Boundedness and Compactness of Toeplitz+Hankel Operators

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Suppose that A is a bounded linear operator on the Hardy space H^p that satisfies

$$\langle Az^j, z^k \rangle = a_{k-j} \quad (j, k \in \mathbb{N}_0)$$

for some sequence of complex numbers $\{a_n\}_{n\in\mathbb{Z}}$. By the Brown–Halmos theorem, A must be a Toeplitz operator with bounded symbol, that is, $\{a_n\}_{n\in\mathbb{Z}}$ is the Fourier sequence of a bounded function. Likewise, Nehari's theorem shows that if A satisfies $\langle Az^j, z^k \rangle = a_{k+j+1}$ instead, then A is equal to a Hankel operator with bounded symbol. These results were proven in the 50's and 60's and have become classical in the theory of Hardy spaces.

More recently, due to some applications in mathematical physics, there has been a lot of interest in so-called Toeplitz+Hankel operators. Quite simply put, a Toeplitz+Hankel operator is the sum of a Toeplitz operator $T(a)$ and a Hankel operator $H(b)$. Now clearly, if both $T(a)$ and $H(b)$ are bounded, then $A = T(a) + H(b)$ is necessarily bounded as well. It is therefore natural to ask whether the converse is also true or if the "unboundedness" of $T(a)$ and $H(b)$ can somehow cancel out. I will elaborate on this question and present a Brown–Halmos type result for Toeplitz+Hankel operators for both the Hardy spaces H^p and the sequence spaces $\ell^p(\mathbb{N}_0)$. A similar characterization for compactness will be obtained as well.

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