6.9) and that in the plane these two concepts define the same class of simply connected domains (Corollary 6.10). Whether these classes of domains coincide in space is an open problem. It follows, then, that the next result implies Theorem 3.16, [7], although it is not the most natural generalization of that result.

Theorem 1.8, [10]**.** *Let $G \subseteq \mathbb{R}^n$ be A-uniform and $f\colon G \to \mathbb{R}^n$ be an Apollonian bilipschitz mapping. The following conditions are equivalent:*

(1) *$f(G)$ is A-uniform.*
(2) *f is quasiconformal in G.*

Notice that we are not able to prove the last statement of Theorem 3.16, [7], (that f would be a restriction of a quasiconformal mapping from $\overline{\mathbb{R}^n}$ onto $\overline{\mathbb{R}^n}$) for the case $n \ge 3$.

Our last result along this line of investigation is a generalization of [7, Theorem 3.29] to $\mathbb{R}^n$, which is also proved quite similarly, although the geometry becomes a bit more complicated in space.

Theorem 1.9, [10]**.** *If $G \subseteq \mathbb{R}^n$ is a ball and $f\colon G \to \mathbb{R}^n$ is an Apollonian isometry then*

(1) *$f(G)$ is a ball and*
(2) *$f = g|_G$, where $g\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius mapping.*

As a final result the following theorem summarizes several characterizations of planar quasidisks in terms of the Apollonian metric. These add to the legion of equivalent condition given e.g. in [6].

Theorem 1.10, [10]**.** *Let G be a simply connected planar domain. The following statements are equivalent:*

(1) *The domain G is a quasidisk.*
(2) *There exists a constant K such that $h_G \le K\alpha_G$, where h_G denotes the hyperbolic metric.* [7, Theorem 3.1]
(3) *The domain G is A-uniform, i.e. there exists a constant K such that $k_G \le K\alpha_G$.* [10, Corollary 6.10]
(4) *The metric α_G is quasiconvex, i.e. there exists a constant K such that for every $x, y \in G$ there exists a path γ connecting x and y in G with $\ell_{\alpha_G}(\gamma) \le K\alpha_G(x, y)$.* [10, Corollary 7.4]

Notice that of the conditions in the previous theorem the fourth one involves only the Apollonian metric.

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