# 1 Introduction to scattering theory (lecture 0)

Each scattering problem depends in the detailed characteristics of the scattering particle: its size, shape, and refractive index. The size is usually described by the size parameter

$$x = \frac{2\pi a}{\lambda}, \tag{1}$$

where $a$ is a typical radial distance in the particle and and $\lambda$ is the wavelength of the original electromagnetic field. In the size dependence of scattering, only the ratio $a/\lambda$ is meaningful. Shape is described by suitable elongation, roughness, or angularity parameters. The constitutive material is characterized by the complex-valued refractive index

$$m = n + \mathrm{i}n', \tag{2}$$

where the real and imaginary parts $n$ and $n'$ are responsible for refraction and absorption of light, respectively. The time dependence of the fields has been chosen to be $\exp(-\mathrm{i}\omega t)$ so that, in physically relevant cases, the imaginary part of the refractive index needs to be non-negative.

## 1.1 Electromagnetic formulation of the problem

Electromagnetic scattering and absorption is here being assessed from the view point of classical electromagnetics. The natural foundation is provided by Maxwell's equations

$$\begin{aligned}
\nabla \cdot \boldsymbol{D} &= \rho, \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\
\nabla \cdot \boldsymbol{B} &= 0, \\
\nabla \times \boldsymbol{H} &= \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t},
\end{aligned} \tag{3}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{D}$ is the electric displacement, and $\boldsymbol{H}$ is the magnetic field. $\rho$ and $\boldsymbol{j}$ are, respectively, the densities of free charges and currents. In order for the charge and current densities to determine the fields unambiguously, constitutive relations describing the interaction of matter and fields are introduced,

$$\begin{aligned}
\boldsymbol{j} &= \sigma \boldsymbol{E}, \\
\boldsymbol{D} &= \epsilon \boldsymbol{E}, \\
\boldsymbol{B} &= \mu \boldsymbol{H},
\end{aligned} \tag{4}$$

where $\sigma$ is the electric conductivity, $\epsilon$ is the electric permittivity, and $\mu$ is the magnetic permeability. In what follows, it is assumed that there are no free charges or currents and that the time dependence of the fields is of the harmonic type $\exp(-\mathrm{i}\omega t)$. Maxwell's equations then reduce to the form

$$\begin{aligned} \nabla \cdot \epsilon \boldsymbol{E} &= 0, \\ \nabla \times \boldsymbol{E} &= \mathrm{i}\omega\mu \boldsymbol{B}, \\ \nabla \cdot \boldsymbol{H} &= 0, \\ \nabla \times \boldsymbol{H} &= -\mathrm{i}\omega\epsilon \boldsymbol{E}, \end{aligned} \tag{5}$$

so that the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ fulfil the vector wave equations vektoriaaltoyhtälöt

$$\begin{aligned} \nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} &= 0, \\ \nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} &= 0, \end{aligned} \tag{6}$$

where $k^2 = \omega^2 m^2 / c^2$ and $m$ is the relative refractive index of the scatterer, $m^2 = \epsilon\mu/\epsilon_0\mu_0$.

Denote the internal field of the particle by $(\boldsymbol{E}_1, \boldsymbol{H}_1)$. The external field $(\boldsymbol{E}_2, \boldsymbol{H}_2)$ is the superposition of the original field $(\boldsymbol{E}_i, \boldsymbol{H}_i)$ and the scattered field $(\boldsymbol{E}_s, \boldsymbol{H}_s)$,

$$\begin{aligned} \boldsymbol{E}_2 &= \boldsymbol{E}_i + \boldsymbol{E}_s, \\ \boldsymbol{H}_2 &= \boldsymbol{H}_i + \boldsymbol{H}_s. \end{aligned} \tag{7}$$

In what follows, let us assume that the original field is a plane wave,

$$\begin{aligned} \boldsymbol{E}_i &= \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)], \\ \boldsymbol{H}_i &= \boldsymbol{H}_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)], \boldsymbol{H}_0 = \frac{1}{\omega\mu_0} \boldsymbol{k} \times \boldsymbol{E}_0, \end{aligned} \tag{8}$$

where $\boldsymbol{k}$ is the wave vector of the medium surrounding the particle. Since there are no free currents according to our hypothesis, the tangential components of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous across the boundary between the particle and the surrounding medium:

$$\begin{aligned} (\boldsymbol{E}_2 - \boldsymbol{E}_1) \times \boldsymbol{n} &= 0, \\ (\boldsymbol{H}_2 - \boldsymbol{H}_1) \times \boldsymbol{n} &= 0, \end{aligned} \tag{9}$$

at the boundary with an outward normal vector $\boldsymbol{n}$. It is our fundamental goal to solve Maxwell's equations everywhere in space with the boundary conditions given.

## 1.2 Amplitude scattering matrix

Let us place an arbitrary particle in a plane wave field according to the figure (cf. Bohren & Huffman). The propagation directions of the original and scattered fields $\boldsymbol{e}_z$ and $\boldsymbol{e}_r$ define a scattering plane, and the original field is divided into components perpendicular and parallel to that plane,

$$\boldsymbol{E}_i = (E_{0\perp}\boldsymbol{e}_{i\perp} + E_{0\|}\boldsymbol{e}_{i\|})\exp[\mathrm{i}(kz-\omega t)] = E_{i\perp}\boldsymbol{e}_{i\perp} + E_{i\|}\boldsymbol{e}_{i\|}. \tag{10}$$

In the radiation zone, that is, far away from the scattering particle, the scattered field is a transverse spherical wave (cf. Jackson),

$$\boldsymbol{E}_s = \frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\boldsymbol{A}, \boldsymbol{e}_r\cdot\boldsymbol{A} = 0, \tag{11}$$

so that

$$\boldsymbol{E}_s = E_{s\perp}\boldsymbol{e}_{s\perp} + E_{s\|}\boldsymbol{e}_{s\|}, \tag{12}$$

where

$$\begin{aligned} \boldsymbol{e}_{s\perp} &= -\boldsymbol{e}_\phi, \\ \boldsymbol{e}_{s\|} &= \boldsymbol{e}_\theta. \end{aligned} \tag{13}$$

Due to the linearity of the boundary conditions, the amplitude of the scattered field depends linearly on the amplitude of the original field. In a matrix form,

$$\begin{bmatrix} E_{s\perp} \\ E_{s\|} \end{bmatrix} = \frac{\exp[\mathrm{i}(kr-kz)]}{-\mathrm{i}kr}\begin{bmatrix} S_1 & S_4 \\ S_3 & S_2 \end{bmatrix}\begin{bmatrix} E_{i\perp} \\ E_{i\|} \end{bmatrix}, \tag{14}$$

where the complex-valued amplitude-scattering-matrix elements $S_j$ $(j = 1,2,3,4)$ generally depend on the scattering angle $\theta$ and the azimuthal angle $\phi$. Since only the relative phases are important, the amplitude scattering matrix has seven free parameters.

## 1.3 Stokes parameters and scattering matrix

In the medium surrounding the particle, the time-averaged Poynting vector $\boldsymbol{S}$ can be divided into the Poynting vectors of the original field, scattered field, and that showing the interaction of the original and scattered fields,

$$\boldsymbol{S} = \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_2\times\boldsymbol{H}_2^*) = \boldsymbol{S}_i + \boldsymbol{S}_s + \boldsymbol{S}_e, \tag{15}$$

where

$$
\begin{aligned}
\boldsymbol{S}_i &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_i \times \boldsymbol{H}_i^*), \\
\boldsymbol{S}_s &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s^*), \\
\boldsymbol{S}_e &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_i \times \boldsymbol{H}_s^* + \boldsymbol{E}_s \times \boldsymbol{H}_i^*). \qquad (16)
\end{aligned}
$$

In the radiation zone, the power incident on a surface element $\Delta A$ perpendicular to the radial direction is

$$
\boldsymbol{S}_s \cdot \boldsymbol{e}_r = \frac{k}{2\omega\mu}\frac{|\boldsymbol{A}|^2}{k^2}\Delta\Omega, \Delta\Omega = \frac{\Delta A}{r^2} \qquad (17)
$$

and $|\boldsymbol{A}|^2$ can be measures as a function of angles. By placing polarizers in between the scattering particle and the detector, we can measure the Stokes parameters of the scattered field (Bohren & Huffman),

$$
\begin{aligned}
I_s &= \langle |E_{s\perp}|^2 + |E_{s\parallel}|^2 \rangle, \\
Q_s &= \langle -|E_{s\perp}|^2 + |E_{s\parallel}|^2 \rangle, \\
U_s &= 2\mathrm{Re}E_{s\perp}^* E_{s\parallel}, \\
V_s &= -2\mathrm{Im}E_{s\perp}^* E_{s\parallel}. \qquad (18)
\end{aligned}
$$

Thus, $I_s$ gives the scattered intensity, $Q_s$ gives the difference between the intensities in the scattering plane and perpendicular to the scfattering plane, $U_s$ gives the difference between $+\pi/4$ and $-\pi/4$ -polarized intensities and, lastly, $V_s$ gives the difference between right-handed and left-handed circularly polarized intensities. The factor $k/2\omega\mu_0$ has been omitted from the intensities; it is not needed since, in practice, relative intensities are measured instead of absolute ones. The Stokes parameters fully describe the polarization state of an electromagnetic field.

The scattering matrix $S$ interrelates the Stokes parameters of the original field and the scattered field, and can be derived from the amplitude scattering matrix:

$$
\boldsymbol{I}_s = \frac{1}{k^2 r^2} S \boldsymbol{I}_i, \qquad (19)
$$

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$$
\begin{aligned}
\boldsymbol{I}_s &= (I_s, Q_s, U_s, V_s)^T, \\
\boldsymbol{I}_i &= (I_i, Q_i, U_i, V_i)^T. \qquad (20)
\end{aligned}
$$

The information about the angular dependence of scattering is fully contained in the 16 elements of the scattering matrix. For a single scattering particle, it has seven free parameters whereas, for an ensemble of particles, all 16 elements can be free. Symmetries reduce the number of free parameters: for example, for a spherical particle, there are three free parameters.

For an unpolarized incident field, the Stokes parameters of the scattered field are

$$\begin{aligned} I_s &= \frac{1}{k^2r^2} S_{11} I_i, \\ Q_s &= \frac{1}{k^2r^2} S_{21} I_i, \\ U_s &= \frac{1}{k^2r^2} S_{31} I_i, \\ V_s &= \frac{1}{k^2r^2} S_{41} I_i. \end{aligned} \tag{21}$$

Thus, $S_{11}$ gives the angular distribution of scattered intensity and the total degree of polarization is

$$P_{\text{tot}} = \frac{\sqrt{S_{21}^2 + S_{31}^2 + S_{41}^2}}{S_{11}}. \tag{22}$$

Scattering polarizes light.

## 1.4 Extinction, scattering and absorption

Let us assume that medium surrounding the scattering particle is non-absorbing. The total or extinction cross section is then the sum of the absorption and scattering cross sections:

$$\sigma_e = \sigma_s + \sigma_a, \tag{23}$$

where

$$\begin{aligned} \sigma_e &= -\frac{1}{I_i} \int_A dA \boldsymbol{S}_e \cdot \boldsymbol{e}_r, \\ \sigma_s &= \frac{1}{I_i} \int_A dA \boldsymbol{S}_s \cdot \boldsymbol{e}_r, \end{aligned} \tag{24}$$

when $A$ is a spherical envelope of radius $r$ containing the scattering particle.

Let the original field be of $\boldsymbol{e}_x$-polarized form $\boldsymbol{E}_0 = E\boldsymbol{e}_x$. In the radiation zone,

$$\begin{aligned} \boldsymbol{E}_s &\propto \frac{\exp[\mathrm{i}k(r-z)]}{-\mathrm{i}kr} \boldsymbol{X} E, \boldsymbol{e}_r \cdot \boldsymbol{X} = 0, \\ \boldsymbol{H}_s &\propto \frac{k}{\omega\mu} \boldsymbol{e}_r \times \boldsymbol{E}_s, \end{aligned} \tag{25}$$

where the vector scattering amplitude $\boldsymbol{X}$ is related to the amplitude scattering matrix as follows:

$$\boldsymbol{X} = (S_4 \cos\phi + S_1 \sin\phi)\boldsymbol{e}_{s\perp} + (S_2 \cos\phi + S_3 \sin\phi)\boldsymbol{e}_{s\|}. \tag{26}$$

By making use of the asymptotic forms of the scattered field shown above and $\boldsymbol{e}_x$-polarized original field, the so-called optical theorem can be derived: extinction depends only on scattering in the exact forward direction,

$$\sigma_e = \frac{4\pi}{k^2}\mathrm{Re}[(\boldsymbol{X} \cdot \boldsymbol{e}_x)_{\theta=0}]. \tag{27}$$

In addition,

$$\sigma_s = \int_{4\pi} d\Omega \frac{d\sigma_s}{d\Omega}, \tag{28}$$

where the differential scattering cross section is

$$\frac{d\sigma_s}{d\Omega} = \frac{|\boldsymbol{X}|^2}{k^2}. \tag{29}$$

The extinction, scattering, and absorption efficiencies are defined as the ratios of the corresponding cross sections to the geometric cross section of the particle $A_\perp$ as projected in the propagation direction of the original field:

$$\begin{aligned} q_e &= \frac{\sigma_e}{A_\perp}, \\ q_s &= \frac{\sigma_s}{A_\perp}, \\ q_a &= \frac{\sigma_a}{A_\perp}. \end{aligned} \tag{30}$$

For an unpolarized original field, the cross sections are

$$\begin{aligned} \sigma_e &= \frac{1}{2}(\sigma_e^{(1)} + \sigma_e^{(2)}), \\ \sigma_s &= \frac{1}{2}(\sigma_s^{(1)} + \sigma_s^{(2)}), \end{aligned} \tag{31}$$

where the indices 1 and 2 refer to two polarization states of the original field perpendicular to one another.