Computational light scattering (PAP315)

Lecture 4a

Karri Muinonen[1]
Professor of Astronomy
[1]Department of Physics, University of Helsinki, Finland

1 Introduction to scattering theory

1.1 Extinction, scattering and absorption

Let us assume that medium surrounding the scattering particle is non-absorbing. The total or extinction cross section is then the sum of the absorption and scattering cross sections:

$$\sigma_e = \sigma_s + \sigma_a, \tag{1}$$

where

$$\begin{aligned} \sigma_e &= -\frac{1}{I_i}\int_A dA\boldsymbol{S}_e \cdot \boldsymbol{e}_r, \\ \sigma_s &= \frac{1}{I_i}\int_A dA\boldsymbol{S}_s \cdot \boldsymbol{e}_r, \end{aligned} \tag{2}$$

when A is a spherical envelope of radius r containing the scattering particle.

Let the original field be of $\boldsymbol{e}_x$-polarized form $\boldsymbol{E}_0 = E\boldsymbol{e}_x$. In the radiation zone,

$$\begin{aligned} \boldsymbol{E}_s &\propto \frac{\exp[\mathrm{i}k(r-z)]}{-\mathrm{i}kr}\boldsymbol{X}E, \boldsymbol{e}_r \cdot \boldsymbol{X} = 0, \\ \boldsymbol{H}_s &\propto \frac{k}{\omega\mu}\boldsymbol{e}_r \times \boldsymbol{E}_s, \end{aligned} \tag{3}$$

where the vector scattering amplitude $\boldsymbol{X}$ is related to the amplitude scattering matrix as follows:

$$\boldsymbol{X} = (S_4\cos\phi + S_1\sin\phi)\boldsymbol{e}_{s\perp} + (S_2\cos\phi + S_3\sin\phi)\boldsymbol{e}_{s\parallel}. \tag{4}$$

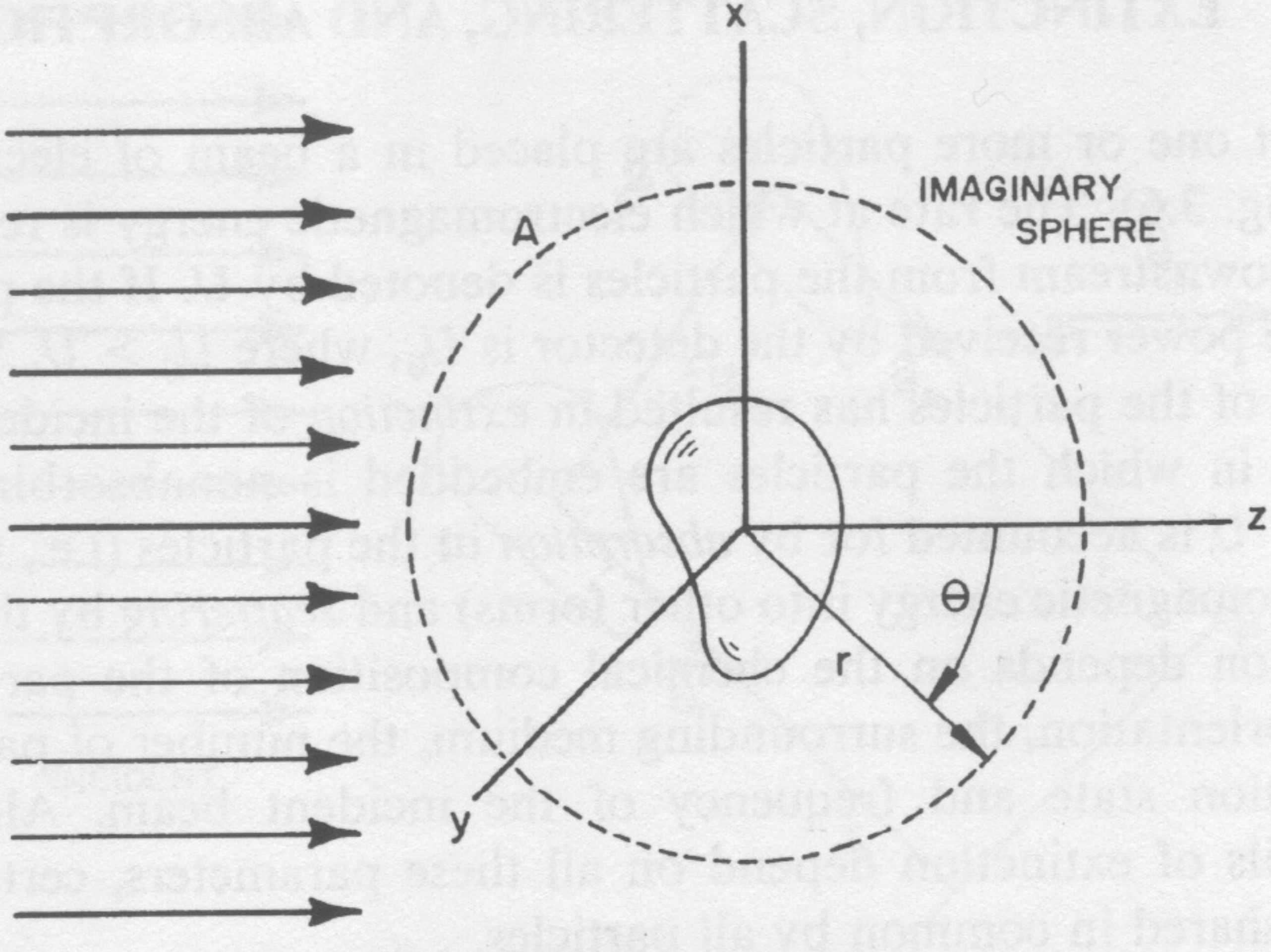

Figure 3.7 Extinction by a single particle.

By making use of the asymptotic forms of the scattered field shown above and $\boldsymbol{e}_x$-polarized original field, the so-called optical theorem can be derived: extinction depends only on scattering in the exact forward direction,

$$\sigma_e \;=\; \frac{4\pi}{k^2}\mathrm{Re}[(\boldsymbol{X}\cdot\boldsymbol{e}_x)_{\theta=0}]. \tag{5}$$

In addition,

$$\sigma_s \;=\; \int_{4\pi} d\Omega \frac{d\sigma_s}{d\Omega}, \tag{6}$$

where the differential scattering cross section is

$$\frac{d\sigma_s}{d\Omega} \;=\; \frac{|\boldsymbol{X}|^2}{k^2}. \tag{7}$$

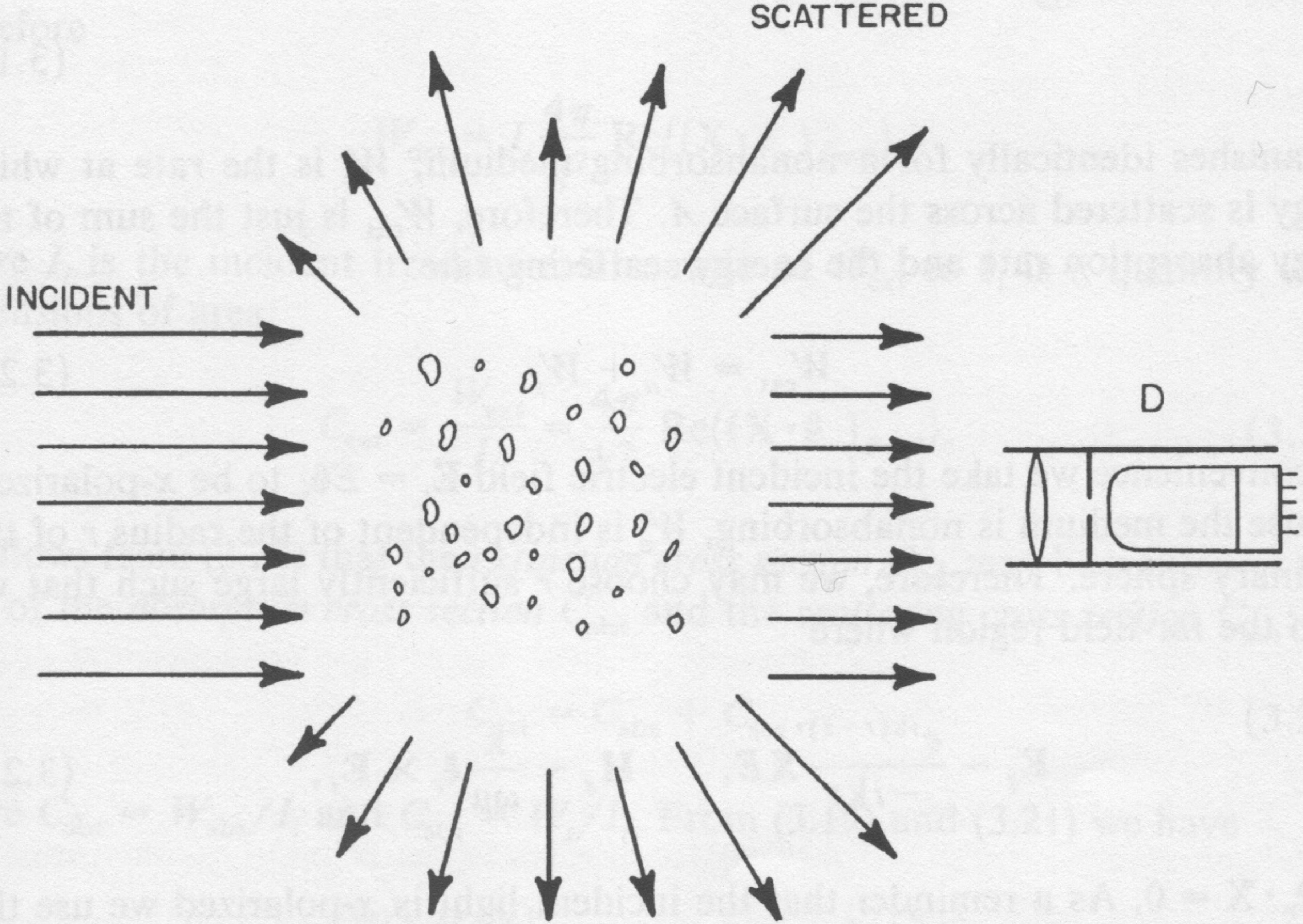

Figure 3.6 Extinction by a collection of particles.

The extinction, scattering, and absorption efficiencies are defined as the ratios of the corresponding cross sections to the geometric cross section of the particle $A_\perp$ as projected in the propagation direction of the original field:

$$\begin{aligned} q_e &= \frac{\sigma_e}{A_\perp}, \\ q_s &= \frac{\sigma_s}{A_\perp}, \\ q_a &= \frac{\sigma_a}{A_\perp}. \end{aligned} \tag{8}$$

For an unpolarized original field, the cross sections are

$$\begin{aligned} \sigma_e &= \frac{1}{2}(\sigma_e^{(1)} + \sigma_e^{(2)}), \\ \sigma_s &= \frac{1}{2}(\sigma_s^{(1)} + \sigma_s^{(2)}), \end{aligned} \tag{9}$$

where the indices 1 and 2 refer to two polarization states of the original field perpendicular to one another.

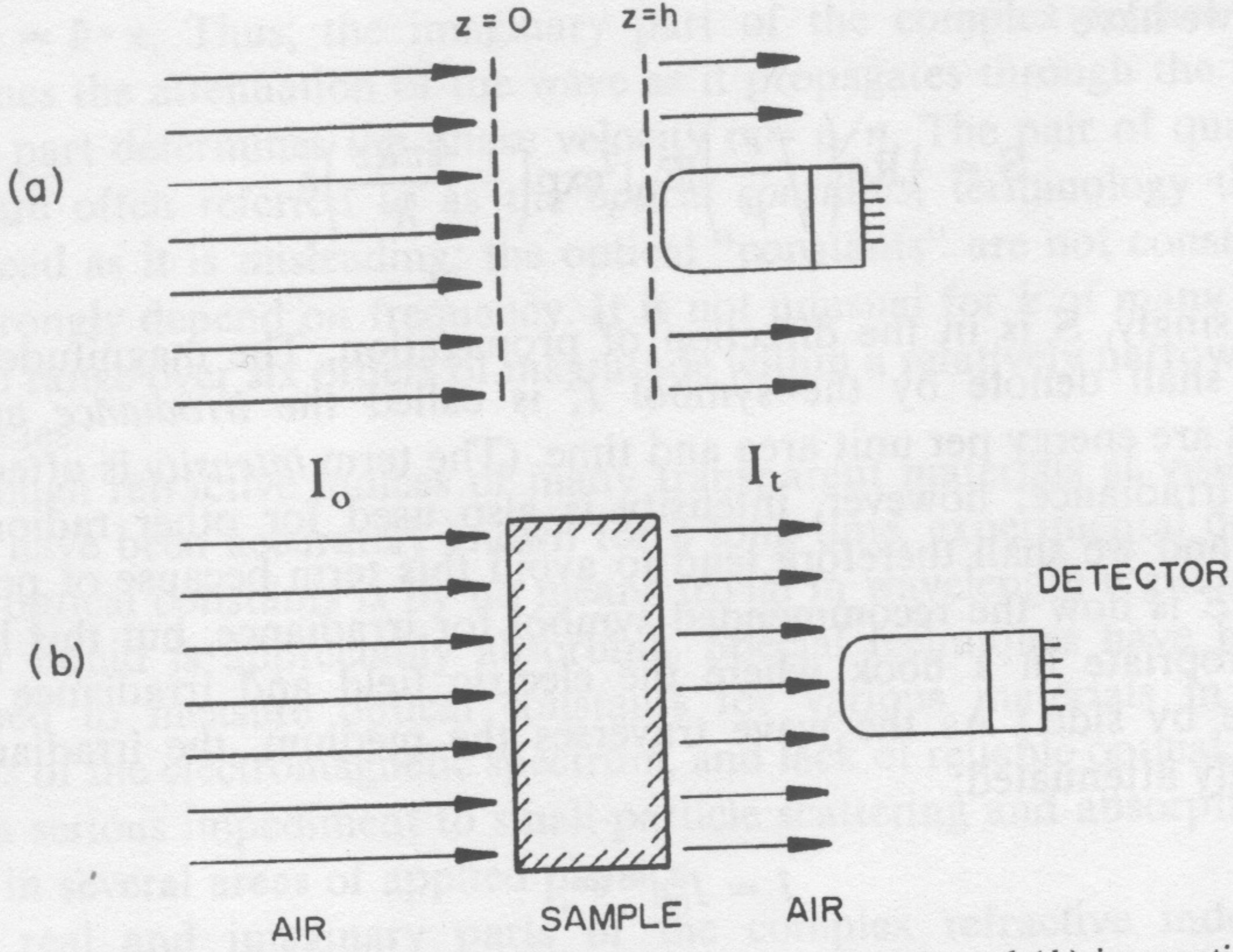

Figure 2.3 Measurement of absorption: (*a*) in principle and (*b*) in practice.

2 Plane waves

The electromagnetic plane wave

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t} \end{aligned} \tag{10}$$

can, under certain conditions, fulfil Maxwell's equations. The physical fields correspond to the real parts of the complex-valued fields. The vectors $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ above are constant vectors and can be complex-valued. Similarly, the wave vector $\boldsymbol{k}$ can be complex-valued:

$$\boldsymbol{k} = \boldsymbol{k}' + i\boldsymbol{k}'', \qquad \boldsymbol{k}', \boldsymbol{k}'' \in \mathbb{R}^n \tag{11}$$

Inserting (11) into equation (10), we obtain

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t} \end{aligned} \tag{12}$$

In Eq. (12), $\boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ and $\boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ are amplitudes and $\boldsymbol{k}'\cdot\boldsymbol{x} - \omega t = \phi$ is the phase of the wave.

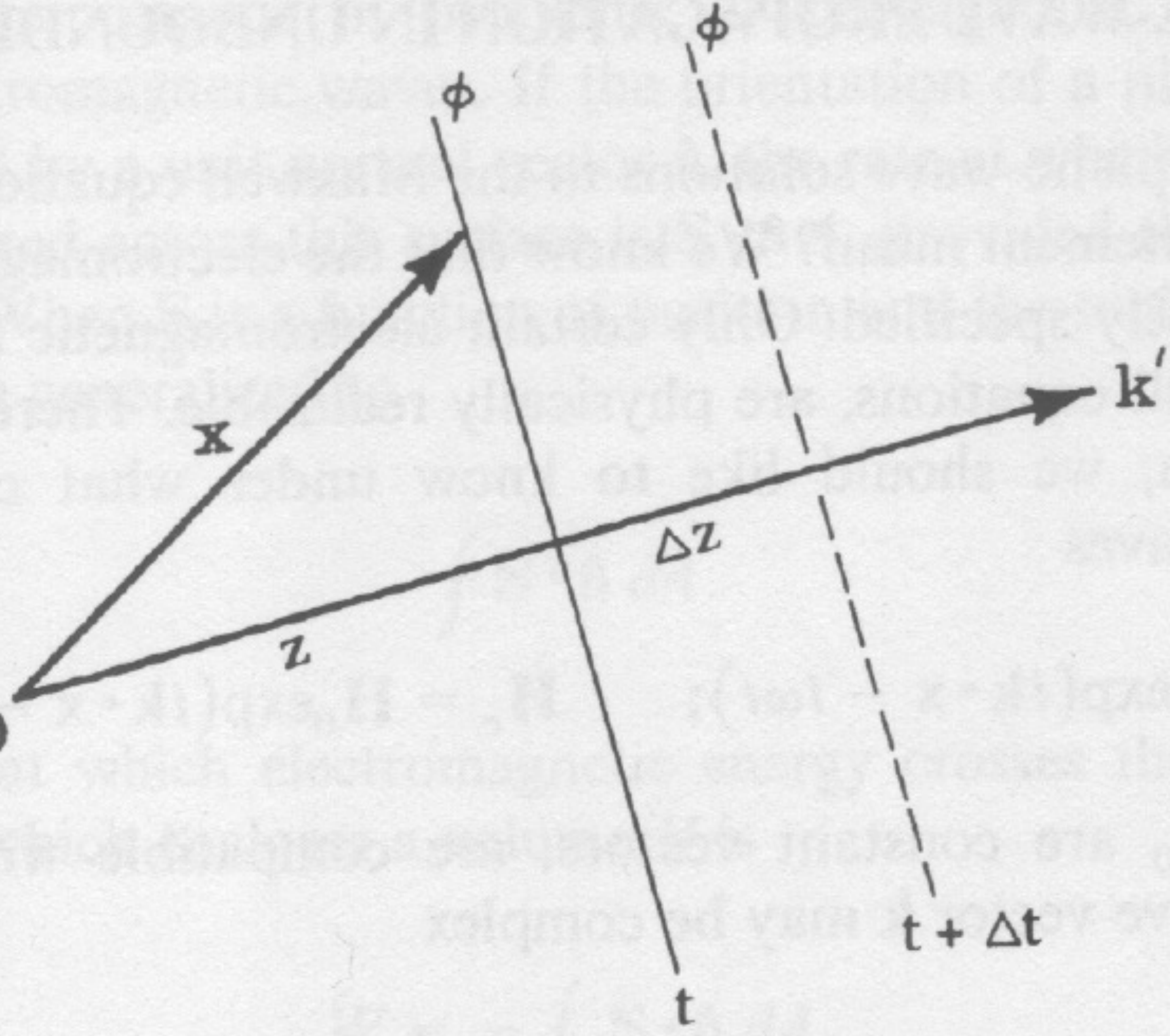

Figure 2.2 Propagation of constant phase surfaces.

An equation of the form $\boldsymbol{k} \cdot \boldsymbol{x}$ =constant defines, in the case of a real-valued vector $\boldsymbol{k}$, a planar surface, whose normal is just the vector $\boldsymbol{k}$. Thus, $\boldsymbol{k}'$ is perpendicular to the planes of constant phase and $\boldsymbol{k}''$ is perpendicular to the planes of constant amplitude. If $\boldsymbol{k}' \parallel \boldsymbol{k}''$, the planes coincide and the wave is *homogeneous*. If $\boldsymbol{k}' \nparallel \boldsymbol{k}''$, the wave is *inhomogeneous*. A plane wave propagating in vacuum is homogeneous.

In the case of plane waves, Maxwell's equaitons can be written as

$$
\begin{aligned}
\boldsymbol{k} \cdot \boldsymbol{E}_0 &= 0 \\
\boldsymbol{k} \cdot \boldsymbol{H}_0 &= 0 \\
\boldsymbol{k} \times \boldsymbol{E}_0 &= \omega\mu \boldsymbol{H}_0 \\
\boldsymbol{k} \times \boldsymbol{H}_0 &= -\omega\epsilon \boldsymbol{E}_0
\end{aligned}
\tag{13}
$$

The two upmost equations are conditions for the transverse nature of the waves: $\boldsymbol{k}$ is perpendicular to both $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. The two lowermost equations show that $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are perpendicular to each other. Since $\boldsymbol{k}$, $\boldsymbol{E}_0$, and $\boldsymbol{H}_0$ are complex-valued, the geometric interpretation is not simple unless the waves are homogeneous.

It follows from Maxwell's equations (13) that, on one hand,

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) = \omega\mu\boldsymbol{k} \times \boldsymbol{H}_0 = -\omega^2\epsilon\mu\boldsymbol{E}_0 \tag{14}$$

and, on the other hand,

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) = \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}_0) - \boldsymbol{E}_0(\boldsymbol{k} \cdot \boldsymbol{k}) = -\boldsymbol{E}_0(\boldsymbol{k} \cdot \boldsymbol{k}), \tag{15}$$

so that

$$\boldsymbol{k} \cdot \boldsymbol{k} = \omega^2\epsilon\mu. \tag{16}$$

Plane wave solutions are in agreement with Maxwell's equations if

$$\boldsymbol{k} \cdot \boldsymbol{E}_0 = \boldsymbol{k} \cdot \boldsymbol{H}_0 = \boldsymbol{E}_0 \cdot \boldsymbol{H}_0 = 0 \tag{17}$$

and if

$$k'^2 - k''^2 + 2i\boldsymbol{k}' \cdot \boldsymbol{k}'' = \omega^2 \epsilon \mu. \tag{18}$$

Note that ϵ and μ are properties of the medium, whereas $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are properties of the wave. Thus, ϵ and μ do not unambiguously determine the details of wave propagation.
In the case of a homogeneous plane wave $(\boldsymbol{k}' \| \boldsymbol{k}'')$,

$$\boldsymbol{k} = (k' + ik'')\hat{\mathbf{e}}, \tag{19}$$

where k' and k'' are non-negative and $\hat{\mathbf{e}}$ is an arbitrary real-valued unit vector.

According to Eq. (16),

$$(k' + ik'')^2 = \omega^2 \epsilon\mu = \frac{\omega^2 m^2}{c^2}, \tag{20}$$

where $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum and m is the complex-valued refractive index

$$m = \sqrt{\frac{\epsilon\mu}{\epsilon_0 \mu_0}} = m_r + im_i, \qquad m_r, m_i \geq 0. \tag{21}$$

In vacuum, the wave number is $\omega/c = 2\pi/\lambda$, where λ is the wavelength. The general homogeneous plane wave takes the form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\frac{2\pi m_i s}{\lambda}} e^{i\frac{2\pi m_r s}{\lambda} - i\omega t} \tag{22}$$

where $s = \boldsymbol{e} \cdot \boldsymbol{x}$. The imaginary and real parts of the refractive index determine the attenuation and phase velocity $v = c/m_r$ of the wave, respectively.

3 Poynting vector

Let us study the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ that is time harmonic. For the physical fields (the real parts of the complex-valued fields), the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{23}$$

describes the direction and amount of energy transfer everywhere in the space.

Let $\boldsymbol{n}$ be the unit normal vector of the planar surface element A. Electromagnetic energy is transferred through the planar surface with power $\boldsymbol{S} \cdot \boldsymbol{n}\ A$, where $\boldsymbol{S}$ is assumed constant on the surface. For an arbitrary surface and $\boldsymbol{S}$ depending on location, the power is

$$W = -\int_A \boldsymbol{S} \cdot \boldsymbol{n} dA, \tag{24}$$

where $\boldsymbol{n}$ is the outward unit normal vector and the sign has been chosen so that positive W corresponds to absorption in the case of a closed surface.
The time-averaged Poynting vector

$$\langle \boldsymbol{S} \rangle = \frac{1}{\tau} \int_t^{t+\tau} \boldsymbol{S}(t') dt' \qquad \tau >> 1/\omega \tag{25}$$

is more important than the momentary Poynting vector (cf. measurements).

The time-averaged Poynting vector for time-harmonic fields is

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\mathrm{\mathbf{Re}}\{\boldsymbol{E} \times \boldsymbol{H}^*\} \tag{26}$$

and, in what follows, this is the Poynting vector meant even though the averaging is not always shown explicitly.
For a plane wave field, the Poynting vector is

$$\boldsymbol{S} = \frac{\mathbf{1}}{\mathbf{2}}\mathrm{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} = \mathrm{Re}\Big\{\frac{\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*)}{\mathbf{2}\boldsymbol{\omega\mu}^*}\Big\}, \tag{27}$$

where

$$\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*) = \boldsymbol{k}^*(\boldsymbol{E} \cdot \boldsymbol{E}^*) - \boldsymbol{E}^*(\boldsymbol{k}^* \cdot \boldsymbol{E}). \tag{28}$$

For a homogeneous plane wave,

$$\boldsymbol{k} \cdot \boldsymbol{E} = \boldsymbol{k}^* \cdot \boldsymbol{E} = 0 \tag{29}$$

and

$$\boldsymbol{S} = \frac{1}{2}\mathbf{Re}\Big\{\frac{\sqrt{\epsilon\mu}}{\mu^*}\Big\}|\boldsymbol{E}_0|^2 e^{-\frac{\mathbf{4\pi Im}(m)z}{\lambda}}\hat{\mathbf{e}}_z. \tag{30}$$

4 Stokes parameters

Consider the following experiment for an arbitrary monochromatic light source (see Bohren & Huffman p. 46). In the experiment, we make use of a measuring apparatus and polarizers with ideal performance: the measuring apparatus detects energy flux density independently of the state of polarization and the polarizers do not change the amplitude of the transmitted wave.

Denote

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{ikz-i\omega t}, \qquad && \boldsymbol{E}_0 = E_\perp \hat{\mathbf{e}}_\perp + E_\parallel \hat{\mathbf{e}}_\parallel \\ E_\perp &= a_\perp e^{-i\delta_\perp} && \\ E_\parallel &= a_\parallel e^{-i\delta_\parallel} && a_\perp, a_\parallel \geq 0, \delta_\perp, \delta_\parallel \in \mathbb{R} \end{aligned} \tag{31}$$

Experiment I
No polarizer: the flux density is proportional to

$$|\boldsymbol{E}_0|^2 = E_\parallel E_\parallel^* + E_\perp E_\perp^* \tag{32}$$

Experiment II
Linear polarizers $\parallel$ and $\perp$:
1) $\parallel$: the amplitude of the transmitted wave is $E_\parallel$ and the flux density is $E_\parallel E_\parallel^*$
2) $\perp$: the amplitude of the transmitted wave is $E_\perp$ and the flux density is $E_\perp E_\perp^*$

The difference of the two measurements is $I_\parallel - I_\perp = E_\parallel E_\parallel^* - E_\perp E_\perp^*$.

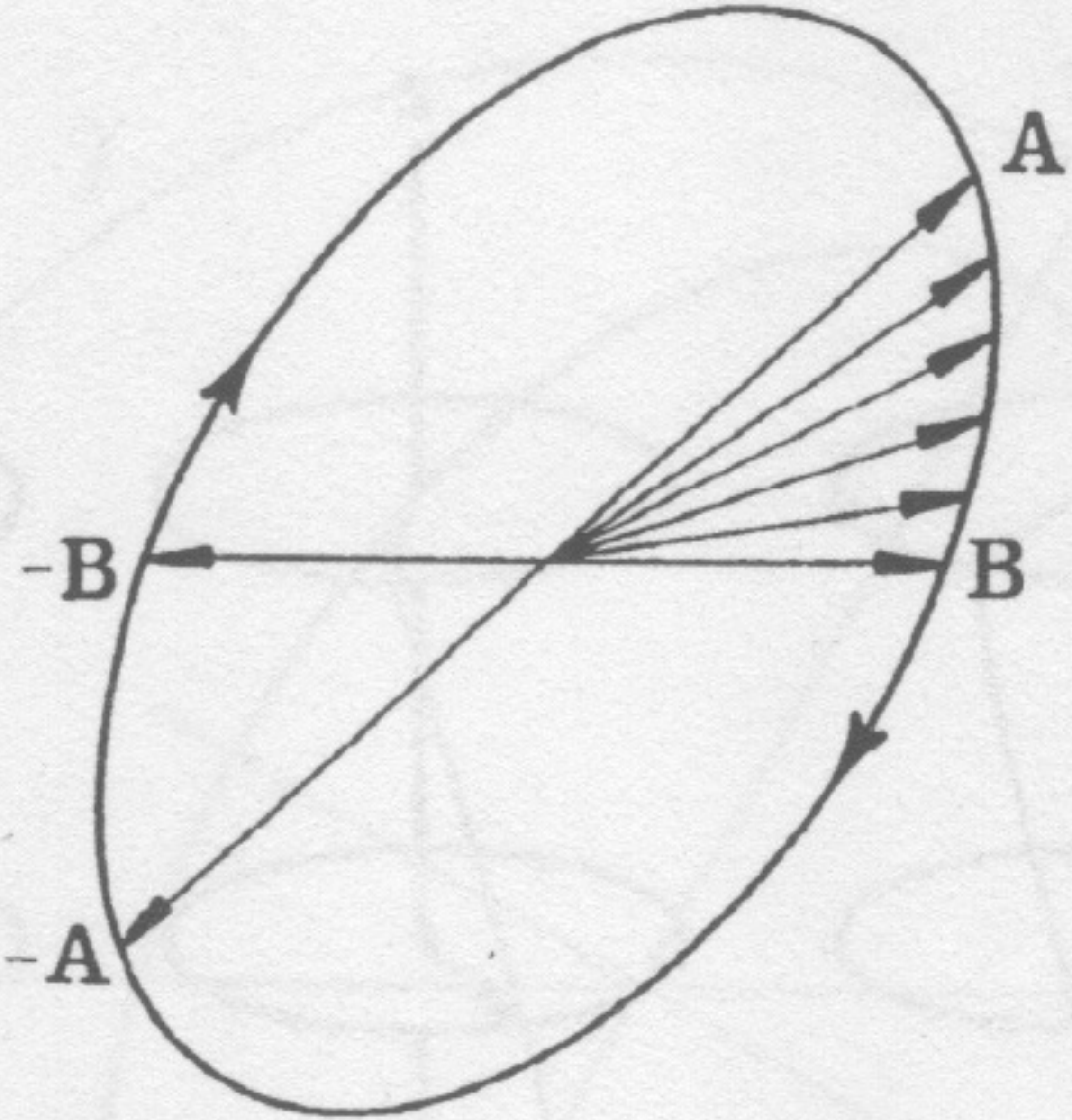

Figure 2.11 Vibration ellipse.

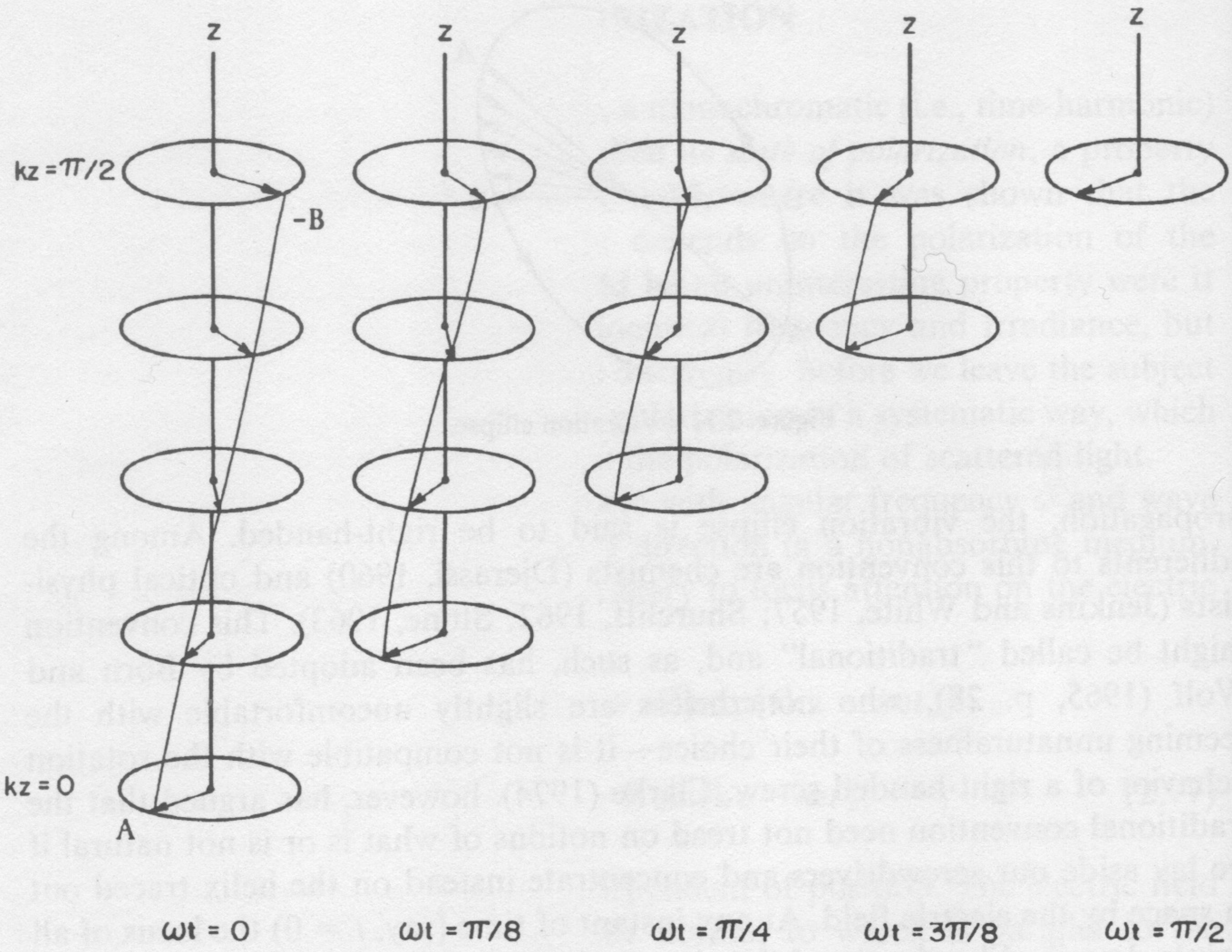

Figure 2.12 A series of snapshots of the electric field.

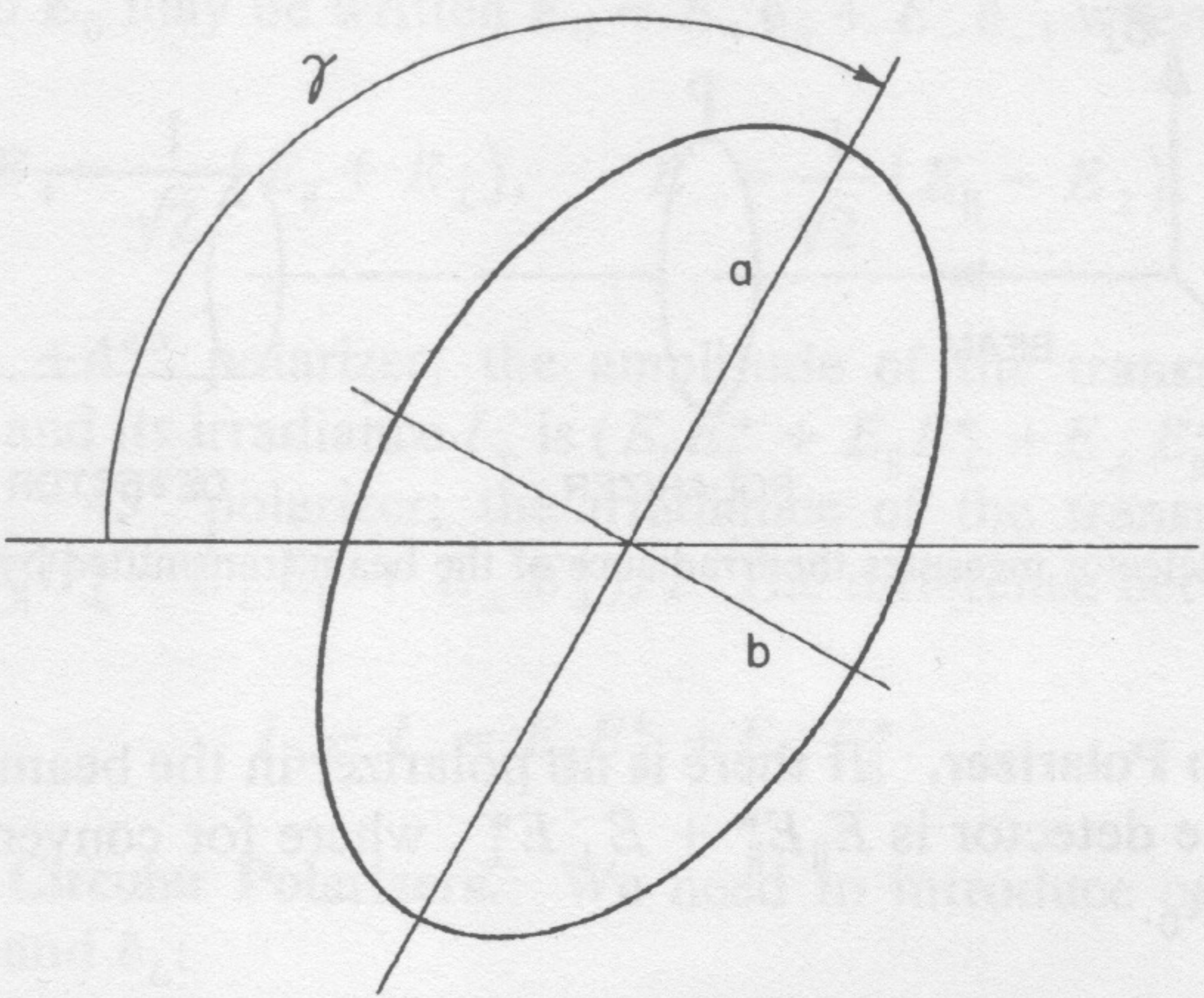

Figure 2.13 Vibration ellipse with ellipticity b/a and azimuth γ.

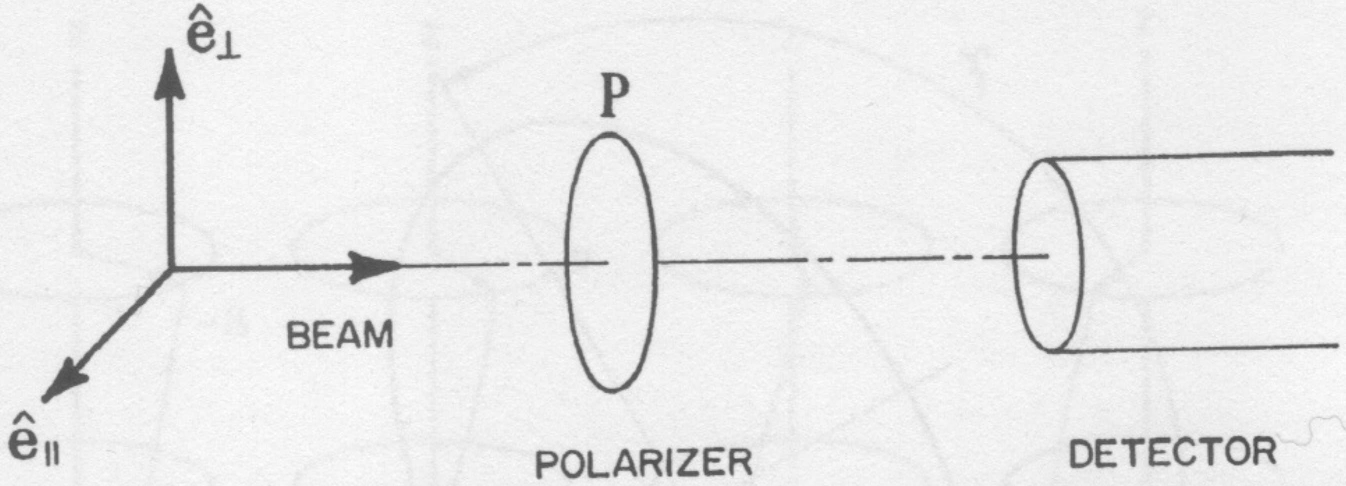

Figure 2.14 The detector measures the irradiance of the beam transmitted by the polarizer P.

Experiment III

Linear polarizers $+45°$ ja $-45°$: The new basis vectors are

$$\begin{cases} \hat{\mathbf{e}}_+ = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + \hat{\mathbf{e}}_\perp) \\ \hat{\mathbf{e}}_- = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - \hat{\mathbf{e}}_\perp) \end{cases}$$

and

$$\begin{aligned} \boldsymbol{E}_0 &= E_+\hat{\mathbf{e}}_+ + E_-\hat{\mathbf{e}}_- \\ E_+ &= \frac{1}{\sqrt{2}}(E_\parallel + E_\perp) \\ E_- &= \frac{1}{\sqrt{2}}(E_\parallel - E_\perp). \end{aligned}$$

1) $+45°$: the amplitude of the transmitted wave is E_+ and the flux density is $E_+E_+^* = \frac{1}{2}(E_\parallel E_\parallel^* + E_\parallel E_\perp^* + E_\perp E_\parallel^* + E_\perp E_\perp^*)$

2) $-45°$: the amplitude of the transmitted wave is E_- and the flux density is $E_-E_-^* = \frac{1}{2}(E_\parallel E_\parallel^* - E_\parallel E_\perp^* - E_\perp E_\parallel^* + E_\perp E_\perp^*)$

The difference os the measurements is $I_+ - I_- = E_\parallel E_\perp^* + E_\perp E_\parallel^*$.

Experiment IV

Circular polarizers R and L:

$$\begin{aligned}
\hat{\mathbf{e}}_R &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_R^* = 1 \\
\hat{\mathbf{e}}_L &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_L \cdot \hat{\mathbf{e}}_L^* = 1 \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_L^* = 0
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{E}_0 &= E_R\hat{\mathbf{e}}_R + E_L\hat{\mathbf{e}}_L \\
E_R &= \frac{1}{\sqrt{2}}(E_\parallel - iE_\perp) \\
E_L &= \frac{1}{\sqrt{2}}(E_\parallel + iE_\perp).
\end{aligned}$$

1) R: the amplitude of the transmitted wave is E_R and the flux density is $E_R E_R^* = \frac{1}{2}(E_\parallel E_\parallel^* - iE_\parallel^* E_\perp + iE_\perp^* E_\parallel + E_\perp E_\perp^*)$

2) L: the amplitude of the transmitted wave is E_L and the flux density is $E_L E_L^* = \frac{1}{2}(E_\parallel E_\parallel^* + iE_\parallel^* E_\perp - iE_\perp^* E_\parallel + E_\perp E_\perp^*)$

The difference of the measurements is $I_R - I_L = i(E_\perp^* E_\parallel - E_\parallel^* E_\perp)$.

With the help of Experiments I-IV, we have determined the Stokes parameters I, Q, U, and V:

$$\begin{aligned}
I &= E_{\parallel}E_{\parallel}^{*} + E_{\perp}E_{\perp}^{*} = a_{\parallel}^{2} + a_{\perp}^{2} \\
Q &= E_{\parallel}E_{\parallel}^{*} - E_{\perp}E_{\perp}^{*} = a_{\parallel}^{2} - a_{\perp}^{2} \\
U &= E_{\parallel}E_{\perp}^{*} + E_{\perp}E_{\parallel}^{*} = 2a_{\parallel}a_{\perp}\cos\delta \\
V &= i(E_{\parallel}E_{\perp}^{*} - E_{\perp}E_{\parallel}^{*}) = 2a_{\parallel}a_{\perp}\sin\delta \qquad \delta = \delta_{\parallel} - \delta_{\perp}
\end{aligned} \tag{33}$$

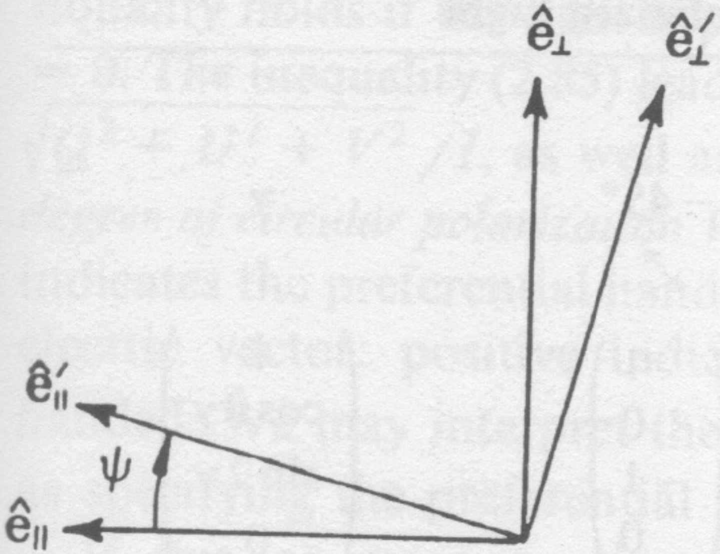

Figure 2.16 Rotation of basis vectors.

are rotated through an angle ψ (Fig. 2.16), the transformation from (I, Q, U, V) to Stokes parameters (I', Q', U', V') relative to the rotated axes $\hat{\mathbf{e}}'_{\parallel}$ and $\hat{\mathbf{e}}'_{\perp}$ is

$$\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & \cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}. \qquad (2.83)$$

Table 2.2 Stokes Parameters for Polarized Light

Linearly Polarized

0°	90°	+45°	−45°	γ
↔	↕	⤡	⤢	
$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$	$\begin{pmatrix}1\\\cos 2\gamma\\\sin 2\gamma\\0\end{pmatrix}$

Circularly Polarized

Right	Left
↺	∩
$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$

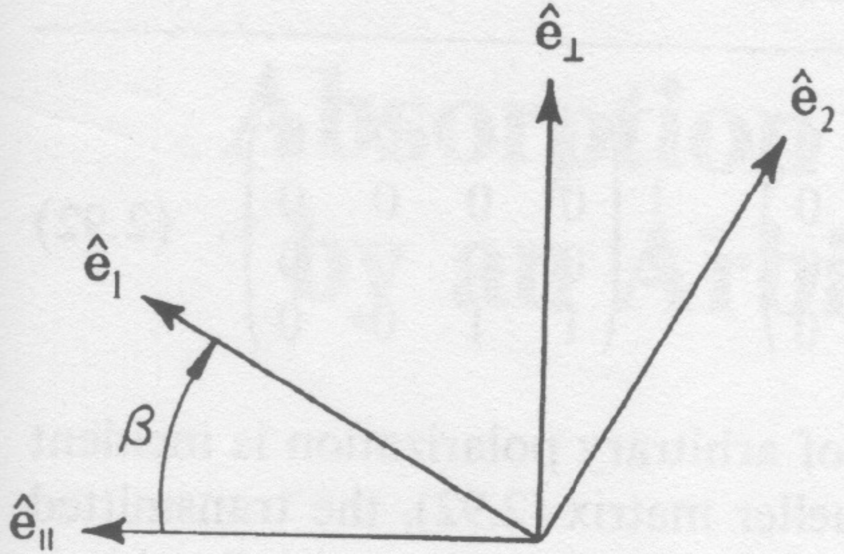

Figure 2.17 $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ specify the axes of an ideal linear retarder.

retarder:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C^2 + S^2\cos\delta & SC(1-\cos\delta) & -S\sin\delta \\ 0 & SC(1-\cos\delta) & S^2 + C^2\cos\delta & C\sin\delta \\ 0 & S\sin\delta & -C\sin\delta & \cos\delta \end{pmatrix}, \tag{2.90}$$

where $C = \cos 2\beta$, $S = \sin 2\beta$, and the *retardance* δ is $\delta_1 - \delta_2$.

The usefulness of the Mueller ...