1 Electromagnetic field by a localized source (lecture 4)

Consider the electromagnetic fields caused by time-dependent charge and current densities localized in a constrained region of space. Here we will mainly study the fields by an electric dipole. Later, the analysis is extended to the full multipole expansion.
Assume harmonic time dependence $e^{-i\omega t}$—arbitrary time dependences can be dealt with using Fourier analysis of their components. The charge density ρ and current density $\mathbf{j}$ are

$$\rho(\mathbf{x}, t) = \rho(\mathbf{x})e^{-i\omega t}$$
$$\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x})e^{-i\omega t}$$

and the physical quantities correspond to the real parts of the comelx quantities. The electromagnetic potentials and fields are also time-harmonic and the sources are assumed to be located in an otherwise empty space.

Let us start from the vector potential $\mathbf{A}$ in Lorentz gauge,

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}' \int dt' \frac{\mathbf{j}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}\delta(t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t) \quad (1)$$

and, by writing $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$, we obtain

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}, \qquad k = \frac{\omega}{c} \quad (2)$$

The magnetic field is, according to definitions, $\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}$ and, outside the source region, the electric field equals $\mathbf{E} = \frac{i\zeta_0}{k}\nabla \times \mathbf{H}$, where $\zeta_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. When the current density $\mathbf{j}(\mathbf{x}')$ is given, the electromagnetic field can be calculated from the integral above, at least in principle. Let us study the case where the source region (size d) is much smaller than the wavelength: $d \ll \lambda = 2\pi c/\omega$. We can distinguish three regimes of interest:

(i) Near zone (static regime): $d \ll r \ll \lambda$
(ii) Intermediate zone (induction regime): $d \ll r \sim \lambda$
(iii) Far zone (radiation regime): $d \ll \lambda \ll r$

In the near zone (i) $kr \ll 1$ and the exponential part of the integrand for the vector potential can be set to unity, and the inverse distance can be presented using series of spherical harmonics Y_{lm}:

$$\lim_{kr\to 0}\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\frac{\mathrm{Y}_{lm}(\theta, \varphi)}{r^{l+1}}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(r')^l\mathrm{Y}^*_{lm}(\theta', \varphi') \quad (3)$$

We can see that the near fields vary harmonically in time but are static in their character: no wave solution follows for the spatial dependence. Above, we have made use of the relation

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \mathrm{Y}_{lm}^*(\theta', \varphi')\mathrm{Y}_{lm}(\theta, \varphi) \tag{4}$$

In the far zone (iii), $kr \gg 1$ and the exponential part of the vector potential varies strongly and dictates the character of the vector potential. We can approximate

$$|\mathbf{x}-\mathbf{x}'| \approx r - \hat{\mathbf{n}} \cdot \mathbf{x}', \qquad \hat{\mathbf{n}} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{r} \tag{5}$$

When the leading term is desired in kr, the inverse distance can be replaced by r. The vector potential is of the form

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'} \tag{6}$$

Therefor, the vector potential behaves like an outgoing spherical wave (e^{ikr}/r) with angular dependence. It can be shown that the electromagnetic field is also of the form of a spherical wave and thus is a radiation field. (Note that this part of the analysis is vlid for localized source regions of arbitrary size.)

Now that $kd \ll 1$ the integral can further be developed into series:

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_n \frac{(-ik)^n}{n!} \int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(\hat{\mathbf{n}}\cdot\mathbf{x}')^n \tag{7}$$

where the magnitude for the nth term is $(1/n!) \int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(k\hat{\mathbf{n}}\cdot\mathbf{x}')^n$ and thus becomes rapidly smaller with increasing n. In this case, the main contribution to radiation comes from the first non-vanishing term in the sum.

In the intermediate zone (ii), all powers of kr need to be accounted for, and no simple limits can be taken. The vector potential is then written with the help of the expansion for the exact Green's function in the form

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr)\mathrm{Y}_{lm}(\theta, \varphi) \int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')j_l(kr')\mathrm{Y}_{lm}^*(\theta', \varphi') \tag{8}$$

where we have made use of the expansion

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik \sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>) \sum_{m=-l}^{l} \mathrm{Y}_{lm}^*(\theta', \varphi')\mathrm{Y}_{lm}(\theta, \varphi) \tag{9}$$

where $r_< = \min(r, r')$, $r_> = \max(r, r')$, and j_l and $h_l^{(1)}$ are the spherical Bessel and Hankel functions.

Again when $kd \ll 1$, the j_l-functions can be approximated and the result is of the same form as the near zone result, when the following replacement is carried out:

$$\frac{1}{r^{l+1}} \to \frac{e^{ikr}}{r^{l+1}}[1 + a_1(ikr) + a_2(ikr)^2 + \ldots + a_l(ikr)^l] \tag{10}$$

The coefficients a_i derive from the explicit expansions of the Hankel functions. This end result allows us to see the transition from the near-zone $kr \ll 1$ static field to the far-zone $kr \gg 1$ radiation field.

2 Electromagnetic field of an electric dipole

If only the first term in kd is kept in the expansion of the vector potential, one obtains

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}') \tag{11}$$

which holds everywhere outside the source region (this follows from the intermediate-zone results above). With the help of partial integration,

$$\int d^3\mathbf{x}'\mathbf{j} = -\int d^3\mathbf{x}'\mathbf{x}'(\nabla\cdot\mathbf{j}) = -i\omega\int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}') \tag{12}$$

where the substitution term disappears (the source region is constrained) and, according to the continuity equation, $i\omega\rho(\mathbf{x}') = \nabla\cdot\mathbf{j}(\mathbf{x}')$. The vector potential is thus

$$\mathbf{A}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}, \tag{13}$$

where $\mathbf{p}$ is the electric dipole moment $\mathbf{p} = \int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}')$.

The electromagnetic fields are

$$\begin{aligned}
\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}}\times\mathbf{p})\frac{e^{ikr}}{r}(1 - \frac{1}{ikr}) \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\Big(k^2(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r} + (3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}) - \mathbf{p})(\frac{1}{r^2} - \frac{ik}{r})\frac{e^{ikr}}{r}\Big)
\end{aligned}$$

We note that the magnetic field is always transverse but that the electric field has both longitudinal and transverse components.

In the far zone,

$$\begin{aligned}\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r} \\ \mathbf{E} &= \zeta_0 \mathbf{H} \times \hat{\mathbf{n}}\end{aligned}$$

which shows the typical form of a spherical wave.

In the near zone,

$$\begin{aligned}\mathbf{H} &= \frac{i\omega}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{1}{r^2} \\ \mathbf{E} &= \frac{1}{4\pi\epsilon_0}(3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p})\frac{1}{r^3}\end{aligned}$$

The electric field is, except for the harmonic time dependence, that of a static lectric dipole. The field $\zeta_0 \mathbf{H}$ is smaller, by a factor of kr, than the field $\mathbf{E}$ so, in the near zone, the field is electric in its nature. In the static limit $k \to 0$, the magnetic field disappears and the near zone extends to infinity.

The power radiated by the vibrating dipole moment $\mathbf{p}$ as per solid angle is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2} Re(r^2 \hat{\mathbf{n}} \cdot \mathbf{E} \times \mathbf{H}^*) \\ &= \frac{c^2\zeta_0}{32\pi^2} k^4 |(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}|^2,\end{aligned}$$

where $\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}$ gives the polarization state. If all components of $\mathbf{p}$ are in the same phase,

$$\frac{dP}{d\Omega} = \frac{c^2\zeta_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta \tag{14}$$

which is the typical radiation pattern of an electric dipole (θ is here measured from the direction of $\mathbf{p}$). Independently of the phases of the components for $\mathbf{p}$:n, the total radiated power is

$$P = \frac{c^2\zeta_0 k^4}{12\pi} |\mathbf{p}|^2 \tag{15}$$

3 Scattering by small spherical particles in the electric dipole approximation

Light scattering by particles clearly smaller than the wavelength can be studied in the approximation, where the incident field induces an electric dipole moment to the particle. The dipole fluctuates in a certain phase with the incident field and thus scatters radiation in directions differing from the propagation direction of the incident field. In this case, the dipole moments can be computed using electrostatic methods.

Assume that a monochromatic plane wave is incident on a small scatterer located in free space. Let the propagation direction and polarization vector of the incident field be $\hat{\mathbf{n}}_0$ and $\hat{\epsilon}_0$:

$$\begin{aligned}\mathbf{E}_i &= \hat{\epsilon}_0 E_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}} \\ \mathbf{H}_i &= \hat{\mathbf{n}}_0 \times \mathbf{E}_i/\zeta_0\end{aligned}$$

where $k = \omega/c$ and the time dependence has been assumed harmonic ($e^{-i\omega t}$). These fields induce a dipole momentn $\mathbf{p}$ in the small particle and the particle radiates energy in (almost) all directions. In the far zone, the scattered fields are of the form

$$\begin{aligned}\mathbf{E}_s &= \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}((\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}) \\ \mathbf{H}_s &= \hat{\mathbf{n}}\times\mathbf{E}_s/\zeta_0\end{aligned}$$

where $\hat{\mathbf{n}}$ is the dirction of the observer and r the distance from the scatterer. The power scattered in direction $\hat{\mathbf{n}}$ with polarization $\hat{\epsilon}$ per unit solid angle divided by the incident flux density is the so-called differential cross section

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}},\hat{\epsilon},\hat{\mathbf{n}}_0,\hat{\epsilon}_0) = \frac{r^2\frac{1}{2\zeta_0}|\hat{\epsilon}^*\cdot\mathbf{E}_s|^2}{\frac{1}{2\zeta_0}|\hat{\epsilon}_0^*\cdot\mathbf{E}_i|^2} \tag{16}$$

where the complex conjugation of the polarization vectors is important for proper treatment of circular polarization. Furthermore,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}},\hat{\epsilon},\hat{\mathbf{n}}_0,\hat{\epsilon}_0) = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\hat{\epsilon}^*\cdot\mathbf{p}|^2, \tag{17}$$

where the $\hat{\mathbf{n}}_0, \hat{\epsilon}_0$ -dependence is implicit in $\mathbf{p}$. We can see that the differential and total cross sections of the dipole scatterer are both proportional to k^4:een and λ^{-4}:een (Rayleigh's law). Assume that the scatterer is a small sphere (radius a) with the relative permittivity $\epsilon_r = \epsilon/\epsilon_0$. According to electrostatics, the dipole moment of the sphere is

$$\mathbf{p} = 4\pi\epsilon_0\Big(\frac{\epsilon_r - 1}{\epsilon_r + 2}\Big)a^3\mathbf{E}_i \tag{18}$$

so that

$$\frac{d\sigma}{d\Omega} = k^4a^6|\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2|\hat{\epsilon}^*\cdot\hat{\epsilon}_0|^2 \tag{19}$$

The polarization dependence is purely that of electric dipole scattering. The scattered radiation is polarized in the plane defined by the dipole moment $\hat{\epsilon}_0$ and the vector $\hat{\mathbf{n}}$.

For unpolarized incident radiation, the differential cross sections in different polarization states of the scattered field are

$$
\begin{aligned}
\frac{d\sigma_{\parallel}}{d\Omega} &= \frac{k^4 a^6}{2}\left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 \cos^2\theta \\
\frac{d\sigma_{\perp}}{d\Omega} &= \frac{k^4 a^6}{2}\left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2
\end{aligned}
$$

where θ is now the scattering angle.

The degree of polarization is

$$
P(\theta) = \frac{\frac{d\sigma_{\perp}}{d\Omega} - \frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta} = -\frac{S_{21}(\theta)}{S_{11}(\theta)} \tag{20}
$$

and the differential cross section summed over the polarization states of the scattered field is

$$
\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 \frac{1}{2}(1+\cos^2\theta) \propto S_{11}(\theta) \tag{21}
$$

where $S_{11}(\theta)$ and $S_{21}(\theta)$ are elements of the scattering matrix. The total scattering cross section is

$$
\sigma = \int_{(4\pi)} \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 \tag{22}
$$

The scattered radiation is 100% positively polarized at the scattering angle $\theta = 90°$. It was the polarization characteristics of the blue sky that got Rayleigh interested in scattering by small particles.