# 1 (lecture 12)

The scattering amplitude due to the illuminated side of the scatterer cannot be calculated without defining the shape and optical properties of the particle. Let us assume in the following example that the illuminated region is perfectly conducting. Then, the tangential components of the fields $\mathbf{E}_s$ and $\mathbf{B}_s$ on $S_1$ are approximately opposite and similar to those of the original fields, respectively. The scattering amplitude due to the illuminated part is then

$$\epsilon^* \cdot \mathbf{F}_{ill} = \frac{E_0}{4\pi i} \int_{ill} dA' \epsilon^* \cdot [-\mathbf{n}' \times (\mathbf{k}_0 \times \epsilon_0) + \mathbf{k} \times (\mathbf{n}' \times \epsilon_0)] \cdot e^{i(\mathbf{k}_0 - \mathbf{k})\cdot \mathbf{x}'}$$

When this is compared with the shadow amplitude, the only notable difference is the sign in the first term. This sign difference results in a completely different scattering amplitude that can also be written in the form muodossa

$$\epsilon^* \cdot \mathbf{F}_{ill} = \frac{E_0}{4\pi i} \int_{ill} dA' \epsilon^* \cdot [(\mathbf{k} - \mathbf{k}_0) \times (\mathbf{n}' \times \epsilon_0) - (\mathbf{n}' \cdot \epsilon_0)\mathbf{k}_0] \cdot e^{i(\mathbf{k}_0 - \mathbf{k})\cdot \mathbf{x}'}$$

When again $kR >> 1$, the exponential factor fluctuates rapidly and one would expect a strong contribution in the forward direction; however, the first term goes to zero in the forward direction and no strong contribution can follow. The illuminated region contributes to scattering in the form of a reflected wave.

Assume next that the scattering particle is spherical (radius $a$). The predominating contribution to the scattering amplitude now derives from a region of integration where the phase of the exponential factor is stationary. If $(\theta, \varphi)$ are the coordinates of $\mathbf{k}$ and $(\alpha, \beta)$ those of $\mathbf{n}'$ (with respect to $\mathbf{k}_0$), the phase factor is

$$\phi(\alpha, \beta) = (\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}' = ka[(1 - \cos\theta)\cos\alpha - \sin\theta \sin\alpha \cos(\beta - \varphi)]$$

The stationary point can be found at angles $\alpha_0, \beta_0$, where $\alpha_0 = \pi/2 + \theta/2$ and $\beta_0 = \varphi$. These angles correspond exactly to the angles of reflection on the surface of the sphere as dictated by geometric optics. At that point, the vector $\mathbf{n}'$ points in the direction of $(\mathbf{k} - \mathbf{k}_0)$. In the proximity of angles $\alpha = \alpha_0$ and $\beta = \beta_0$

$$\phi(\alpha, \beta) = -2ka \sin\frac{\theta}{2}[1 - \frac{1}{2}(x^2 + \cos^2\frac{\theta}{2} y^2) + \ldots]$$

where $x = \alpha - \alpha_0$ and $y = \beta - \beta_0$. The integration can be carried out approximately:

$$\epsilon^* \cdot \mathbf{F}_{ill} \cong ka^2 E_0 \sin\theta e^{-2ika\sin\frac{\theta}{2}} (\epsilon^* \cdot \epsilon_r) \cdot \int dx e^{i[ka \sin\frac{\theta}{2}]x^2} \int dy e^{i[ka \sin\frac{\theta}{2}\cos^2\frac{\theta}{2}]y^2}$$

$$\epsilon_r = -\epsilon_0 + 2(\mathbf{n}_r \cdot \epsilon_0)\mathbf{n}_r, \qquad \mathbf{n}_r = \frac{\mathbf{k} - \mathbf{k}_0}{|\mathbf{k} - \mathbf{k}_0|}$$

When $2ka\sin\frac{\theta}{2} >> 1$, the integrals can be calculated using the result $\int_{-\infty}^{\infty} dx e^{i\alpha x^2} = \sqrt{\pi i/\alpha}$,

$$\epsilon^* \cdot \mathbf{F}_{ill} \cong E_0 \frac{a}{2} e^{-2ika\sin\frac{\theta}{2}} \epsilon^* \cdot \epsilon_r$$

For large $2ka\sin\frac{\theta}{2}$, the intensity of the reflected part of the radiation is constant as a function of the angle, but the part has a rapidly varying phase. When $\theta \to 0$, the intensity vanishes as $\theta^2$ (see the integral above).

Comparison of the amplitudes due to the shadowed and illuminated parts of the surface shows that, in the forward direction, the former amplitude predominates over the latter by a factor $ka >> 1$ whereas, at the scattering angles $2ka\sin\theta >> 1$, the ratio of the amplitudes is of the order of $1/(ka\sin^3\theta)^{1/2}$. The differential scattering cross section (summed over the polarization states of the original and scattered waves) is

$$\frac{d\sigma}{d\Omega} \cong \begin{cases} a^2(ka)^2 \left|\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right|^2, & \theta \lesssim \frac{10}{ka} \quad ; \\ \frac{a^2}{4}, \quad \theta >> \frac{1}{ka} & . \end{cases}$$

The total scattering cross section is twice the geometric cross section of the particle.

# 2 Optical theorem

The optical theorem is a fundamental relation that connects the exticntion cross section to the imaginary part of the forward-scattering amplitude. Consider a plane wave with a wave vector $\mathbf{k}_0$ and field components $\mathbf{E}_i, \mathbf{B}_i$. The plane wave is incident on a finite-sized scatterer inside the surface $S_1$. The scattered field $\mathbf{E}_s, \mathbf{B}_s$ propagates away from the scatterer and is observed in the far zone in the direction $\mathbf{k}$. The total field outside the surface $S_1$ is, by definition,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_i + \mathbf{E}_s \\ \mathbf{B} &= \mathbf{B}_i + \mathbf{B}_s. \end{aligned}$$

In the general case, the scatterer absorbs energy from the original field. The absorbed power can be calculated by integrating the inward-directed Poynting-vector component of the total field over the surface $S_1$:

$$P_{abs} = -\frac{1}{2\mu_0} \oint_{S_1} dA' \mathrm{Re}(\mathbf{E} \times \mathbf{B}^*) \cdot \mathbf{n}'$$

The scattered power is computed in the usual way from the asymptotic form of the Poynting vector for the scattered fields in the regime, where the fields are simple transverse spherical waves that attenuate as $1/r$. But since there are no sources between $S_1$ and infinity, the scattered power can as well be calculated as an integral of the outward-directed component of the Poynting vector for the scattered field over $S_1$:

$$P_{sca} = \frac{1}{2\mu_0} \oint_{S_1} dA' \mathrm{Re}(\mathbf{E}_s \times \mathbf{B}_s^*) \cdot \mathbf{n}'$$

The total power is the sum of the absorbed and scattered power so that, after rearranging,

$$P = P_{abs} + P_{sca} = -\frac{1}{2\mu_0} \oint_{S_1} dA' \text{Re}(\mathbf{E}_s \times \mathbf{B}_i^* + \mathbf{E}_i^* \times \mathbf{B}_s) \cdot \mathbf{n}'$$

When the original field in written explicitly in the form

$$\begin{aligned} \mathbf{E}_i &= E_0 \epsilon_0 e^{i\mathbf{k}_0 \cdot \mathbf{x}} \\ c\mathbf{B}_i &= \frac{1}{k} \mathbf{k}_0 \times \mathbf{E}_i \end{aligned}$$

the total power can be transformed to the form

$$P = \frac{1}{2\mu_0} Re E_0^* \oint_{S_1} dA' e^{-i\mathbf{k}_0 \cdot \mathbf{x}} [\epsilon_0^* \cdot (\mathbf{n}' \times \mathbf{B}_s) + \epsilon_0^* \cdot \frac{\mathbf{k}_0 \times (\mathbf{n}' \times \mathbf{E}_s)}{kc}]$$

By comparing this with the scattering amplitude $\mathbf{F}(\mathbf{k}, \mathbf{k}_0)$ derived earlier, we can recognize that the total power is proportional to the value of $\mathbf{F}$ in the forward-scattering direction $\mathbf{k} = \mathbf{k}_0$ in the polarization state coinciding with that of the original field:

$$P = \frac{2\pi}{kZ_0} \text{Im}[E_0^* \epsilon_0^* \cdot \mathbf{F}(\mathbf{k} = \mathbf{k}_0)],$$

which is the basic form of the optical theorem.

The total or extinction cross section $\sigma_e$ is defined as the ratio of the total and original flux densities ($|E_0|^2/2Z_0$, power as per unit surface area).

In a corresponding way, one can define a normalized scattering amplitude $\mathbf{f}$ (against the original field value at origin)

$$f(\mathbf{k} = \mathbf{k}_0) = \frac{\mathbf{F}(\mathbf{k}, \mathbf{k}_0)}{E_0}$$

The final form of the optical theorem is then

$$\sigma_e = \frac{4\pi}{k} \text{Im}[\epsilon_0^* \cdot \mathbf{f}(\mathbf{k} = \mathbf{k}_0)].$$