1 Lecture 10

We have solved for the scattered field completely for two circular polarization states of the original field $\epsilon_1 \pm i\epsilon_2$. We thus know the elements $S_j^{(c)}$ (j=1,2,3,4) of the amplitude scattering matrix that relates the original field to the scattered field in the circular-polarization representation.

$$\begin{pmatrix} E_{s-} \\ E_{s+} \end{pmatrix} = \frac{e^{ik(r-z)}}{-ikr} \begin{pmatrix} S_2^{(c)} & S_3^{(c)} \\ S_4^{(c)} & S_1^{(c)} \end{pmatrix} \begin{pmatrix} E_{i-} \\ E_{i+} \end{pmatrix} \tag{1}$$

The amplitude scattering matrix elements of the circular-polarization representation relate linearly to the commonly used ones of the linear-polarization representation.

$$\begin{pmatrix} S_1^{(c)} \\ S_2^{(c)} \\ S_3^{(c)} \\ S_4^{(c)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & -i & i \\ -1 & 1 & i & i \\ -1 & 1 & -i & -i \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}, \tag{2}$$

where, in the case of Lorenz-Mie scattering ($S_3 = S_4 = 0$),

$$\left.\begin{array}{c} S_1^{(c)} = (S_1 + S_2)/2 \\ S_2^{(c)} = (S_1 + S_2)/2 = S_1^{(c)} \\ S_3^{(c)} = (-S_1 + S_2)/2 \\ S_4^{(c)} = S_3^{(c)} \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} S_1 = S_1^{(c)} - S_3^{(c)} \\ S_2 = S_1^{(c)} + S_3^{(c)} \end{array}\right. \tag{3}$$

The complete scattering matrix follows now from the elements S_1, S_2 in a standard manner. The elements $S_1^{(c)}$, $S_2^{(c)}$, $S_3^{(c)}$, $S_4^{(c)}$ follow from the far-zone expressions for the scattered fields, for which the vector spherical harmonics expansions reduce to the form utlized earlier in the expression for the differential scattering cross section. A more detailed assessment is left for an exercise.

2 Scattering at the short-wavelength limit. Scalar diffraction theory.

Traditionally, diffraction entails those deviations from geometric optics that derive from the finite wavelength of the waves. Thereby, diffraction is connected to objects (e.g., holes, obstacles) that are large compared to the wavelength. The possible geometries are described in the figure below (see Jackson). The sources of the radiation are located in region I and we want to derive the diffracted fields in the diffraction region II. The regions are bounded by the interfaces S_1 and S_2. Kirchhoff was the first one to treat this topic systematically.

For simplicity, we will first study scalar fields, whereafter we will extend the analysis to vector fields. Let $\psi(\mathbf{x}, t)$ be a scalar field, for which we assume a harmonic time dependence

$e^{-i\omega t}$. In essence, ψ is one of the components of the $\mathbf{E}$ or $\mathbf{B}$ fields. We assume that ψ fulfils the scalar Helmholtz wave equation

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = 0$$

in the volume V bounded by S_1 and S_2. We introduce the Green's function $G(\mathbf{x}, \mathbf{x}')$,

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$$

and start from Green's theorem

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3\mathbf{x}' = \oint_S [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}]dA'$$

$$\frac{\partial\psi}{\partial n} \equiv \mathbf{n}' \cdot \nabla\psi$$

where $\mathbf{n}'$ is the unit inward normal vector of S. Let us now set $\psi = G$ and $\phi = \psi$ so that, with the help of the wave equations for ψ and G,

$$\psi(\mathbf{x}) = \oint_S dA'[\psi(\mathbf{x}')\mathbf{n}' \cdot \nabla' G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\mathbf{n}' \cdot \nabla'\psi(\mathbf{x}')]$$

Kirchhoff's diffraction integral follows from this relation when G is chosen to be the free-space Green's function describing outgoing waves,

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{ikR}}{4\pi R}, \qquad \mathbf{R} = \mathbf{x} - \mathbf{x}', R = |\mathbf{R}|$$

Then

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S dA'\frac{e^{ikR}}{R}\mathbf{n}' \cdot [\nabla'\psi + ik(1 + \frac{i}{kR})\frac{\mathbf{R}}{R}\psi]$$

The surface S is composed of S_1 and S_2 and the integration can be divided into two parts. In the proximity of S_2, ψ is an outgoing wave and fulfils the so-called radiation condition

$$\psi \to f(\theta, \varphi)\frac{e^{ikr}}{r}, \qquad \frac{1}{\psi}\frac{\partial\psi}{\partial r} \to (ik - \frac{1}{r}).$$

By inserting these results into the integral above, it is possible to show that the integral over S_2 vanishes at least as the inverse of the radius of the sphere when the radius approaches infinity. There remains the integral over S_1, giving the final form of the Kirchhoff integral relation,

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{S_1} dA'\frac{e^{ikR}}{R}\mathbf{n}' \cdot [\nabla'\psi + ik(1 + \frac{i}{kR})\frac{\mathbf{R}}{R}\psi]$$

In applying the integral relation, it is necessary to know both ψ and $\partial\psi/\partial n$ on the surface S_1. In general, these are not known, at least not precisely. Kirchhoff's approach was based on the idea that ψ and $\partial\psi/\partial n$ are approximated on S_1 for the computation of the diffracted wave. This so-called Kirchhoff's approximation consists of the following assumptions:

1. ψ and $\partial\psi/\partial n$ vanish everywhere else but the holes of S_1

2. ψ and $\partial\psi/\partial n$ in the holes are equal to the original field values when there are no diffracting elements in space.

These assumptions contain a serious mathemtical inconsistency: if ψ and $\partial\psi/\partial n$ are zero on a finite surface, then $\psi = 0$ everywhere. In spite of the inconsistency, the Kirchhoff approximation works in an excellent way in practical problems and constitutes the basis of all diffraction calculus in classical optics.

The mathematical inconsistencies can be removed by a proper choice of the Green's function. In the setup of the figure below (see Jackson), (both P and P' are located several wavelengths away from the hole) we obtain

$$\psi(P) = \frac{k}{2\pi i}\int_{S_1} dA' \frac{e^{ikr}}{r}\frac{e^{ikr'}}{r'}\mathcal{O}(\theta,\theta')$$

$$\mathcal{O}(\theta,\theta') = \begin{cases} \cos\theta, & ; \\ \cos\theta', & ; \\ \frac{1}{2}(\cos\theta + \cos\theta'), & \text{(Kirchhoffin approksimaatio).} \end{cases}$$

The obliquity factor $\mathcal{O}(\theta,\theta')$ assumes less significance than the phase factors, which partly explains the success of the Kirchhoff approximation.

3 Vector Kirchhoff integral relation

The scalar Kirchhoff integral relation is an exact relation between the scalar fields on the surface and at infinity. In a corresponding way, the vector Kirchhoff integral relation is an exact relation between the $\mathbf{E}, \mathbf{B}$ fields on the surface S and the diffracted or scattered fields at infinity. Such a relation is interesting in itself and it is a correct guess that the relation carries practical significance, too.

In what follows, we derive the vector relation for the electric field $\mathbf{E}$, starting from the generalization of Green's theorem already appearing in the scalar case for all components of the $\mathbf{E}$-field,

$$\mathbf{E}(\mathbf{x}) = \oint_S dA'[\mathbf{E}(\mathbf{n}'\cdot\nabla' G) - G(\mathbf{n}'\cdot\nabla')\mathbf{E}],$$

when $\mathbf{x} \in V$ and V is the volume bounded by S. Again, $\mathbf{n}'$ is the unit normal vector pointing into the volume V. Since G is singular at $\mathbf{x}' = \mathbf{x}$ and we make use of vector calculus valid for smooth functions, we assume that S is composed of the outer surface S' and an infinitesimally small inner surface S'' so that the point $\mathbf{x}' = \mathbf{x}$ is left out from volume V (but the point is inside S''). In such a case, the left-hand side of the previous equation disappears, but the integration over S'' on the right-hand side returns $-\mathbf{E}(\mathbf{x})$ when the radius of S'' goes to zero.

The vector relation can now be written in the form

$$0 = \oint_S dA'[2\mathbf{E}(\mathbf{n}' \cdot \nabla' G) - \mathbf{n}' \cdot \nabla'(G\mathbf{E})]$$

and, with the help of the divergence theorem ja divergenssiteoreeman

$$\int_V dV' \nabla \cdot A = \oint_S dA' \mathbf{A} \cdot \mathbf{n},$$

the latter term can be transformed to a volume integral

$$0 = \oint_S dA' 2\mathbf{E}(\mathbf{n}' \cdot \nabla' G) + \int_V dV' \nabla'^2 (G\mathbf{E})$$

Now

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

$$\int_V dV \nabla \phi = \oint_S dA \mathbf{n} \phi, \qquad (\mathbf{n} \text{ ulkonormaali})$$

$$\int_V dV \nabla \times \mathbf{A} = \oint_S dA (\mathbf{n} \times \mathbf{A})$$

and the volume integral can be returned back to a surface integral

$$0 = \oint_S dA'[2\mathbf{E}(\mathbf{n}' \cdot \nabla' G) - \mathbf{n}'(\nabla' \cdot (G\mathbf{E})) + \mathbf{n}' \times (\nabla' \times (G\mathbf{E}))]$$

When the ∇-operations are carried out for $G\mathbf{E}$ and use is made of Maxwell's equations $\nabla' \cdot \mathbf{E} = 0, \nabla' \times \mathbf{E} = i\omega \mathbf{B}$, one obtains

$$0 = \oint_S dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G]$$

and, furthermore,

$$\mathbf{E}(\mathbf{x}) = \oint_S dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G]$$

where the volume bounded by S now again includes the point $\mathbf{x}$.

As in the case of the scalar relation, we can now derive the vector Kirchhoff integral relation

$$\mathbf{E}(\mathbf{x}) = \oint_{S_1} dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G],$$

where the integration extends over S_1 only.

Finally, we derive a relation between the scattering amplitude and the near fields. For the fields in the vector Kirchhoff integral relation, we choose the scattered fields $\mathbf{E}_s, \mathbf{B}_s$, that is, the total fields $\mathbf{E}, \mathbf{B}$ minus the original fields $\mathbf{E}_i, \mathbf{B}_i$. If the observation point is far away from the scatterer, both the Green's function and the scattered electric field can be given in their asymptotic forms

$$G(\mathbf{x}, \mathbf{x}') \Longrightarrow \frac{1}{4\pi} \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

$$\mathbf{E}_s(\mathbf{x}) \rightarrow \frac{e^{ikr}}{r} \mathbf{F}(\mathbf{k}, \mathbf{k}_0)$$

where $\mathbf{k}$ is a wave vector pointing in the direction of the observer, $\mathbf{k}_0$ is the wave vector of the original field, and $\mathbf{F}(\mathbf{k}, \mathbf{k}_0)$ is the vector scattering amplitude. In this limit, $\nabla' G = -i\mathbf{k}G$ and we obtain an integral relation for the scattering amplitude,

$$\mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{i}{4\pi} \oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'} [\omega(\mathbf{n}' \cdot \mathbf{B}_s) + \mathbf{k} \times (\mathbf{n}' \times \mathbf{E}_s) - \mathbf{k}(\mathbf{n}' \cdot \mathbf{E}_s)]$$

The relation depends explicitly on the direction of $\mathbf{k}$ and the dependence on $\mathbf{k}_0$ is implicit in $\mathbf{E}_s$ and $\mathbf{B}_s$. Since $\mathbf{k} \cdot \mathbf{F} = 0$, we can reduce the relation to

$$\mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{1}{4\pi i} \mathbf{k} \times \oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'} [\frac{c\mathbf{k} \times (\mathbf{n}' \times \mathbf{B}_s)}{k} - \mathbf{n}' \times \mathbf{E}_s]$$

Alternatively, one may want the scattering amplitude in direction $\mathbf{k}$ for a specific polarization state ϵ^*,

$$\epsilon^* \cdot \mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{i}{4\pi} \oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'} [\omega\epsilon^* \cdot (\mathbf{n}' \times \mathbf{B}_s) + \epsilon^* \cdot (\mathbf{k} \times (\mathbf{n}' \times \mathbf{E}_s))]$$

These integral relations are useful in scattering problems entailing short wavelengths and in the derivation of the optical theorem.