1 Diffraction by a circular aperture (lecture 11)

Diffraction is divided into Fraunhofer and Fresnel diffraction depending on the geometry under consideration. There are three length scales involved: the size of the diffracting system d, the distance from the system to the observation point r and the wavelength λ. The diffraction pattern is generated when $r >> d$. In tht case, the slowly changing parts of the vector integral relation can be kept constant. Particular attention needs to be paid to the phase factor e^{ikR}. When $r >> d$, we obtain

$$kR = kr - k\mathbf{n}\cdot\mathbf{x}' + \frac{k}{2r}[r'^2 - (\mathbf{n}\cdot\mathbf{x}')^2] + \ldots, \qquad \mathbf{n} = \frac{\mathbf{x}}{r}, \tag{1}$$

where $\mathbf{n}$ is a unit vector pointing in the direction of the observer. The magnitudes of the terms in the expansion are $kr, kd, (kd)^2/kr$. In Fraunhofer diffraction, the terms from the third one (inclusive) onwards are negligible. When the third term becomes significant (e.g., large diffracting systems), we enter the domain of Fresnel diffraction. Far enough from any diffracting system, we end up in the domain of Fraunhofer diffraction.

If the observation point is far away from the diffracting system, Kirchhoff's scalar integral relation assumes the form

$$\Psi(\mathbf{x}) = -\frac{e^{ikr}}{4\pi r}\int_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}\Big[\mathbf{n}\cdot\nabla'\Psi(\mathbf{x}') + i\mathbf{k}\cdot\mathbf{n}\Psi(\mathbf{x}'), \tag{2}$$

where $\mathbf{n}$ now is the unit normal vectoron, $\mathbf{x}'$ denotes the position of the element dA', and $r = |\mathbf{x}|, \mathbf{k} = k(\mathbf{x}/r)$. The so-called Smythe-Kirchhoff integral relation is an improved version of the pure Kirchhoff relation and, in the present limit, takes the form

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}}{2\pi r}\mathbf{k}\times\int_{S_1} dA'\mathbf{n}\times\mathbf{E}(\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}'} \tag{3}$$

Let us study next what the different diffraction formulae give for a circular hole (radius a) in an infinitesimally thin perfectly conducting slab.

Figure (see Jackson)

In the vector relation,

$$(\mathbf{n}\times\mathbf{E}_i)_{z=0} = E_0\epsilon_2\cos\alpha e^{ik\sin\alpha x'} \tag{4}$$

and, in polar coordinates,

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}E_0\cos\alpha}{2\pi r}(\mathbf{k}\times\epsilon_2)\int_0^a d\zeta\zeta\int_0^{2\pi} d\beta e^{ik\zeta[\sin\alpha\cos\beta - \sin\theta\cos(\varphi-\beta)]} \tag{5}$$

Define

$$\xi \equiv \frac{1}{k}|\mathbf{k}_\perp - \mathbf{k}_{0,\perp}| = \sqrt{\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\varphi}, \tag{6}$$

in which case the integral takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta' e^{-ik\zeta\xi\cos\beta'} = J_0(k\zeta\xi) \tag{7}$$

that is, the result is the Bessel function J_0. Hereafter, the integration over the radial part can be calculated analytically, and

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}}{r} a^2 E_0 \cos\alpha (\mathbf{k}\times\epsilon_2)\frac{J_1(ka\xi)}{ka\xi} \tag{8}$$

The time-averaged power as per unit solid angle is then

$$\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}(\cos^2\theta + \cos^2\varphi\sin^2\theta)|\frac{2J_1(ka\xi)}{ka\xi}|^2 \tag{9}$$

$$P_i = (E_0^2/2z_0)\pi a^2 \cos\alpha, \tag{10}$$

where P_i is the total power normally incident on the hole. If $ka >> 1$, the function $[(2J_1(ka\xi)/ka\xi)^2]$ peaks sharply at 1 with the argument $\xi = 0$ and falls down to zero at $\Delta\xi \approx 1/ka$. The main part of the wave propagates according to geometric optics and only modest diffraction effects show up. If, however, $ka \approx 1$, the Bessel function varies slowly as a function of the angles and the transmitted wave bends into directions considerably deviting from the propagation direction of the incident field. In the extreme limit $ka << 1$, the angular dependence derives from the polarization factor $\mathbf{k}\times\epsilon_2$, but the analysis fails because the field in the hole can no longer be the original undisturbed field as assumed earlier.

let us study the scalar solution assuming that Ψ corresponds the magnitude of the $\mathbf{E}$ field,

$$\begin{aligned}\Psi(\mathbf{x}) &= -ik\frac{e^{ikr}}{r}a^2E_0\frac{1}{2}(\cos\alpha+\cos\theta)\frac{J_1(ka\xi)}{ka\xi} \\ \frac{dP}{d\Omega} &\cong P_i\frac{(ka)^2}{4\pi}\cos\alpha(\frac{\cos\alpha+\cos\theta}{2\cos\alpha})|\frac{2J_1(ka\xi)}{ka\xi}|^2\end{aligned} \tag{11}$$

Both the vector and scalar results include the Bessel part $[(2J_1(ka\xi)/ka\xi)^2]$ and the same wave number dependence. But whereas there is no azimuthal dependence in the scalar result, the vector result is significantly affected by the azimuthal dependence. The dependence derives from the polarization of the vector field. For an original field propagating in the direction of the normal vector, the polarization effects are not important, when additionally $ka >> 1$. Then, all the results reduce into the familiar expression

$$\frac{dP}{d\Omega} \cong P_i\frac{(ka)^2}{\pi}|\frac{J_i(ka\sin\theta)}{ka\sin\theta}|^2 \tag{12}$$

However, for oblique directions, there are large deviations and, for very small holes, the analysis fails completely.

2 Scattering in detail

Let us now consider a small particle that is much larger than the wavelength and study what kind of tools the vector Kirchhoff integral relation offers, if the fields close to the surface can be estimated somehow.

For example, the surface of the scatterer is divided into the illuminated and shadowed parts. The boundary between the two parts is sharp only in the limit of geometric optics and, in the transition zone, the breadth of the boundary is of the order of $(2/kR)^{1/3} \cdot R$, where R is a typical radius of curvature on the surface of the particle.

On the shadow side, the scattered field must be equal to the original field but opposite in sign, in which case the total field vanishes. On the illuminated side, the field depends in a detailed way on the properties of the scattering particle. If the curvature radii are large compared to the wavelength, we can make use of Fresnel's coefficients and geometric optics in general. The analysis can be generalized into the case of a transparent particle and the method is known as the physical-optics approximation (or Kirchhoff approximation).

Let us write the scattering amplitude explicitly in two parts,

$$\epsilon^* \cdot \mathbf{F} = \epsilon^* \cdot \mathbf{F}_{sh} + \epsilon^* \cdot \mathbf{F}_{ill} \tag{13}$$

and assume that the incident fields is a plane wave

$$\begin{aligned} \mathbf{E}_i &= E_0 \epsilon_0 e^{i\mathbf{k}_0 \cdot \mathbf{x}} \\ \mathbf{B}_i &= \mathbf{k}_0 \times \mathbf{E}_i / kc \end{aligned}$$

The shadow scattering amplitude is then ($\mathbf{E}_s \approx -\mathbf{E}_i, \mathbf{B}_s \approx -\mathbf{B}_i$)

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{E_0}{4\pi i} \int_{sh} dA' \epsilon^* \cdot [\mathbf{n}' \times (\mathbf{k}_0 \times \epsilon_0) + \mathbf{k} \times (\mathbf{n}' \times \epsilon_0)] \cdot e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'} \tag{14}$$

where the integration is over the shadowed region. The amplitude can be rearranged into the form

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{E_0}{4\pi i} \int_{sh} dA' \epsilon^* \cdot [(\mathbf{k} + \mathbf{k}_0) \times (\mathbf{n}' \times \epsilon_0) + (\mathbf{n}' \cdot \epsilon_0)\mathbf{k}_0] \cdot e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'} \tag{15}$$

In the short-wavelength limit, $\mathbf{k}_0 \cdot \mathbf{x}'$ and $\mathbf{k} \cdot \mathbf{x}'$ vary across a large regime and the exponential factor fluctuates rapidly and eliminates the integral everywhere else but the forward-scattering direction $\mathbf{k} \approx \mathbf{k}_0$. In that direction ($\theta \lesssim 1/kR$), the second factor is negligible compared to the first one since $(\epsilon^* \cdot \mathbf{k}_0)/k$ is of the order of $\sin\theta << 1, (\epsilon^* \cdot \mathbf{k} = 0, \mathbf{k}_0 \approx \mathbf{k})$. Thus,

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{iE_0}{2\pi} (\epsilon^* \cdot \epsilon_0) \int_{sh} dA' (\mathbf{k}_0 \cdot \mathbf{n}') e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'} \tag{16}$$

In this approximation, the integral over the shadow side only depends on the projected area against the propagation direction of the original field. This can be seen from the fact that

$\mathbf{k}_0 \cdot \mathbf{n}' dA' = k dx' dy' = k d^2\mathbf{x}'_\perp$ ja $(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}' = k(1 - \cos\theta) z' - \mathbf{k}_\perp \cdot \mathbf{x}'_\perp \approx -\mathbf{k}_\perp \cdot \mathbf{x}'_\perp$. The final form of the shadow scattering amplitude is thus

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{ik}{2\pi} E_0 (\epsilon^* \cdot \epsilon_0) \int_{sh} d^2\mathbf{x}'_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{x}'_\perp} \tag{17}$$

In this limit, all scatterers producing the same prohjected area will have the same shadow scattering amplitude. For example, in the case of a circular cylindrical slab (radius a)

$$\int_{sh} d^2\mathbf{x}'_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{x}'_\perp} = 2\pi a^2 \frac{J_1(ka\sin\theta)}{ka\sin\theta}, \tag{18}$$

$$\epsilon^* \cdot \mathbf{F}_{sh} \cong ika^2 E_0 (\epsilon^* \cdot \epsilon_0) \frac{J_1(ka\sin\theta)}{ka\sin\theta}. \tag{19}$$

This explains nicely the forward diffraction pattern in scattering by small particles.