1 (lecture 8)

In order to solve the vector wave equation, we return one more time to the angular part of the scalar wave equation and introduce useful auxiliary tools. The spherical harmonics are solutions of the following equation:

$$-\Big[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Big]Y_{lm} = l(l+1)Y_{lm},$$

which can be written in the form (cf. quantum mechanics)

$$L^2 Y_{lm} = l(l+1)Y_{lm}$$

where

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$\mathbf{L} = \frac{1}{i}(\mathbf{r}\times\nabla)$$

so that $\mathbf{L}$ is $\hbar^{-1}$ times the orbital impulse momentum operator in wave mechanics. $\mathbf{L}$ can be presented conveniently using the operators L_+, L_-, and L_z,

$$\begin{aligned} L_+ &= L_x + iL_y = e^{i\varphi}\Big(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\Big) \\ L_- &= L_x - iL_y = e^{-i\varphi}\Big(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\Big) \qquad (1) \\ L_z &= -i\frac{\partial}{\partial\varphi} \qquad (2) \end{aligned}$$

$\mathbf{L}$ only operates on the angular variables and $\mathbf{r}\cdot\mathbf{L} = 0$. For what follows, it is useful to notice that, based on the recursive relations of the spherical harmonics,

$$\begin{aligned} L_+ Y_{lm} &= \sqrt{(l-m)(l+m+1)}Y_{l,m+1} \\ L_- Y_{lm} &= \sqrt{(l+m)(l-m+1)}Y_{l,m-1} \qquad (3) \\ L_z Y_{lm} &= mY_{lm} \qquad (4) \end{aligned}$$

In addition, $\mathbf{L}, L^2$ and ∇^2 fulfil the following commutation rules:

$$\begin{aligned} L^2\mathbf{L} &= \mathbf{L}L^2 \\ \mathbf{L}\times\mathbf{L} &= i\mathbf{L} \qquad (5) \\ L_j\nabla^2 &= \nabla^2 L_j \qquad (6) \end{aligned}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r) - \frac{L^2}{r^2}$$

2 Multipole expansions of electromagnetic fields

In free space, Maxwell's equations take the form (time dependence $e^{-i\omega t}$)

$$\nabla \times \mathbf{E} = ik\zeta_0 \mathbf{H}, \qquad \nabla \times \mathbf{H} = -ik\mathbf{E}/\zeta_0 \tag{7}$$

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0 \tag{8}$$

where $k = \omega/c$. If the $\mathbf{E}$-field is eliminated, one obtains

$$(\nabla^2 + k^2)\mathbf{H} = 0, \qquad \nabla \cdot \mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{k\zeta_0}\nabla \times \mathbf{E}$$

Alternatively, eliminating the $\mathbf{H}$-field yields

$$(\nabla^2 + k^2)\mathbf{E} = 0, \qquad \nabla \cdot \mathbf{E} = 0$$

$$\mathbf{E} = \frac{i\zeta_0}{k}\nabla \times \mathbf{H}.$$

Both groups of three equations are equivalent to the original Maxwell's equations. We attempt to find multipole solutions for the vector fields $\mathbf{E}$ and $\mathbf{H}$. It is clear that each Cartesian component of $\mathbf{E}$ and $\mathbf{H}$ fulfil the scalar wave equation so that each component could be developed into series in multipoles of the scalar wave equation. However, the conditions about the sourceless nature of both $\mathbf{E}$ and $\mathbf{H}$ would be difficult to account for and it would be difficult to construct pure multipoles for the vector wave equation.

Instead, we start from the scalar quantity $\mathbf{r} \cdot \mathbf{A}$, where $\mathbf{A}$ is a regularly behaving vector field. First,

$$\nabla^2(\mathbf{r} \cdot \mathbf{A}) = \mathbf{r} \cdot (\nabla^2 \mathbf{A}) + 2\nabla \cdot \mathbf{A}$$

so that

$$\nabla^2(\mathbf{r} \cdot \mathbf{E}) = \mathbf{r} \cdot (-k^2 \mathbf{E}) \Leftrightarrow (\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{E}) = 0$$

and, in a corresponding way,

$$(\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{H}) = 0$$

Therefor, the general solution for $\mathbf{r} \cdot \mathbf{E}$ and $\mathbf{r} \cdot \mathbf{H}$:n can be presented as series of basis functions of the scalar wave equation.

We define the magnetic multipole of order (l, m) by the conditions

$$\mathbf{r} \cdot \mathbf{H}_{lm}^{(M)} = \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta, \varphi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^{(M)} = 0 \tag{9}$$

where $g_l(kr) = A_l^{(1)} h_l^{(1)}(kr) + A_l^{(2)} h_l^{(2)}(kr)$ (the coefficient $l(l+1)/k$ has been introduced for convenience).

Now

$$\zeta_0 k\mathbf{r}\cdot\mathbf{H} = \frac{1}{i}\mathbf{r}\cdot(\nabla\times\mathbf{E}) = \frac{1}{i}(\mathbf{r}\times\nabla)\cdot\mathbf{E} = \mathbf{L}\cdot\mathbf{E}$$

where $\mathbf{L}$ is the operator showin g up when solving the scalar wave equation. When $\mathbf{r}\cdot\mathbf{H} = \mathbf{r}\cdot H_{lm}^{(M)}$, it must be true that

$$\mathbf{L}\cdot\mathbf{E}_{lm}^{(M)}(r,\theta,\varphi) = l(l+1)\zeta_0 g_l(kr)Y_{lm}(\theta,\varphi)$$

and

$$\mathbf{r}\cdot\mathbf{E}_{lm}^{(M)} = 0$$

Since $\mathbf{L}$ only operates on the angular variables (θ,φ), the r-dependence of $\mathbf{E}_{lm}^{(M)}$ is $g_l(kr)$. In order for $\mathbf{L}\cdot\mathbf{E}_{lm}^{(M)}$ to produce a pure $Y_{lm}(\theta,\varphi)$ angular dependence, $\mathbf{E}_{lm}^{(M)}$ need to be prepared using the L_z, L_+, and L_--operators so that, ultimately,

$$\begin{aligned}\mathbf{E}_{lm}^{(M)} &= \zeta_0 g_l(kr)\mathbf{L}Y_{lm}(\theta,\varphi)\\ \mathbf{H}_{lm}^{(M)} &= -\frac{1}{k\zeta_0}\nabla\times\mathbf{E}_{lm}^{(M)}\end{aligned} \tag{10}$$

This is the definition for the electromagnetic fields of the magnetic multipole of order (l,m). Occasionally, this is also called the transverse electric multipole (TE).

The electromagnetic fields of an electric or transverse magnetic (TM) multipole of order (l,m) follow from the conditions

$$\begin{aligned}\mathbf{r}\cdot\mathbf{E}_{lm}^{(E)} &= -\zeta_0\frac{l(l+1)}{k}f_l(kr)Y_{lm}(\theta,\varphi)\\ \mathbf{r}\cdot\mathbf{H}_{lm}^{(E)} &= 0\end{aligned}$$

and are of the form

$$\begin{aligned}\mathbf{H}_{lm}^{(E)} &= f_l(kr)\mathbf{L}Y_{lm}(\theta,\varphi)\\ \mathbf{E}_{lm}^{(E)} &= \frac{i\zeta_0}{k}\nabla\times\mathbf{H}_{lm}^{(E)}\end{aligned} \tag{11}$$

where the r-dependent part $f_l(kr)$ is again a combination of the spherical Hankel or Bessel and Neumann functions.

It can be shown that the electric and magnetic multipole fields constitute a complete vectorial set of solutions for Maxwell's equations in source-free space. In what follows, the terminology of electric and magnetic multipoles is being used as, physically, the sources are the electric charge density and the magnetic moment density, respectively.

In the consideration of vector spherical harmonics, the vector spherical harmonics functions $\mathbf{L}Y_{lm}$ assume a central role. For convenience, the vector functions are normalized so that the final vector spherical harmonics are

$$\mathbf{X}_{lm}(\theta,\varphi) \equiv \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\varphi)$$

We define $\mathbf{X}_{00} \equiv 0$, since spherically symmetric solutions to Maxwell's equations only exist in source-free space at the static limit $k \to 0$. For $\mathbf{X}_{lm}$, the following orthogonality relations can be ascertained,

$$\int_{(4\pi)} d\Omega \mathbf{X}^*_{l',m'} \cdot \mathbf{X}_{lm} = \delta_{ll'}\delta_{mm'}$$

$$\int_{(4\pi)} d\Omega \mathbf{X}^*_{l',m'} \cdot (\mathbf{r} \times \mathbf{X}_{lm}) = 0$$

The proof is left for an exercise.

The general solution for Maxwell's equations can now be written as an expansion of electric and magnetic multipoles,

$$\mathbf{H} = \sum_{l,m} \left[a_E(l,m) f_l(kr) \mathbf{X}_{lm} - \frac{i}{k} a_M(l,m) \nabla \times g_l(kr) \mathbf{X}_{lm} \right]$$

$$\mathbf{E} = \zeta_0 \sum_{l,m} \left[\frac{i}{k} a_E(l,m) \nabla \times f_l(kr) \mathbf{X}_{lm} + a_M(l,m) g_l(kr) \mathbf{X}_{lm} \right]$$

where the coefficients $a_E(l,m)$ and $a_M(l,m)$ give the amount of electric and magnetic multipoles of order (l,m). The functions $f_l(kr)$ and $g_l(kr)$ are linear combinations of $h_l^{(1,2)}$ or j_l and n_l. The coefficients $a_E(l,m)$ and $a_M(l,m)$ are determined by the sources and the boundary conditions. Explicitly, this is seen by the scalar quantitites $\mathbf{r} \cdot \mathbf{H}$ and $\mathbf{r} \cdot \mathbf{E}$ being sufficient to determine the unknown coefficients:

$$a_M(l,m) g_l(kr) = \frac{k}{\sqrt{l(l+1)}} \int_{(4\pi)} d\Omega Y^*_{lm} \mathbf{r} \cdot \mathbf{H}$$

$$\zeta_0 a_E(l,m) f_l(kr) = -\frac{k}{\sqrt{l(l+1)}} \int_{(4\pi)} d\Omega Y^*_{lm} \mathbf{r} \cdot \mathbf{E}$$

When $\mathbf{r} \cdot \mathbf{H}$ and $\mathbf{r} \cdot \mathbf{E}$ are known at two distances differing from one another in the source-free region, the fields can be unambiguously determined, all the way to the mutual proportions of the two parts in the radial dependences f_l and g_l.

3 Energy in multipole fields

Consider multipole fields in the near zone $kr << 1$. Then, the leading contribution derives from the Neumann function so that $f_l \propto n_l$; assume that the coefficient of the multipole in question differs from zero. We obtain

$$\mathbf{H}^{(E)}_{lm} \to -\frac{k}{l} \mathbf{L} \frac{Y_{lm}}{r^{l+1}}$$

where the factor $-k/l$ is introduced for convenience. In order to calculate the electric field, we must compute the curl of the right-hand side of the equation; in doing this, we make use of the result

$$i\nabla \times \mathbf{L} = \mathbf{r}\nabla^2 - \nabla(1 + r\frac{\partial}{\partial r})$$

The electric field is

$$\mathbf{E}_{lm}^{(E)} \rightarrow -\frac{i}{l}\zeta_0 \nabla \times \mathbf{L}\Big(\frac{Y_{lm}}{r^{l+1}}\Big)$$

and, since Y_{lm}/r^{l+1} obeys the Laplace equation,

$$\nabla^2 \frac{Y_{lm}}{r^{l+1}} = 0$$

and, for the electric field, we obtain

$$\mathbf{E}_{lm}^{(E)} \rightarrow -\zeta_0 \nabla \frac{Y_{lm}}{r^{l+1}}$$

which is the multipole field of electrostatics. The magnetic field $\mathbf{H}_{lm}^{(E)}$ is smaller than $\mathbf{E}_{lm}^{(E)}/\zeta_0$ by a factor of kr so that, in the near zone, the magnetic field of the electric multipole is considerably smaller than the electric field (cf. earlier treatment for an electric dipole moment).

By exchanging $\mathbf{E}$ and $\mathbf{H}$ in the previous analysis, we can obtain the case of the magnetic multipole,

$$\mathbf{E}^{(E)} \rightarrow -\zeta_0 \mathbf{H}^{(M)}, \qquad \mathbf{H}^{(E)} \rightarrow \mathbf{E}^{(M)}/\zeta_0$$

Let us study the multipole fields in the far zone $kr >> 1$. The fields depend on the boundary conditions set and, as an example, we study outgoing waves that are applicable to the case of radiation by a localized source, too. Now $f_l(kr) \propto h_l^{(1)}(kr)$ and

$$\mathbf{H}_{lm}^{(E)} \rightarrow (-i)^{l+1} \frac{e^{ikr}}{kr} \mathbf{L} Y_{lm}$$

and the electric field is of the form

$$\mathbf{E}_{lm}^{(E)} = \zeta_0 \frac{(-i)^l}{k^2} \Big[\nabla\Big(\frac{e^{ikr}}{r}\Big) \times \mathbf{L}Y_{lm} + \frac{e^{ikr}}{r} \nabla \times \mathbf{L}Y_{lm}\Big]$$

The asymptotic form of $h_l^{(1)}$ is already used in the expression of the electric field so only factors proportional to r^{-1} can be conserved in the expressions. By using, again, the aforedescribed result for $\nabla \times \mathbf{L}$, we obtain

$$\mathbf{E}_{lm}^{(E)} = -\zeta_0 (-i)^{l+1} \frac{e^{ikr}}{kr} \Big[\mathbf{n} \times \mathbf{L}Y_{lm} - \frac{1}{k}(\mathbf{r}\nabla^2 - \nabla)Y_{lm}\Big]$$

where $\mathbf{n} = \mathbf{r}/r$. The second term on the right is of the order of $1/kr$ and can be omitted from the expression in parentheses in the limit $kr >> 1$. We obtain

$$\mathbf{E}_{lm}^{(E)} = \zeta_0 \mathbf{H}_{lm}^{(E)} \times \mathbf{n}$$

where $\mathbf{H}_{lm}^{(E)}$ is asymptotic form given above.

The multipole fields can be utilized in the computation of the energy transported by the radiation. As an example, consider the linear superposition of electric multipoles of order (l, m) with different values of m, when l is kept constant. The fields are of the form

$$\mathbf{H}_l = \sum_m a_E(l,m) \mathbf{X}_{lm} h_l^{(1)}(kr) e^{-i\omega t}$$

$$\mathbf{E}_l = \frac{i}{k} \zeta_0 \nabla \times \mathbf{H}_l$$

The time-averaged energy density of time-harmonic fields is

$$u = \frac{\epsilon_0}{4} (\mathbf{E} \cdot \mathbf{E}^* + \zeta_0^2 \mathbf{H} \cdot \mathbf{H}^*)$$

In the far zone, the two terms of the energy density are equal and, in a spherical shell $r, r+dr$, there is the following amount of energy:

$$dU = \frac{\mu_0 dr}{2k^2} \sum_{m,m'} a_E^*(l,m') a_E(l,m) \int_{(4\pi)} d\Omega \mathbf{X}_{lm'}^* \cdot \mathbf{X}_{lm}$$

and, due to the orthogonality,

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2} \sum_m |a_E(l,m)|^2$$

which is independent of r. In the general case of electric and magnetic multipoles, the summation goes over both l and m and $|a_E|^2 \to |a_E|^2 + |a_M|^2$. In the spherical shell in the radiation zone, the total energy is thus the incoherent sum over all multipoles.

4 Angular dependence of multipole radiation

For an arbitrary localized source distribution, the fields in the radiation zone are obtained as a superposition

$$\mathbf{H} \to \frac{e^{ikr - i\omega t}}{kr} \sum_{lm} (-i)^{l+1} \Big[a_E(l,m) \mathbf{X}_{lm} + a_M(l,m) \mathbf{n} \times \mathbf{X}_{lm} \Big]$$

$$\mathbf{E} \to \zeta_0 \mathbf{H} \times \mathbf{n}, \qquad \mathbf{n} = \frac{\mathbf{r}}{r}$$

where the coefficients $a_E(l,m)$ and $a_M(l,m)$ are connected to the properties of the source. The time-averaged power as per solid angle is

$$\frac{dP}{d\Omega} = \frac{\zeta_0}{2k^2} |\sum_{l,m} (-i)^{l+1} \left[a_E(l,m)\mathbf{X}_{lm} \times \mathbf{n} + a_M(l,m)\mathbf{X}_{lm} \right] |^2$$

The dimension of the expression inside the || marks is the dimension of the magnetic field. The directions of the vectors determine the polarization of the radiation. The angular dependence of the electric and magnetic multipoles of order (l,m) coincide but the polarizations are perpendicular to one another. It then follows that the order of the multipoles can be determined from the angualr dependence but the electric or magnetic nature can be determined only after a polarization measurement.

The angular dependence of a pure multipole of order (l,m) is

$$\frac{dP(l,m)}{d\Omega} = \frac{\zeta_0}{2k^2} |a(l,m)|^2 |\mathbf{X}_{lm}|^2$$

Based on the definition of $\mathbf{X}_{lm}$ and the rules of calculus for L_+ and L_-,

$$\frac{dP(l,m)}{d\Omega} = \frac{\zeta_0 |a(l,m)|^2}{2k^2 l(l+1)} \left[\frac{1}{2}(l-m)(l+m+1)|Y_{l,m+1}|^2 + \frac{1}{2}(l+m)(l-m+1)|Y_{l,m-1}|^2 + m^2 |Y_{lm}|^2 \right]$$

Examples of angular dependences $|\mathbf{X}_{lm}(\theta,\varphi)|^2$ follow:
Dipole: (dipole vibrating in the direction of the z-axis)

$$l = 1, m = 0 \qquad \frac{3}{8\pi} \sin^2\theta$$

(dipoles vibrating along the x- and y-axes with a phase difference $\frac{\pi}{2}$)

$$l = 1, m = \pm 1 \qquad \frac{3}{16\pi}(1 + \cos^2\theta)$$

Quadrupole:

$$l = 2, m = 0 \qquad \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$l = 2, m = \pm 1 \qquad \frac{5}{16\pi}(1 - 3\cos^2\theta + 4\cos^4\theta)$$

$$l = 2, m = \pm 2 \qquad \frac{5}{16\pi}(1 - \cos^4\theta)$$

With the help of the addition rule for spherical harmonics, one can show that

$$\sum_{m=-l}^{l} |\mathbf{X}_{lm}(\theta,\varphi)|^2 = \frac{2l+1}{4\pi}$$

so that the vector spherical harmonics have their own addition rule. This implies that the angular dependence of radiation is isotropic when the source is composed of incoherently radiating multipoles of order l with coefficients $a(l, m)$ independent of m.

The total power radiated by a pure multipole can be obtained via integration and, due to the orthonormality,

$$P(l, m) = \frac{\zeta_0}{2k^2} |a(l, m)|^2$$

For a general source, the angular dependence follows from the coherent that has been shown above. When computing the total power, due to the orthgonality, the interference terms do not contribute, and the total power is the incoherent sum of the contributions from the different multipoles:

$$P = \frac{\zeta_0}{2k^2} \sum_{l,m} \left[|a_E(l, m)|^2 + |a_M(l, m)|^2 \right]$$