ELECTROMAGNETIC SCATTERING I

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Contents

List of Lectures

The course is based on

- M. I. Mishchenko, L. D. Travis, & A. A. Lacis, *Scattering, Absorption, and Emission of Light by Small Particles*, Cambridge University Press, 2002
- A. Doicu, Y. Eremin, & T. Wriedt, *Acoustic & Electromagnetic Scattering Analysis Using Discrete Sources*, Academic Press, 2000
- M. I. Mishchenko, J. W. Hovenier, & L. D. Travis, *Light Scattering by Nonspherical Particles*, Academic Press, 2000
- C. F. Bohren & D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, Wiley & Sons, 1983
- J. D. Jackson, *Classical Electrodynamics*, Wiley & Sons, 1975
- H. C. van de Hulst, *Light Scattering by Small Particles*, Wiley & Sons, 1957 (Dover, 1981)

The intention is to cover the fundamentals in electromagnetic scattering by single small particles. For example, we include the following:

Jackson (2nd Ed.): 7.1–4, 9.1–3, 9.6–14, 16.1–5, 16.8–10
Bohren & Huffman: 1.1–5, 2.1–2, 2.5–7, 2.11, 3.1–3.4, 4.1–8, 5.1–2, 6.1–2, 8.6, 13.6, 13.9, 14.5

1 Introduction to scattering theory

(Lecture 0)

Each scattering problem depends in the detailed characteristics of the scattering particle: its size, shape, and refractive index. The size is usually described by the size parameter

$$x = \frac{2\pi a}{\lambda}, \tag{1}$$

where a is a typical radial distance in the particle and and λ is the wavelength of the original electromagnetic field. In the size dependence of scattering, only the ratio a/λ is meaningful. Shape is described by suitable elongation, roughness, or angularity parameters. The constitutive material is characterized by the complex-valued refractive index

$$m = n + \mathrm{i}n', \tag{2}$$

where the real and imaginary parts n and n' are responsible for refraction and absorption of light, respectively. The time dependence of the fields has been chosen to be $\exp(-\mathrm{i}\omega t)$ so that, in physically relevant cases, the imaginary part of the refractive index needs to be non-negative.

1.1 Electromagnetic formulation of the problem

Electromagnetic scattering and absorption is here being assessed from the view point of classical electromagnetics. The natural foundation is provided by Maxwell's equations

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \rho, \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \times \boldsymbol{H} &= \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}, \end{aligned} \tag{3}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{D}$ is the electric displacement, and $\boldsymbol{H}$ is the magnetic field. ρ and $\boldsymbol{j}$ are, respectively, the densities of free charges and currents. In order for the charge and current densities to determine the fields unambiguously, constitutive relations describing the interaction of matter and fields are introduced,

$$\begin{aligned} \boldsymbol{j} &= \sigma \boldsymbol{E}, \\ \boldsymbol{D} &= \epsilon \boldsymbol{E}, \\ \boldsymbol{B} &= \mu \boldsymbol{H}, \end{aligned} \tag{4}$$

where σ is the electric conductivity, ϵ is the electric permittivity, and μ is the magnetic permeability. In what follows, it is assumed that there are no free charges or currents and that the time dependence of the fields is of the harmonic type $\exp(-\mathrm{i}\omega t)$. Maxwell's equations then reduce to the form

$$\begin{aligned} \nabla \cdot \epsilon \boldsymbol{E} &= 0, \\ \nabla \times \boldsymbol{E} &= \mathrm{i}\omega\mu \boldsymbol{B}, \\ \nabla \cdot \boldsymbol{H} &= 0, \\ \nabla \times \boldsymbol{H} &= -\mathrm{i}\omega\epsilon \boldsymbol{E}, \end{aligned} \tag{5}$$

so that the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ fulfil the vector wave equations vektoriaaltoyhtälöt

$$\begin{aligned} \nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} &= 0, \\ \nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} &= 0, \end{aligned} \tag{6}$$

where $k^2 = \omega^2 m^2/c^2$ and m is the relative refractive index of the scatterer, $m^2 = \epsilon\mu/\epsilon_0\mu_0$.

Denote the internal field of the particle by $(\boldsymbol{E}_1, \boldsymbol{H}_1)$. The external field $(\boldsymbol{E}_2, \boldsymbol{H}_2)$ is the superposition of the original field $(\boldsymbol{E}_i, \boldsymbol{H}_i)$ and the scattered field $(\boldsymbol{E}_s, \boldsymbol{H}_s)$,

$$\begin{aligned} \boldsymbol{E}_2 &= \boldsymbol{E}_i + \boldsymbol{E}_s, \\ \boldsymbol{H}_2 &= \boldsymbol{H}_i + \boldsymbol{H}_s. \end{aligned} \tag{7}$$

In what follows, let us assume that the original field is a plane wave,

$$\begin{aligned} \boldsymbol{E}_i &= \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)], \\ \boldsymbol{H}_i &= \boldsymbol{H}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)], \boldsymbol{H}_0 = \frac{1}{\omega\mu_0}\boldsymbol{k}\times\boldsymbol{E}_0, \end{aligned} \tag{8}$$

where $\boldsymbol{k}$ is the wave vector of the medium surrounding the particle. Since there are no free currents according to our hypothesis, the tangential components of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous across the boundary between the particle and the surrounding medium:

$$\begin{aligned} (\boldsymbol{E}_2 - \boldsymbol{E}_1)\times\boldsymbol{n} &= 0, \\ (\boldsymbol{H}_2 - \boldsymbol{H}_1)\times\boldsymbol{n} &= 0, \end{aligned} \tag{9}$$

at the boundary with an outward normal vector $\boldsymbol{n}$. It is our fundamental goal to solve Maxwell's equations everywhere in space with the boundary conditions given.

1.2 Amplitude scattering matrix

Let us place an arbitrary particle in a plane wave field according to the figure (cf. Bohren & Huffman). The propagation directions of the original and scattered fields $\boldsymbol{e}_z$ and $\boldsymbol{e}_r$ define a scattering plane, and the original field is divided into components perpendicular and parallel to that plane,

$$\boldsymbol{E}_i = (E_{0\perp}\boldsymbol{e}_{i\perp} + E_{0\|}\boldsymbol{e}_{i\|})\exp[\mathrm{i}(kz - \omega t)] = E_{i\perp}\boldsymbol{e}_{i\perp} + E_{i\|}\boldsymbol{e}_{i\|}. \tag{10}$$

In the radiation zone, that is, far away from the scattering particle, the scattered field is a transverse spherical wave (cf. Jackson),

$$\boldsymbol{E}_s = \frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\boldsymbol{A}, \boldsymbol{e}_r\cdot\boldsymbol{A} = 0, \tag{11}$$

so that

$$\boldsymbol{E}_s = E_{s\perp}\boldsymbol{e}_{s\perp} + E_{s\|}\boldsymbol{e}_{s\|}, \tag{12}$$

where

$$\begin{aligned} \boldsymbol{e}_{s\perp} &= -\boldsymbol{e}_\phi, \\ \boldsymbol{e}_{s\|} &= \boldsymbol{e}_\theta. \end{aligned} \tag{13}$$

Due to the linearity of the boundary conditions, the amplitude of the scattered field depends linearly on the amplitude of the original field. In a matrix form,

$$\left[\begin{array}{c} E_{s\perp} \\ E_{s\|} \end{array}\right] = \frac{\exp[\mathrm{i}(kr-kz)]}{-\mathrm{i}kr}\left[\begin{array}{cc} S_1 & S_4 \\ S_3 & S_2 \end{array}\right]\left[\begin{array}{c} E_{i\perp} \\ E_{i\|} \end{array}\right], \quad (14)$$

where the complex-valued amplitude-scattering-matrix elements S_j ($j = 1, 2, 3, 4$) generally depend on the scattering angle θ and the azimuthal angle ϕ. Since only the relative phases are important, the amplitude scattering matrix has seven free parameters.

1.3 Stokes parameters and scattering matrix

In the medium surrounding the particle, the time-averaged Poynting vector $\boldsymbol{S}$ can be divided into the Poynting vectors of the original field, scattered field, and that showing the interaction of the original and scattered fields,

$$\boldsymbol{S} = \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_2 \times \boldsymbol{H}_2^*) = \boldsymbol{S}_i + \boldsymbol{S}_s + \boldsymbol{S}_e, \quad (15)$$

where

$$\begin{aligned} \boldsymbol{S}_i &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_i \times \boldsymbol{H}_i^*), \\ \boldsymbol{S}_s &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s^*), \\ \boldsymbol{S}_e &= \frac{1}{2}\mathrm{Re}(\boldsymbol{E}_i \times \boldsymbol{H}_s^* + \boldsymbol{E}_s \times \boldsymbol{H}_i^*). \end{aligned} \quad (16)$$

In the radiation zone, the power incident on a surface element ΔA perpendicular to the radial direction is

$$\boldsymbol{S}_s \cdot \boldsymbol{e}_r = \frac{k}{2\omega\mu}\frac{|\boldsymbol{A}|^2}{k^2}\Delta\Omega, \Delta\Omega = \frac{\Delta A}{r^2} \quad (17)$$

and $|\boldsymbol{A}|^2$ can be measures as a function of angles. By placing polarizers in between the scattering particle and the detector, we can measure the Stokes parameters of the scattered field (Bohren & Huffman),

$$\begin{aligned} I_s &= \langle |E_{s\perp}|^2 + |E_{s\|}|^2 \rangle, \\ Q_s &= \langle -|E_{s\perp}|^2 + |E_{s\|}|^2 \rangle, \\ U_s &= 2\mathrm{Re}E_{s\perp}^* E_{s\|}, \\ V_s &= -2\mathrm{Im}E_{s\perp}^* E_{s\|}. \end{aligned} \quad (18)$$

Thus, I_s gives the scattered intensity, Q_s gives the difference between the intensities in the scattering plane and perpendicular to the scfattering plane, U_s gives the difference between $+\pi/4$ and $-\pi/4$ -polarized intensities and, lastly, V_s gives the difference between right-handed and left-handed circularly polarized intensities. The factor $k/2\omega\mu_0$ has been omitted from the intensities; it is not needed since, in practice, relative intensities are measured instead of absolute ones. The Stokes parameters fully describe the polarization state of an electromagnetic field.

The scattering matrix S interrelates the Stokes parameters of the original field and the scattered field, and can be derived from the amplitude scattering matrix:

$$\boldsymbol{I}_s = \frac{1}{k^2 r^2} S \boldsymbol{I}_i, \quad (19)$$

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$$\begin{aligned} \boldsymbol{I}_s &= (I_s, Q_s, U_s, V_s)^T, \\ \boldsymbol{I}_i &= (I_i, Q_i, U_i, V_i)^T. \end{aligned} \tag{20}$$

The information about the angular dependence of scattering is fully contained in the 16 elements of the scattering matrix. For a single scattering particle, it has seven free parameters whereas, for an ensemble of particles, all 16 elements can be free. Symmetries reduce the number of free parameters: for example, for a spherical particle, there are three free parameters.

For an unpolarized incident field, the Stokes parameters of the scattered field are

$$\begin{aligned} I_s &= \frac{1}{k^2r^2} S_{11} I_i, \\ Q_s &= \frac{1}{k^2r^2} S_{21} I_i, \\ U_s &= \frac{1}{k^2r^2} S_{31} I_i, \\ V_s &= \frac{1}{k^2r^2} S_{41} I_i. \end{aligned} \tag{21}$$

Thus, S_{11} gives the angular distribution of scattered intensity and the total degree of polarization is

$$P_{\text{tot}} = \frac{\sqrt{S_{21}^2 + S_{31}^2 + S_{41}^2}}{S_{11}}. \tag{22}$$

Scattering polarizes light.

1.4 Extinction, scattering and absorption

Let us assume that medium surrounding the scattering particle is non-absorbing. The total or extinction cross section is then the sum of the absorption and scattering cross sections:

$$\sigma_e = \sigma_s + \sigma_a, \tag{23}$$

where

$$\begin{aligned} \sigma_e &= -\frac{1}{I_i} \int_A dA \boldsymbol{S}_e \cdot \boldsymbol{e}_r, \\ \sigma_s &= \frac{1}{I_i} \int_A dA \boldsymbol{S}_s \cdot \boldsymbol{e}_r, \end{aligned} \tag{24}$$

when A is a spherical envelope of radius r containing the scattering particle.

Let the original field be of $\boldsymbol{e}_x$-polarized form $\boldsymbol{E}_0 = E\boldsymbol{e}_x$. In the radiation zone,

$$\begin{aligned} \boldsymbol{E}_s &\propto \frac{\exp[\mathrm{i}k(r-z)]}{-\mathrm{i}kr} \boldsymbol{X} E, \boldsymbol{e}_r \cdot \boldsymbol{X} = 0, \\ \boldsymbol{H}_s &\propto \frac{k}{\omega\mu} \boldsymbol{e}_r \times \boldsymbol{E}_s, \end{aligned} \tag{25}$$

where the vector scattering amplitude $\boldsymbol{X}$ is related to the amplitude scattering matrix as follows:

$$\boldsymbol{X} = (S_4 \cos\phi + S_1 \sin\phi)\boldsymbol{e}_{s\perp} + (S_2 \cos\phi + S_3 \sin\phi)\boldsymbol{e}_{s\parallel}. \tag{26}$$

By making use of the asymptotic forms of the scattered field shown above and $\boldsymbol{e}_x$-polarized original field, the so-called optical theorem can be derived: extinction depends only on scattering in the exact forward direction,

$$\sigma_e \;=\; \frac{4\pi}{k^2}\mathrm{Re}[(\boldsymbol{X}\cdot\boldsymbol{e}_x)_{\theta=0}]. \tag{27}$$

In addition,

$$\sigma_s \;=\; \int_{4\pi} d\Omega \frac{d\sigma_s}{d\Omega}, \tag{28}$$

where the differential scattering cross section is

$$\frac{d\sigma_s}{d\Omega} \;=\; \frac{|\boldsymbol{X}|^2}{k^2}. \tag{29}$$

The extinction, scattering, and absorption efficiencies are defined as the ratios of the corresponding cross sections to the geometric cross section of the particle $A_\perp$ as projected in the propagation direction of the original field:

$$\begin{aligned} q_e &= \frac{\sigma_e}{A_\perp}, \\ q_s &= \frac{\sigma_s}{A_\perp}, \\ q_a &= \frac{\sigma_a}{A_\perp}. \end{aligned} \tag{30}$$

For an unpolarized original field, the cross sections are

$$\begin{aligned} \sigma_e &= \frac{1}{2}(\sigma_e^{(1)} + \sigma_e^{(2)}), \\ \sigma_s &= \frac{1}{2}(\sigma_s^{(1)} + \sigma_s^{(2)}), \end{aligned} \tag{31}$$

where the indices 1 and 2 refer to two polarization states of the original field perpendicular to one another.

2 Plane waves

(Lecture 1 and 2)

The electromagnetic plane wave

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t} \end{aligned} \tag{32}$$

can, under certain conditions, fulfil Maxwell's equations. The physical fields correspond to the real parts of the complex-valued fields. The vectors $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ above are constant vectors and can be complex-valued. Similarly, the wave vector $\boldsymbol{k}$ can be complex-valued:

$$\boldsymbol{k} = \boldsymbol{k}' + i\boldsymbol{k}'', \qquad \boldsymbol{k}', \boldsymbol{k}'' \in \mathbb{R}^n \tag{33}$$

Inserting (33) into equation (32), we obtain

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t} \\ \boldsymbol{H} &= \boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\omega t}\end{aligned} \tag{34}$$

In Eq. (34), $\boldsymbol{E}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ and $\boldsymbol{H}_0 e^{-\boldsymbol{k}''\cdot\boldsymbol{x}}$ are amplitudes and $\boldsymbol{k}'\cdot\boldsymbol{x} - \omega t = \phi$ is the phase of the wave.

An equation of the form $\boldsymbol{k}\cdot\boldsymbol{x}$ =constant defines, in the case of a real-valued vector $\boldsymbol{k}$, a planar surface, whose normal is just the vector $\boldsymbol{k}$. Thus, $\boldsymbol{k}'$ is perpendicular to the planes of constant phase and $\boldsymbol{k}''$ is perpendicular to the planes of constant amplitude. If $\boldsymbol{k}' \parallel \boldsymbol{k}''$, the planes coincide and the wave is *homogeneous*. If $\boldsymbol{k}' \nparallel \boldsymbol{k}''$, the wave is *inhomogeneous*. A plane wave propagating in vacuum is homogeneous.

In the case of plane waves, Maxwell's equaitons can be written as

$$\begin{aligned}\boldsymbol{k}\cdot\boldsymbol{E}_0 &= 0 \\ \boldsymbol{k}\cdot\boldsymbol{H}_0 &= 0 \\ \boldsymbol{k}\times\boldsymbol{E}_0 &= \omega\mu\boldsymbol{H}_0 \\ \boldsymbol{k}\times\boldsymbol{H}_0 &= -\omega\epsilon\boldsymbol{E}_0\end{aligned} \tag{35}$$

The two upmost equations are conditions for the transverse nature of the waves: $\boldsymbol{k}$ is perpendicular to both $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. The two lowermost equations show that $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are perpendicular to each other. Since $\boldsymbol{k}$, $\boldsymbol{E}_0$, and $\boldsymbol{H}_0$ are complex-valued, the geometric interpretaion is not simple unless the waves are homogeneous.
It follows from Maxwell's equations (35) that, on one hand,

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \omega\mu\boldsymbol{k}\times\boldsymbol{H}_0 = -\omega^2\epsilon\mu\boldsymbol{E}_0 \tag{36}$$

and, on the other hand,

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}_0) - \boldsymbol{E}_0(\boldsymbol{k}\cdot\boldsymbol{k}) = -\boldsymbol{E}_0(\boldsymbol{k}\cdot\boldsymbol{k}), \tag{37}$$

so that

$$\boldsymbol{k}\cdot\boldsymbol{k} = \omega^2\epsilon\mu. \tag{38}$$

Plane waves solutions are in agreement with Maxwell's equations if

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = \boldsymbol{k}\cdot\boldsymbol{H}_0 = \boldsymbol{E}_0\cdot\boldsymbol{H}_0 = 0 \tag{39}$$

and if

$$k'^2 - k''^2 + 2i\boldsymbol{k}'\cdot\boldsymbol{k}'' = \omega^2\epsilon\mu. \tag{40}$$

Note that ϵ and μ are properties of the medium, whereas $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are properties of the wave. Thus, ϵ and μ do not unambiguously determine the details of wave propagation.
In the case of a homogeneous plane wave ($\boldsymbol{k}'\|\boldsymbol{k}''$),

$$\boldsymbol{k} = (k' + ik'')\hat{\mathbf{e}}, \tag{41}$$

where k' and k'' are non-negative and ê is an arbitrary real-valued unit vector.

According to Eq. (38),

$$(k' + ik'')^2 = \omega^2 \epsilon\mu = \frac{\omega^2 m^2}{c^2}, \tag{42}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum and m is the complex-valued refractive index

$$m = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = m_r + im_i, \qquad m_r, m_i \geq 0. \tag{43}$$

In vacuum, the wave number is $\omega/c = 2\pi/\lambda$, where λ is the wavelength. The general homogeneous plane wave takes the form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-\frac{2\pi m_i s}{\lambda}} e^{i\frac{2\pi m_r s}{\lambda} - i\omega t} \tag{44}$$

where $s = \boldsymbol{e} \cdot \boldsymbol{x}$. The imaginary and real parts of the refractive index determine the attenuation and phase velocity $v = c/m_r$ of the wave, respectively.

3 Poynting vector

Let us study the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ that is time harmonic. For the physical fields (the real parts of the complex-valued fields), the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{45}$$

describes the direction and amount of energy transfer everywhere in the space.

Let $\boldsymbol{n}$ be the unit normal vector of the planar surface element A. Electromagnetic energy is transferred through the planar surface with power $\boldsymbol{S} \cdot \boldsymbol{n}\ A$, where $\boldsymbol{S}$ is assumed constant on the surface. For an arbitrary surface and $\boldsymbol{S}$ depending on location, the power is

$$W = -\int_A \boldsymbol{S} \cdot \boldsymbol{n} dA, \tag{46}$$

where $\boldsymbol{n}$ is the outward unit normal vector and the sign has been chosen so that positive W corresponds to absorption in the case of a closed surface.
The time-averaged Poynting vector

$$\langle \boldsymbol{S} \rangle = \frac{1}{\tau} \int_t^{t+\tau} \boldsymbol{S}(t') dt' \qquad \tau >> 1/\omega \tag{47}$$

is more important than the momentary Poynting vector (cf. measurements).

The time-averaged Poynting vector for time-harmonic fields is

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} \tag{48}$$

and, in what follows, this is the Poynting vector meant even though the averaging is not always shown explicitly.
For a plane wave field, the Poynting vector is

$$\boldsymbol{S} = \frac{1}{2}\mathbf{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} = \mathbf{Re}\left\{\frac{\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*)}{2\omega\mu^*}\right\}, \tag{49}$$

where

$$\boldsymbol{E} \times (\boldsymbol{k}^* \times \boldsymbol{E}^*) = \boldsymbol{k}^*(\boldsymbol{E} \cdot \boldsymbol{E}^*) - \boldsymbol{E}^*(\boldsymbol{k}^* \cdot \boldsymbol{E}). \tag{50}$$

For a homogeneous plane wave,

$$\boldsymbol{k} \cdot \boldsymbol{E} = \boldsymbol{k}^* \cdot \boldsymbol{E} = 0 \tag{51}$$

and

$$\boldsymbol{S} = \frac{1}{2}\mathbf{Re}\left\{\frac{\sqrt{\epsilon\mu}}{\mu^*}\right\}|\boldsymbol{E}_0|^2 e^{-\frac{4\pi \mathbf{Im}(m) z}{\lambda}}\hat{\mathbf{e}}_z. \tag{52}$$

4 Stokes parameters

Consider the following experiment for an arbitrary monochromatic light source (see Bohren & Huffman p. 46). In the experiment, we make use of a measuring apparatus and polarizers with ideal performance: the measuring apparatus detects energy flux density independently of the state of polarization and the polarizers do not change the amplitude of the transmitted wave.

Denote

$$\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_0 e^{ikz - i\omega t}, \qquad && \boldsymbol{E}_0 = E_\perp \hat{\mathbf{e}}_\perp + E_\| \hat{\mathbf{e}}_\| \\
E_\perp &= a_\perp e^{-i\delta_\perp} && \\
E_\| &= a_\| e^{-i\delta_\|} && a_\perp, a_\| \geq 0, \delta_\perp, \delta_\| \in \mathbb{R}
\end{aligned} \tag{53}$$

<u>Experiment I</u>
No polarizer: the flux density is proportional to

$$|\boldsymbol{E}_0|^2 = E_\| E_\|^* + E_\perp E_\perp^* \tag{54}$$

<u>Experiment II</u>
Linear polarizers $\|$ and $\perp$:
1) $\|$: the amplitude of the transmitted wave is $E_\|$ and the flux density is $E_\| E_\|^*$
2) $\perp$: the amplitude of the transmitted wave is $E_\perp$ and the flux density is $E_\perp E_\perp^*$

The difference of the two measurements is $I_\parallel - I_\perp = E_\parallel E_\parallel^* - E_\perp E_\perp^*$.

Experiment III
Linear polarizers $+45°$ ja $-45°$: The new basis vectors are

$$\begin{cases} \hat{\mathbf{e}}_+ = & \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + \hat{\mathbf{e}}_\perp) \\ \hat{\mathbf{e}}_- = & \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - \hat{\mathbf{e}}_\perp) \end{cases}$$

and

$$\begin{aligned} \boldsymbol{E}_0 &= E_+\hat{\mathbf{e}}_+ + E_-\hat{\mathbf{e}}_- \\ E_+ &= \frac{1}{\sqrt{2}}(E_\parallel + E_\perp) \\ E_- &= \frac{1}{\sqrt{2}}(E_\parallel - E_\perp). \end{aligned}$$

1) $+45°$: the amplitude of the transmitted wave is E_+ and the flux density is $E_+E_+^* = \frac{1}{2}(E_\parallel E_\parallel^* + E_\parallel E_\perp^* + E_\perp E_\parallel^* + E_\perp E_\perp^*)$
2) $-45°$: the amplitude of the transmitted wave is E_- and the flux density is $E_-E_-^* = \frac{1}{2}(E_\parallel E_\parallel^* - E_\parallel E_\perp^* - E_\perp E_\parallel^* + E_\perp E_\perp^*)$

The difference os the measurements is $I_+ - I_- = E_\parallel E_\perp^* + E_\perp E_\parallel^*$.

Experiment IV
Circular polarizers R and L:

$$\begin{aligned} \hat{\mathbf{e}}_R &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel + i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_R^* = 1 \\ \hat{\mathbf{e}}_L &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_\parallel - i\hat{\mathbf{e}}_\perp) \qquad \hat{\mathbf{e}}_L \cdot \hat{\mathbf{e}}_L^* = 1 \qquad \hat{\mathbf{e}}_R \cdot \hat{\mathbf{e}}_L^* = 0 \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{E}_0 &= E_R\hat{\mathbf{e}}_R + E_L\hat{\mathbf{e}}_L \\ E_R &= \frac{1}{\sqrt{2}}(E_\parallel - iE_\perp) \\ E_L &= \frac{1}{\sqrt{2}}(E_\parallel + iE_\perp). \end{aligned}$$

1) R: the amplitude of the transmitted wave is E_R and the flux density is $E_RE_R^* = \frac{1}{2}(E_\parallel E_\parallel^* - iE_\parallel^* E_\perp + iE_\perp^* E_\parallel + E_\perp E_\perp^*)$
2) L: the amplitude of the transmitted wave is E_L and the flux density is $E_LE_L^* = \frac{1}{2}(E_\parallel E_\parallel^* + iE_\parallel^* E_\perp - iE_\perp^* E_\parallel + E_\perp E_\perp^*)$

The difference of the measurements is $I_R - I_L = i(E_\perp^* E_\| - E_\|^* E_\perp)$.

With the help of Experiments I-IV, we have determined the Stokes parameters I, Q, U, and V:

$$\begin{aligned}
I &= E_\| E_\|^* + E_\perp E_\perp^* = a_\|^2 + a_\perp^2 \\
Q &= E_\| E_\|^* - E_\perp E_\perp^* = a_\|^2 - a_\perp^2 \\
U &= E_\| E_\perp^* + E_\perp E_\|^* = 2a_\| a_\perp \cos\delta \\
V &= i(E_\| E_\perp^* - E_\perp E_\|^*) = 2a_\| a_\perp \sin\delta \qquad \delta = \delta_\| - \delta_\perp
\end{aligned} \tag{55}$$

5 Scattering of light at the plane interface between two media

(Lecture 3)

Two kinds of features can be distinguished in the reflection and refraction of light at the plane interface between two media:
i) Kinematical properties:
a) the angle of reflection coincides with the angle of incidence
b) the angle of refraction relates to the angle of incidence and the refractive indices of the media via Snel's law
ii) Dynamical properties:
a) the intensitities of reflected and refracted radiation
b) phase shifts and polarization

The kinematical properties follow from the wave nature of the phenomena and the existence of the boundary conditions. The dynamical properties depend fully of the characteristics of the waves and their boundary conditions.
The coordinate systems and symbols are defined in Fig. 1. The original plane wave (wave vector $\mathbf{k}$, angular frequency ω) is incident on the interface from the medium μ, ϵ (refractive index $m = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$). The refracted plane wave propagates in the medium μ', ϵ' ($m' = \sqrt{\epsilon'\mu'/\epsilon_0\mu_0}$) with wave vector $\mathbf{k}_t$ and the reflected plane wave in the medium μ, ϵ with wave vector $\mathbf{k}_r$.

The kinematics are described by the angles of incidence θ_i, reflection θ_r, and refraction θ_t. Assume first that μ, ϵ, μ', ϵ' and therefor also m and m' are real-valued.

Based on what has already been described before, we can write the incident, reflected, and refracted fields as follows:

$$\begin{aligned}
\mathbf{E}_i &= \mathbf{E}_{0i} e^{i\mathbf{k}_i \cdot \mathbf{x} - i\omega t} \\
\mathbf{B}_i &= \sqrt{\epsilon\mu} \frac{\mathbf{k}_i \times \mathbf{E}_i}{k_i}
\end{aligned} \tag{56}$$

$$\begin{aligned}
\mathbf{E}_r &= \mathbf{E}_{0r} e^{i\mathbf{k}_r \cdot \mathbf{x} - i\omega t} \\
\mathbf{B}_r &= \sqrt{\epsilon\mu} \frac{\mathbf{k}_r \times \mathbf{E}_r}{k_r}
\end{aligned} \tag{57}$$

$$
\begin{aligned}
\mathbf{E}_t &= \mathbf{E}_{0t} e^{i\mathbf{k}_t\cdot\mathbf{x}-i\omega t} \\
\mathbf{B}_t &= \sqrt{\epsilon'\mu'}\frac{\mathbf{k}_t\times\mathbf{E}_t}{k_t}
\end{aligned} \tag{58}
$$

The lengths of the wave vectors are

$$
\begin{aligned}
|\mathbf{k}_i| &= |\mathbf{k}_r| = k_i = k_r = \omega\sqrt{\epsilon\mu} \\
|\mathbf{k}_t| &= k_t = \omega\sqrt{\epsilon'\mu'}
\end{aligned} \tag{59}
$$

The boundary conditions are to be valid at the interface $z = 0$ at all times. Therefor, the spatial dependences of the fields need to coincide at the interface and, in particular, the arguments of the phase factors

$$
(\mathbf{k}_i\cdot\mathbf{x})_{z=0} = (\mathbf{k}_r\cdot\mathbf{x})_{z=0} = (\mathbf{k}_t\cdot\mathbf{x})_{z=0} \tag{60}
$$

independently of the detailed properties of the boundary conditions. It follows, first, that the wave vectors must be confined to a single plane. Second, it follows that $\theta_i = \theta_r$ and, third, we obtain Snel's law

$$
\begin{aligned}
k_i\sin\theta_i &= k_t\sin\theta_t \\
\Leftrightarrow\qquad m\sin\theta_i &= m'\sin\theta_t.
\end{aligned} \tag{61}
$$

According to the boundary conditions of electromagnetic fields, the normal components of $\mathbf{D}$ and $\mathbf{B}$ and the tangential components of $\mathbf{E}$ and $\mathbf{H}$ must be continuous across the boundary. Then, at the interface $z = 0$, we have

$$
\begin{aligned}
\hat{\mathbf{n}} &\cdot [\epsilon(\mathbf{E}_{0i}+\mathbf{E}_{0r}) - \epsilon'\mathbf{E}_{0t}] = 0 \\
\hat{\mathbf{n}} &\cdot [\mathbf{k}_i\times\mathbf{E}_{0i} + \mathbf{k}_r\times\mathbf{E}_{0r} - \mathbf{k}_t\times\mathbf{E}_{0t}] = 0 \\
\hat{\mathbf{n}} &\times [\mathbf{E}_{0i}+\mathbf{E}_{0r}-\mathbf{E}_{0t}] = 0 \\
\hat{\mathbf{n}} &\times [\frac{1}{\mu}(\mathbf{k}_i\times\mathbf{E}_{0i} + \mathbf{k}_r\times\mathbf{E}_{0r}) - \frac{1}{\mu'}(\mathbf{k}_t\times\mathbf{E}_{0t})] = 0
\end{aligned} \tag{62}
$$

Let us divide the scattering problem into two cases: first, the incident field is linearly poalrized so that the electric field is perpendicular to the plane defined by $\mathbf{k}_i$ and $\hat{\mathbf{n}}$; second, the electric field is within that plane. An arbitrary elliptic polarization can be treated as a linear sum of the results following for the two cases defined above.

First, let the elctric field be perpendicular to the plane of incidence (see Fig. 2). The choice of $\mathbf{B}$-vectors guarantees a positive flow of energy in the direction of the wave vectors. With the help of the third and fourth boundary conditions above, we obtain

$$
\begin{aligned}
E_{0i} + E_{0r} - E_{0t} &= 0 \\
\sqrt{\frac{\epsilon}{\mu}}(E_{0i}-E_{0r})\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}E_{0t}\cos\theta_t &= 0
\end{aligned} \tag{63}
$$

Denote the Fresnel coefficients by

$$r_\perp = \frac{E_{0r}}{E_{0i}}, \qquad t_\perp = \frac{E_{0t}}{E_{0i}}.$$

Then,

$$\begin{aligned} 1 + r_\perp - t_\perp = 0 \\ \sqrt{\frac{\epsilon}{\mu}}(1 - r_\perp)\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}t_\perp \cos\theta_t = 0 \end{aligned} \tag{64}$$

and it follows that

$$\begin{aligned} t_\perp &= 1 + r_\perp \\ \sqrt{\frac{\epsilon}{\mu}}\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t &= (\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t)r_\perp \end{aligned} \tag{65}$$

and, furthermore, we obtain, for the Fresnel coefficients,

$$\begin{aligned} r_\perp &= \frac{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i - \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t}{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t} \\ t_\perp &= \frac{2\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i}{\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i + \sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_t} \end{aligned} \tag{66}$$

Second, let the electric field be within the plane of incidence (see Fig. 3). Again, based on the third and fourth boundary conditions above, we have

$$\begin{aligned} (E_{0i} - E_{0r})\cos\theta_i - E_{0t}\cos\theta_t &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(E_{0i} + E_{0r}) - \sqrt{\frac{\epsilon'}{\mu'}}E_{0t} &= 0 \end{aligned} \tag{67}$$

Denote the Fresnel coefficients by

$$r_\parallel = \frac{E_{0r}}{E_{0i}}, \qquad t_\parallel = \frac{E_{0t}}{E_{0i}}.$$

Then,

$$\begin{aligned} (1 - r_\parallel)\cos\theta_i - t_\parallel \cos\theta_t &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(1 + r_\parallel) - \sqrt{\frac{\epsilon'}{\mu'}}t_\parallel &= 0 \end{aligned} \tag{68}$$

and we obtain the following pair of equations,

$$
\begin{aligned}
t_{\parallel} &= \frac{\cos\theta_i}{\cos\theta_t}(1 - r_{\parallel}) \\
\sqrt{\frac{\epsilon}{\mu}} - \sqrt{\frac{\epsilon'}{\mu'}}\frac{\cos\theta_i}{\cos\theta_t} &= -\Big(\sqrt{\frac{\epsilon}{\mu}} + \sqrt{\frac{\epsilon'}{\mu'}}\frac{\cos\theta_i}{\cos\theta_t}\Big) r_{\parallel}
\end{aligned} \tag{69}
$$

allowing for the Fresnel coefficients to be explicitly solved for:

$$
\begin{aligned}
r_{\parallel} &= \frac{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i - \sqrt{\frac{\epsilon}{\mu}}\cos\theta_t}{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i + \sqrt{\frac{\epsilon}{\mu}}\cos\theta_t} \\
t_{\parallel} &= \frac{2\sqrt{\frac{\epsilon}{\mu}}\cos\theta_i}{\sqrt{\frac{\epsilon'}{\mu'}}\cos\theta_i + \sqrt{\frac{\epsilon}{\mu}}\cos\theta_t}
\end{aligned} \tag{70}
$$

In the case of a plane wave normally incident on the interface ($\theta_i = 0$), we obtain

$$
\begin{aligned}
r_{\parallel} &= -r_{\perp} = \frac{\sqrt{\frac{\epsilon'}{\mu'}} - \sqrt{\frac{\epsilon}{\mu}}}{\sqrt{\frac{\epsilon'}{\mu'}} + \sqrt{\frac{\epsilon}{\mu}}} \to \frac{m' - m}{m' + m}, \mu = \mu' \\
t_{\parallel} &= t_{\perp} = \frac{2\sqrt{\frac{\epsilon}{\mu}}}{\sqrt{\frac{\epsilon'}{\mu'}} + \sqrt{\frac{\epsilon}{\mu}}} \to \frac{2m}{m' + m}, \mu = \mu'
\end{aligned} \tag{71}
$$

The Fresnel coefficients derived above are also valid for complex-valued ϵ, μ, ϵ', and μ'. Usually, for visible light, $\mu = \mu' = \mu_0$. The generalization of Snel's law for complex m' is left for an exercise. In addition, the derivation of the 4×4 reflection and refraction matrices relating the Stokes parameters of incident, reflected, and refracted light is left for an exercise.

In the case of incident electric field polarized in the plane of incidence, we can find the so-called Brewster angle, at which there is no reflected wave. Let $\mu = \mu'$. At the Brewster angle,

$$
\begin{aligned}
m'\cos\theta_{iB} &= m\sqrt{1 - \frac{m^2}{m'^2}\sin^2\theta_{iB}} \\
(\frac{m'}{m})^2\cos^2\theta_{iB} &= 1 - (\frac{m}{m'})^2\sin^2\theta_{iB} \\
(\frac{m'}{m})^2 &= 1 + \tan^2\theta_{iB} - (\frac{m}{m'})^2\tan^2\theta_{iB} \\
\tan^2\theta_{iB} &= \frac{(\frac{m'}{m})^2 - 1}{1 - (\frac{m}{m'})^2} = (\frac{m'}{m})^2
\end{aligned}
$$

The physical solution is

$$
\theta_{iB} = \arctan(\frac{m'}{m}) \tag{72}
$$

As a rule for other angles of incidence, too, the reflected light tends to be polarized perpendicular to the plane of incidence.

Total internal reflection can occur when $m > m'$ (the incident wave is "internals"). If $m > m'$, $\theta_t > \theta_{i0}$ according to Snel's law and

$$\theta_{i0} = \arcsin \frac{m'}{m} \tag{73}$$

When the angle of incidence is θ_{i0}, the refracted wave is propagating parallel to the interface and there is no energy flow across the interface. Thus, all the incident energy is reflected back. When $\theta_i > \theta_{i0}$, $\sin\theta_t > 1$ and θ_t must be a complex-valued angle that has a purely imaginary cosine,

$$\cos\theta_t = i\sqrt{(\frac{\sin\theta_i}{\sin\theta_{i0}})^2 - 1} \tag{74}$$

The refracted wave is of the form

$$\begin{aligned} e^{i\mathbf{k}_t\cdot\mathbf{x}} &= e^{ik_t(x\sin\theta_t - z\cos\theta_t)} \\ &= e^{-k_t\sqrt{(\frac{\sin\theta_i}{\sin\theta_{i0}})^2-1}|\mathbf{z}|} e^{ik_t(\frac{\sin\theta_i}{\sin\theta_{i0}})x} \end{aligned} \tag{75}$$

and, thus, attenuates exponentially in the medium m' and propagates only in the direction of the interface.

6 Electromagnetic field by a localized source

(Lecture 4)

Consider the electromagnetic fields caused by time-dependent charge and current densities localized in a constrained region of space. Here we will mainly study the fields by an electric dipole. Later, the analysis is extended to the full multipole expansion.
Assume harmonic time dependence $e^{-i\omega t}$—arbitrary time dependences can be dealt with using Fourier analysis of their components. The charge density ρ and current density $\mathbf{j}$ are

$$\rho(\mathbf{x}, t) = \rho(\mathbf{x})e^{-i\omega t}$$
$$\mathbf{j}(\mathbf{x}, t) = \mathbf{j}(\mathbf{x})e^{-i\omega t}$$

and the physical quantities correspond to the real parts of the comelx quantities. The electromagnetic potentials and fields are also time-harmonic and the sources are assumed to be located in an otherwise empty space.

Let us start from the vector potential $\mathbf{A}$ in Lorentz gauge,

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}' \int dt' \frac{\mathbf{j}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}\delta(t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t)$$

and, by writing $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$, we obtain

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|}, \qquad k = \frac{\omega}{c}$$

The magnetic field is, according to definitions, $\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}$ and, outside the source region, the electric field equals $\mathbf{E} = \frac{i\zeta_0}{k}\nabla \times \mathbf{H}$, where $\zeta_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space.
When the current density $\mathbf{j}(\mathbf{x}')$ is given, the electromagnetic field can be calculated from the integral above, at least in principle. Let us study the case where the source region (size d) is much smaller than the wavelength: $d \ll \lambda = 2\pi c/\omega$. We can distinguish three regimes of interest:

(i) Near zone (static regime): $d \ll r \ll \lambda$

(ii) Intermediate zone (induction regime): $d \ll r \sim \lambda$

(iii) Far zone (radiation regime): $d \ll \lambda \ll r$

In the near zone (i) $kr \ll 1$ and the exponential part of the integrand for the vector potential can be set to unity, and the inverse distance can be presented using series of spherical harmonics Y_{lm}:

$$\lim_{kr\to 0} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\sum_{l,m}\frac{4\pi}{2l+1}\frac{\mathrm{Y}_{lm}(\theta,\varphi)}{r^{l+1}}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(r')^l\mathrm{Y}^*_{lm}(\theta',\varphi')$$

We can see that the near fields vary harmonically in time but are static in their character: no wave solution follows for the spatial dependence. Above, we have made use of the relation

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r^l_<}{r^{l+1}_>}\mathrm{Y}^*_{lm}(\theta',\varphi')\mathrm{Y}_{lm}(\theta,\varphi)$$

In the far zone (iii), $kr \gg 1$ and the exponential part of the vector potential varies strongly and dictates the character of the vector potential. We can approximate

$$|\mathbf{x}-\mathbf{x}'| \approx r - \hat{\mathbf{n}}\cdot\mathbf{x}', \qquad \hat{\mathbf{n}} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{r}$$

When the leading term is desired in kr, the inverse distance can be replaced by r. The vector potential is of the form

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'}$$

Therefor, the vector potential behaves like an outgoing spherical wave (e^{ikr}/r) with angular dependence. It can be shown that the electromagnetic field is also of the form of a spherical wave and thus is a radiation field. (Note that this part of the analysis is vlid for localized source regions of arbitrary size.)

Now that $kd \ll 1$ the integral can further be developed into series:

$$\lim_{kr\to\infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n\frac{(-ik)^n}{n!}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(\hat{\mathbf{n}}\cdot\mathbf{x}')^n$$

where the magnitude for the nth term is $(1/n!)\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')(k\hat{\mathbf{n}}\cdot\mathbf{x}')^n$ and thus becomes rapidly smaller with increasing n. In this case, the main contribution to radiation comes from the first non-vanishing term in the sum.

In the intermediate zone (ii), all powers of kr need to be accounted for, and no simple limits can be taken. The vector potential is then written with the help of the expansion for the exact Green's function in the form

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik\sum_{l,m}h^{(1)}_l(kr)\mathrm{Y}_{lm}(\theta,\varphi)\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')j_l(kr')\mathrm{Y}^*_{lm}(\theta',\varphi')$$

where we have made use of the expansion

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} \mathrm{Y}_{lm}^*(\theta',\varphi')\mathrm{Y}_{lm}(\theta,\varphi)$$

where $r_< = \min(r,r')$, $r_> = \max(r,r')$, and j_l and $h_l^{(1)}$ are the spherical Bessel and Hankel functions.

Again when $kd \ll 1$, the j_l-functions can be approximated and the result is of the same form as the near zone result, when the following replacement is carried out:

$$\frac{1}{r^{l+1}} \to \frac{e^{ikr}}{r^{l+1}}[1 + a_1(ikr) + a_2(ikr)^2 + \ldots + a_l(ikr)^l]$$

The coefficients a_i derive from the explicit expansions of the Hankel functions. This end result allows us to see the transition from the near-zone $kr \ll 1$ static field to the far-zone $kr \gg 1$ radiation field.

7 Electromagnetic field of an electric dipole

If only the first term in kd is kept in the expansion of the vector potential, one obtains

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}')$$

which holds everywhere outside the source region (this follows from the intermediate-zone results above). With the help of partial integration,

$$\int d^3\mathbf{x}'\mathbf{j} = -\int d^3\mathbf{x}'\mathbf{x}'(\nabla\cdot\mathbf{j}) = -i\omega\int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}')$$

where the substitution term disappears (the source region is constrained) and, according to the continuity equation, $i\omega\rho(\mathbf{x}') = \nabla\cdot\mathbf{j}(\mathbf{x}')$. The vector potential is thus

$$\mathbf{A}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r},$$

where $\mathbf{p}$ is the electric dipole moment $\mathbf{p} = \int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}')$.

The electromagnetic fields are

$$\begin{aligned}
\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}}\times\mathbf{p})\frac{e^{ikr}}{r}\big(1-\frac{1}{ikr}\big) \\
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\Big(k^2(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r} + \big(3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}\big)\big(\frac{1}{r^2}-\frac{ik}{r}\big)\frac{e^{ikr}}{r}\Big)
\end{aligned}$$

We note that the magnetic field is always transverse but that the electric field has both longitudinal and transverse components.

In the far zone,

$$\begin{aligned}
\mathbf{H} &= \frac{ck^2}{4\pi}(\hat{\mathbf{n}}\times\mathbf{p})\frac{e^{ikr}}{r} \\
\mathbf{E} &= \zeta_0\mathbf{H}\times\hat{\mathbf{n}}
\end{aligned}$$

which shows the typical form of a spherical wave.

In the near zone,

$$\begin{aligned}\mathbf{H} &= \frac{i\omega}{4\pi}(\hat{\mathbf{n}}\times\mathbf{p})\frac{1}{r^2}\\ \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\big(3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}\big)\frac{1}{r^3}\end{aligned}$$

The electric field is, except for the harmonic time dependence, that of a static lectric dipole. The field $\zeta_0\mathbf{H}$ is smaller, by a factor of kr, than the field $\mathbf{E}$ so, in the near zone, the field is electric in its nature. In the static limit $k\to 0$, the magnetic field disappears and the near zone extends to infinity.

The power radiated by the vibrating dipole moment $\mathbf{p}$ as per solid angle is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2}Re(r^2\hat{\mathbf{n}}\cdot\mathbf{E}\times\mathbf{H}^*)\\ &= \frac{c^2\zeta_0}{32\pi^2}k^4|(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}|^2,\end{aligned}$$

where $\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}$ gives the polarization state. If all components of $\mathbf{p}$ are in the same phase,

$$\frac{dP}{d\Omega}=\frac{c^2\zeta_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta$$

which is the typical radiation pattern of an electric dipole (θ is here measured from the direction of $\mathbf{p}$). Independently of the phases of the components for $\mathbf{p}$:n, the total radiated power is

$$P=\frac{c^2\zeta_0 k^4}{12\pi}|\mathbf{p}|^2$$

8 Scattering by small spherical particles in the electric dipole approximation

Light scattering by particles clearly smaller than the wavelength can be studied in the approximation, where the incident field induces an electric dipole moment to the particle. The dipole fluctuates in a certain phase with the incident field and thus scatters radiation in directions differing from the propagation direction of the incident field. In this case, the dipole moments can be computed using electrostatic methods.

Assume that a monochromatic plane wave is incident on a small scatterer located in free space. Let the propagation direction and polarization vector of the incident field be $\hat{\mathbf{n}}_0$ and $\hat{\epsilon}_0$:

$$\begin{aligned}\mathbf{E}_i &= \hat{\epsilon}_0 E_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}}\\ \mathbf{H}_i &= \hat{\mathbf{n}}_0\times\mathbf{E}_i/\zeta_0\end{aligned}$$

where $k=\omega/c$ and the time dependence has been assumed harmonic ($e^{-i\omega t}$). These fields induce a dipole momentn $\mathbf{p}$ in the small particle and the particle radiates energy in (almost) all directions. In the far zone, the scattered fields are of the form

$$\begin{aligned}\mathbf{E}_s &= \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\big((\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\big)\\ \mathbf{H}_s &= \hat{\mathbf{n}}\times\mathbf{E}_s/\zeta_0\end{aligned}$$

where $\hat{\mathbf{n}}$ is the dirction of the observer and r the distance from the scatterer. The power scattered in direction $\hat{\mathbf{n}}$ with polarization $\hat{\epsilon}$ per unit solid angle divided by the incident flux density is the so-called differential cross section

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \hat{\epsilon}, \hat{\mathbf{n}}_0, \hat{\epsilon}_0) = \frac{r^2 \frac{1}{2\zeta_0}|\hat{\epsilon}^* \cdot \mathbf{E}_s|^2}{\frac{1}{2\zeta_0}|\hat{\epsilon}_0^* \cdot \mathbf{E}_i|^2}$$

where the complex conjugation of the polarization vectors is important for proper treatment of circular polarization. Furthermore,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \hat{\epsilon}, \hat{\mathbf{n}}_0, \hat{\epsilon}_0) = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\hat{\epsilon}^* \cdot \mathbf{p}|^2,$$

where the $\hat{\mathbf{n}}_0, \hat{\epsilon}_0$ -dependence is implicit in $\mathbf{p}$. We can see that the differential and total cross sections of the dipole scatterer are both proportional to k^4:een and λ^{-4}:een (Rayleigh's law). Assume that the scatterer is a small sphere (radius a) with the relative permittivity $\epsilon_r = \epsilon/\epsilon_0$. According to electrostatics, the dipole moment of the sphere is

$$\mathbf{p} = 4\pi\epsilon_0 \Big(\frac{\epsilon_r - 1}{\epsilon_r + 2}\Big) a^3 \mathbf{E}_i$$

so that

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2$$

The polarization dependence is purely that of electric dipole scattering. The scattered radiation is polarized in the plane defined by the dipole moment $\hat{\epsilon}_0$ and the vector $\hat{\mathbf{n}}$.

For unpolarized incident radiation, the differential cross sections in different polarization states of the scattered field are

$$\begin{aligned} \frac{d\sigma_{\parallel}}{d\Omega} &= \frac{k^4 a^6}{2} |\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 \cos^2\theta \\ \frac{d\sigma_{\perp}}{d\Omega} &= \frac{k^4 a^6}{2} |\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 \end{aligned}$$

where θ is now the scattering angle.

The degree of polarization is

$$P(\theta) = \frac{\frac{d\sigma_{\perp}}{d\Omega} - \frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}} = \frac{\sin^2\theta}{1 + \cos^2\theta} = -\frac{S_{21}(\theta)}{S_{11}(\theta)}$$

and the differential cross section summed over the polarization states of the scattered field is

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2 \frac{1}{2}(1 + \cos^2\theta) \propto S_{11}(\theta)$$

where $S_{11}(\theta)$ and $S_{21}(\theta)$ are elements of the scattering matrix. The total scattering cross section is

$$\sigma = \int_{(4\pi)} \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 |\frac{\epsilon_r - 1}{\epsilon_r + 2}|^2$$

The scattered radiation is 100% positively polarized at the scattering angle $\theta = 90°$. It was the polarization characteristics of the blue sky that got Rayleigh interested in scattering by small particles.

9 Scattering by an ensemble of small particles in the dipole approximation

(Lecture 5)

Consider an ensemble of numerous small particles which have fixed locations in space and the scattering amplitudes of which can be expressed in the dipole approximation. Assume presently that the particles do not interact with each other. Since the induced dipole moments are proportional to the incident field, the moments will depend on the phase factor $e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}_j}$, where $\mathbf{x}_j$ is the location of the jth scatterer. When the observer is located far away from the scatterer, the exponential part of the Green's function results in an additional phase factor for the jth scatterer, $e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}_j}$. In the dipole approximation, the ensemble of particles scatters as follows:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\sum_j \hat{\epsilon}^* \cdot \mathbf{p}_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|, \qquad \mathbf{q} = k(\hat{\mathbf{n}}_0 - \hat{\mathbf{n}}) \tag{76}$$

Except for the forward-scattering direction ($\mathbf{q} = 0$), scattering will depend sensitively on how the small particles are located in space.

Assume now that all the particles are identical so that $\mathbf{p} = \mathbf{p}_j$ for all j and

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}|\hat{\epsilon}^* \cdot \mathbf{p}|^2 F(\mathbf{q}), \tag{77}$$

where $F(\mathbf{q})$ is the so-called structure factor,

$$F(\mathbf{q}) = |\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|^2 = \sum_{j,j'} e^{i\mathbf{q}\cdot(\mathbf{x}_j - \mathbf{x}_{j'})} \tag{78}$$

If the small particles are located in random positions, the terms $j \neq j'$ will cause a negligible contribution to the sum. Only the terms $j = j'$ are significant and $F(\mathbf{q}) = N$, where N is the number of scatterers. In this case, the total scattering is the incoherent superposition of the individual contributions.

If the small particles are regularly located in space, the structure factor disappears almost everywhere except for the proximity of the forward-scattering direction. Therefor, large regular arrays of small particles do not scatter (for example, individual transparent crystals of rock salt and quartz).

Consider scatterers located in a regular cubic lattice. The structure factor can be calculated analytically, since

$$\begin{aligned}
\left|\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}\right|^2 &= \left|\sum_{j_1=0}^{N_1-1} e^{iq_1 j_1 a} \sum_{j_2=0}^{N_2-1} e^{iq_2 j_2 a} \sum_{j_3=0}^{N_3-1} e^{iq_3 j_3 a}\right|^2 \\
&= \left|(\frac{1-e^{iq_1 N_1 a}}{1-e^{iq_1 a}})(\frac{1-e^{iq_2 N_2 a}}{1-e^{iq_2 a}})(\frac{1-e^{iq_3 N_3 a}}{1-e^{iq_3 a}})\right|^2 \\
&= N^2[(\frac{\sin^2\frac{1}{2}N_1 q_1 a}{N_1^2 \sin^2\frac{1}{2}q_1 a})(\frac{\sin^2\frac{1}{2}N_2 q_2 a}{N_2^2 \sin^2\frac{1}{2}q_2 a})(\frac{\sin^2\frac{1}{2}N_3 q_3 a}{N_3^2 \sin^2\frac{1}{2}q_3 a})],
\end{aligned} \tag{79}$$

where a is the lattice constant (distance between the lattice points) and where N_1, N_2, and N_3 are the numbers of lattice points in each direction hilapisteiden so that the total number of lattice points equals $N = N_1 N_2 N_3$ (this was utilized to obtain the final result above). The components of the vector $\mathbf{q}$ in each direction are q_1, q_2, and q_3.

We note that, at short wavelengths ($ka \geq \pi$), the structure factor has peaks when the Bragg condition is fulfilled: $q_i a = 0, 2\pi, 4\pi \ldots$, where $i = 1, 2, 3 \ldots$. This is typical in X-ray diffraction. At long wavelengths, only the peak $q_i a = 0$ is relevant, since $\max |q_i a| = 2ka << 1$. In this limit, the structure factor is a product of three $\sin^2 x_i / x_i^2$-type factors ($x_i = \frac{1}{2} N_i q_i a$), and scattering is confined to the region $q_i \leq 2\pi / N_i a$, corresponding to the angles λ / L, where L is the size of the lattice.

10 Volume integral equation for scattering

In a uniform medium, the electromagnetic wave propagates undisturbed and wiythout changing its direction of propagation. If there are fluctuations in the medium depending on space or time, the wave is scattered, and part of its energy is redirected. If the fluctuations in the medium are small, scattering is weak and one may utilize methods based on perturbation series.

Consider a uniform isotropic medium with electric permittivity ϵ_m and magnetic permeability equal to the permeability of vacuum, $\mu_m = \mu_0$. Fluctuations in the medium result in $\mathbf{D} \neq \epsilon_m \mathbf{E}$ in some constrained region. Let us start from Maxwell's equations in sourceless space:

$$\nabla \cdot \mathbf{B} = 0 \quad , \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{D} = 0 \quad , \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

Then

$$\nabla \times (\mathbf{D} - \mathbf{D} + \epsilon_m \mathbf{E}) = -\epsilon_m \frac{\partial \mathbf{B}}{\partial t}, \tag{80}$$

so that

$$\nabla \times (\nabla \times \mathbf{D}) = \nabla \times [\nabla \times (\mathbf{D} - \epsilon_m \mathbf{E})] - \epsilon_m \frac{\partial}{\partial t} \mu_0 \nabla \times \mathbf{H}. \tag{81}$$

Moreover, after further manipulation,

$$-\nabla^2 \mathbf{D} = \nabla \times \nabla \times (\mathbf{D} - \epsilon_m \mathbf{E}) - \epsilon_m \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{D},$$

which can be written in the form

$$\nabla^2 \mathbf{D} - \epsilon_m \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = -\nabla \times \nabla \times (\mathbf{D} - \epsilon_m \mathbf{E}),$$

that is the exact wave equation for the $\mathbf{D}$-field derived without any approximations. Later, the right-hand side of the equation is treated as a small perturbation.

If the right-hand side of the equation were known, the solution of the wave equation could be written an a suitable integral of it. Although the right-hand side is usually unknown, the integral form is useful, since it allows the derivation of important approximations.

Assume again harmonic time dependence $e^{-i\omega t}$, in which case

$$\begin{aligned} (\nabla^2 + k^2) \mathbf{D} &= -\nabla \times \nabla \times (\mathbf{D} - \epsilon_m \mathbf{E}) \\ k^2 &= \mu_0 \epsilon_m \omega^2, \end{aligned} \tag{82}$$

where ϵ_m is the permittivity corresponding to the angular frequency ω. The solution of the undisturbed problem is obtained by setting the right-hand side equal to zero; denote this solution by $\mathbf{D}^{(0)}$. The formal complete solution is then, in an exact way,

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}^{(0)}(\mathbf{x}) + \frac{1}{4\pi}\int d^3\mathbf{x}' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\nabla' \times \nabla' \times (\mathbf{D}(\mathbf{x}') - \epsilon_m \mathbf{E}(\mathbf{x}')) \tag{83}$$

In a scattering problem, the integral on the right-hand side is taken over a constrained region of space and $\mathbf{D}^{(0)}$ describes the incident field. Then, in the far zone,

$$\mathbf{D}(\mathbf{x}) \to \mathbf{D}^{(0)}(\mathbf{x}) + \frac{e^{ikr}}{r}\mathbf{A}_s, \tag{84}$$

where the scattering amplitude $\mathbf{A}_s$ is

$$\mathbf{A}_s = \frac{1}{4\pi}\int d^3\mathbf{x}' e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'}\nabla' \times \nabla' \times (\mathbf{D}(\mathbf{x}') - \epsilon_m \mathbf{E}(\mathbf{x}')). \tag{85}$$

After some partial integration and noticing that the substitution terms diappear, one obtains

$$\mathbf{A}_s = \frac{k^2}{4\pi}\int d^3\mathbf{x}' e^{-ik\hat{\mathbf{n}}\cdot\mathbf{x}'}\{[\hat{\mathbf{n}} \times (\mathbf{D}(\mathbf{x}') - \epsilon_m \mathbf{E}(\mathbf{x}'))] \times \hat{\mathbf{n}}\}.$$

The vector characteristics of the integrand can be compared with the field scattered by an electric dipole: the contribution from the term $\mathbf{D} - \epsilon_m \mathbf{E}$ is precisely the field of the electric dipole so that the scattering amplitude is a vector sum from all induced electric dipole moments. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{|\hat{\epsilon}^* \cdot \mathbf{A}_s|}{|\mathbf{D}^{(0)}|^2}, \tag{86}$$

where $\hat{\epsilon}$ is the polarization vector of scattered radiation. In principle, we have solved the scattering problem for an arbitrary scatterer in an exact way. The caveat is that we do not know the field inside the scatterer.

11 Rayleigh-Gansin or Born approximation

(Lecture 6)

The integral equation derived above allows for a solution via perturbation series, where the internal field of the scatterer is first approximated by the incident field. What follows is the so-called Rayleigh-Gans approximation or the first Born approximation based on the corresponding integral equation in quantum mechanics.

Consider purely spatial fluctuations from an otherwise uniform medium and assume, in addition, that the fluctuations are linear, $\mathbf{D}(\mathbf{x}) = [\epsilon_m + \delta\epsilon(\mathbf{x})]\mathbf{E}(\mathbf{x})$, where $\delta\epsilon(\mathbf{x})$ is small compared to ϵ_m. The difference $\mathbf{D} - \epsilon_m\mathbf{E}$ showing up in the integral equation is proportional to $\delta\epsilon(\mathbf{x})$. In the lowest order,

$$\mathbf{D} - \epsilon_m \mathbf{E} \approx \frac{\delta\epsilon(\mathbf{x})}{\epsilon_m}\mathbf{D}^{(0)}. \tag{87}$$

Let the incident field be a plane wave so that $\mathbf{D}^{(0)}(\mathbf{x}) = \hat{\epsilon}_0 D_0 e^{ik\hat{n}_0\cdot\mathbf{x}}$. Then

$$\begin{aligned} \frac{\hat{\epsilon}^* \cdot \mathbf{A}_s^{(0)}}{D_0} &= \frac{k^2}{4\pi}\int d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}}\hat{\epsilon}^* \cdot \hat{\epsilon}_0 \frac{\delta\epsilon(\mathbf{x})}{\epsilon_m} \\ \mathbf{q} &= k(\hat{n}_0 - \hat{n}), \end{aligned} \tag{88}$$

the square of which, in absolute terms, gives the differential cross section. If the wavelength is much larger than the size of the region where $\delta\epsilon \neq 0$, the exponent in the integral can be set to unity. This results in the dipole approximation that was treated before for a small spherical particle.

Let us study the situation where the particle continues to be spherical and is located in free space. Thus, $\delta\epsilon \neq 0$ inside a sphere of radius a. We obtain

$$\begin{aligned}
\frac{\hat{\epsilon}^* \cdot \mathbf{A}_s^{(1)}}{D_0} &= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}'} \\
&= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \int_0^a dr' r'^2 e^{iqr'\cos\theta'} \\
&= \frac{k^2}{2}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\int_0^a dr' r'^2 /_{-1}^{1} \frac{1}{iqr'} e^{iqr'\mu'}, \qquad \mu' = \cos\theta' \\
&= \frac{k^2}{4\pi}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\frac{\delta\epsilon}{\epsilon_0}\frac{1}{iq}\left\{ /_0^a r' \frac{1}{iq}(e^{iqr'} + e^{-iqr'}) - \int_0^a dr' \frac{1}{iq}(e^{iqr'} + e^{-iqr'}) \right\} \\
&= k^2 \frac{\delta\epsilon}{\epsilon_0}(\hat{\epsilon}^* \cdot \hat{\epsilon}_0)\left(\frac{\sin qa - qa\cos qa}{q^3}\right), \qquad q = |\mathbf{q}| = \sqrt{2}k\sqrt{1 - \hat{n}\cdot\hat{n}_0}.
\end{aligned}$$

In the limit $a \to 0$, the term inside the parentheses approaches $a^3/3$ so that, for scatterers much smaller than the wavelength or for q approaching zero,

$$\lim_{q\to 0}\left(\frac{d\sigma}{d\Omega}\right)_{R-G} = k^4 a^6 \left|\frac{\delta\epsilon}{3\epsilon_0}\right|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2.$$

This is in agreement with the long-wavelength limit studied earlier. The integral $\int_S d^3\mathbf{x}' e^{i\mathbf{q}\cdot\mathbf{x}'}$ is commonly called the form factor.

12 Why is the sky blue?

In the present context, we can consider the blueness of the sky and redness of the sunrises and sunsets. Assume that the atmosphere is composed of individual molecules with locations $\mathbf{x}_j$ and that have the dipole moment $\mathbf{p}_j = \hat{\epsilon}_0 \gamma_{\mathrm{mol}} \mathbf{E}(\mathbf{x}_j)$, where γ_{mol} is the molecular polarizability. Then, the fluctuations of the electric permittivity can be described with the sum

$$\delta\epsilon(\mathbf{x}) = \epsilon_0 \sum_j \gamma_{mol}\delta(\mathbf{x} - \mathbf{x}_j)$$

The differential scattering cross section is of the form

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}|\gamma_{mol}|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2 F(\mathbf{q}),$$

where F is the structure factor treated before. For randomly distributed scatterers, $F(\mathbf{q})$ is directly the number of the molecules. For low-density gas, the relative permittivity is $\epsilon_r = \epsilon/\epsilon_0 = 1 + N\gamma_{\mathrm{mol}}$, where N is now the number of molecules in unit volume. The total scattering cross section as per molecule is

$$\sigma_s \approx \frac{k^4}{6\pi N^2}|\epsilon_r - 1|^2 \cong \frac{2k^4}{3\pi N^2}|m - 1|^2,$$

where m is the refractive index and $|m-1| \ll 1$.

When the radiation propagates a distance dx in the atmoshpere, the relative change in its intensity is $N\sigma dx$ and $I(x) = I_0 e^{-k_e x}$, where k_e is the so-called extinction coefficient:

$$k_e = N\sigma_s \cong \frac{2k^4}{3\pi N}|m-1|^2$$

This is called Rayleigh scattering that is incoherent scattering by gas molecules and other dipole scatterers, where each scatterer scatters radiation based on Rayleigh's $1/\lambda^4$ -law.

The $1/\lambda^4$ -law means that blue light is scattered much more efficiently than red light. In practice, this shows up so that blue color predominates when looking in directions other than the light source whereas, in the direction of the light source, red color predominates.

For visible light, $\lambda = 0.41 - 0.65\mu$m and, under normal conditions, $m - 1 \approx 2.78 \cdot 10^{-4}$. When $N = 2.69 \cdot 10^{19}$ molecules/cm^3, we obtain for the mean free path $1/k_e$ =30, 77, and 188 km at wavelengths 0.41 μm (violet), 0.52 μm (green), and 0.65 μm (red), respectively.

Polarization reaches its maximum of 75 % at the wavelength of 0.55 μm. The deviation from 100 % derives from multiple scattering (6 %), the anisotropy of the molecules (6 %), reflection from the surface (5 %, in particular, for green light in the case of vegetation), and aerosols (8 %).

13 Mie scattering, or scattering by a spherical particle

(Lecture 7)

An exact solution for scattering by electromagnetic waves by a spherical particle was presented by Mie and this kind of scattering is commonly called Mie scattering. Lately, the contribution by Lorenz has also been recognized, but his solution was not based on Maxwell's equations.

The solution of the scattering problem is composed of several fundamental stages. To start with, the scalar Helmholtz equation is solved in spherical coordinates, introducing the spherical harmonics and Bessel, Neumann, and Hankel special functions of fractional order (the so-called spherical Bessel functions, etc.).

In solving the vector Helmholtz wave equation, a general expansion in electric and magnetic multipoles is introduced and, in particular, the vector spherical harmonics. The energy and angular distributions of multipole fields are illustrated with examples, underscoring the power of the multipole analysis. To cope with the boundary conditions in the spherical geometry, the original incident plane wave field must be presented as a multipole expansion.

The actual scattering problem for a spherical particle can then be solved in a straightforward way. With the help of the multipole exansion, we can have a look at the boundary conditions for a nonspherical particle. In this case, the coefficients of the vector spherical harmonics can no longer be obtained analytically.

14 Scalar wave equation in spherical geometry

In order to prepare for the treatment of the vector wave equation, we consider the scalar wave equation for scalar field $\Psi(\mathbf{x}, t)$,

$$\nabla^2 \Psi(\mathbf{x}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(\mathbf{x}, t) = 0 \qquad (89)$$

We can Fourier-transform the wave equation with respect to time,

$$\Psi(\mathbf{x},t) = \int_{-\infty}^{\infty} d\omega \Psi(\mathbf{x},\omega) e^{-i\omega t}, \tag{90}$$

in which case each Fourier-component fulfils the wave equation

$$(\nabla^2 + k^2)\Psi(\mathbf{x},\omega) = 0, \qquad k^2 = \omega^2/c^2 \tag{91}$$

In the case of a single small particle, it is advantageous to search for the solution of the wave equation in the spherical coordinate system. Scattering extends to the full solid angle 4π and the small particle is located in a constrained region near the origin. In the spherical coordinates r, θ, φ, the wave equation is of the form (see Arfken, Jackson)

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \Psi}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial \Psi}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Psi}{\partial \varphi^2} + k^2\Psi = 0 \tag{92}$$

The scalar wave equation can be solved by separating the variables so that the part including the angular coordinates is represented by the scalar spherical harmonics functions and the part including the radial dependence is represented by the spherical Bessel, Neumann, and Hankel functions,

$$\Psi(\mathbf{x},\omega) = \sum_{l,m} f_{lm}(r) Y_{lm}(\theta,\varphi) \tag{93}$$

The radial part ($f_{lm}(r)$) fulfils its differential equation independently of the index m,

$$\left[\frac{d^2}{dr^2}\frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right] f_l(r) = 0. \tag{94}$$

By writing

$$f_l(r) = \frac{1}{\sqrt{r}} u_l(r) \tag{95}$$

we obtain

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{(l+\frac{1}{2})^2}{r^2}\right] u_l(r) = 0, \tag{96}$$

which is the Bessel equation with order $l + \frac{1}{2}$. Then, in the most general way,

$$\begin{aligned} f_{lm}(r) &= A_{lm} j_l(kr) + B_{lm} n_l(kr) \\ &= \tilde{A}_{lm} h_l^{(1)}(kr) + \tilde{B}_{lm} h_l^{(2)}(kr), \end{aligned}$$
$$h_l^{(1)}(x) = j_l(x) + i n_l(x), \qquad h_l^{(2)}(x) = j_l(x) - i n_l(x), \tag{97}$$

where $j_l, n_l, h_l^{(1)}$ and $h_l^{(2)}$ are the spherical Bessel, Neumann, and Hankel functions. For example,

$$\begin{aligned}
j_0(x) &= \frac{\sin x}{x}, \\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, \\
j_2(x) &= (\frac{3}{x^3} - \frac{1}{x})\sin x - \frac{3\cos x}{x^2}, \\
\\
n_0(x) &= -\frac{\cos x}{x}, \\
n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \\
n_2(x) &= -(\frac{3}{x^3} - \frac{1}{x})\cos x - \frac{3\sin x}{x^2}, \\
\\
h_0^{(1)}(x) &= \frac{e^{ix}}{ix}, \\
h_1^{(1)}(x) &= -\frac{e^{ix}}{x}(1 + \frac{i}{x}), \\
h_2^{(1)}(x) &= \frac{ie^{ix}}{x}(1 + \frac{3i}{x} - \frac{3}{x^2}). \qquad (98)
\end{aligned}$$

The functions j_l and n_l can be analytically generated using the so-called Rodriques' formulae

$$j_l(x) = (-x)^l(\frac{1}{x}\frac{d}{dx})^l(\frac{\sin x}{x}) \qquad (99)$$

$$n_l(x) = -(-x)^l(\frac{1}{x}\frac{d}{dx})^l(\frac{\cos x}{x}) \qquad (100)$$

In the limit $x << 1, l$, the functions can be calculated using the leading terms of their series expansions,

$$\begin{aligned}
j_l(x) &= \frac{x^l}{(2l+1)!!}\left(1 - \frac{x^2}{2(2l+3)} + \ldots\right), \\
n_l(x) &= -\frac{(2l-1)!!}{x^{l+1}}\left(1 - \frac{x^2}{2(1-2l)} + \ldots\right). \qquad (101)
\end{aligned}$$

Correspondingly, in the limit $x >> l$, we obtain

$$\begin{aligned}
j_l(x) &\approx \frac{1}{x}\sin(x - \frac{l\pi}{2}), \\
n_l(x) &\approx -\frac{1}{x}\cos(x - \frac{l\pi}{2}), \\
h_l^{(1)}(x) &\approx (-i)^{l+1}\frac{e^{ix}}{x}. \qquad (102)
\end{aligned}$$

The functions obey the recursive relations

$$\begin{aligned}
\frac{2l+1}{x}z_l(x) &= z_{l-1}(x) + z_{l+1}(x), \\
z_l'(x) &= \frac{1}{2l+1}[lz_{l-1}(x) - (l+1)z_{l+1}(x)], \\
\frac{d}{dx}[xz_l(x)] &= xz_{l-1}(x) - lz_l(x), \qquad (103)
\end{aligned}$$

where $z_l(x)$ can be any of the functions j_l, n_l, $h_l^{(1)}$ or $h_l^{(2)}$. In practical numerical computations, special attention needs to paid to numerical stability, for example, to the direction the recursive relations are utilized. The Wronskian determinants are, pair-wise,

$$W(j_l, n_l) = \frac{1}{i} W(j_l, h_l^{(1)}) = -W(n_l, h_l^{(1)}) = \frac{1}{x^2}. \tag{104}$$

Thus, the general solution of the scalar wave equation in spherical coordinates can be presented in the form

$$\Psi(\mathbf{x}) = \sum_{l,m} \left[A_{lm}^{(1)} h_l^{(1)}(kr) + A_{lm}^{(2)} h_l^{(2)}(kr) \right] Y_{lm}(\theta, \varphi) \tag{105}$$

that is, as a sum of outgoing and incoming waves.

Consider next the properties of the spherical-harmonics functions $Y_{lm}(\theta, \varphi)$. According to the definition,

$$\begin{aligned} Y_{lm}(\theta, \varphi) &= \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}, \\ l &= 0, 1, 2, \ldots, \\ m &= -l, -l+1, \ldots, 0, \ldots, l-1, l. \end{aligned} \tag{106}$$

The functions $P_l^m(x)$ are associated Legendre functions that can be derived from the Legendre polynomials $P_l(x)$ by the Rodriques' formula,

$$\begin{aligned} P_l^m(x) &= (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \\ &= (-1)^m \frac{1}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l. \end{aligned} \tag{107}$$

For $P_l^m(x)$, it is generally true that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{108}$$

so that

$$Y_{l,-m}(\theta, \varphi) = (-1)^m Y_{l,m}^*(\theta, \varphi) \tag{109}$$

The spherical-hamonics functions constitute a complete orthonormal set of functions,

$$\int_{4\pi} d\Omega Y_{l',m'}^*(\theta, \varphi) Y_{l,m}(\theta, \varphi) = \delta_{ll'} \delta_{mm'}, \tag{110}$$

with the closure relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{l,m}^*(\theta', \varphi') Y_{l,m}(\theta, \varphi) = \delta(\varphi - \varphi') \delta(\cos\theta - \cos\theta') \tag{111}$$

For example,

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}}, \\ Y_{10} &= \sqrt{\frac{3}{4\pi}} \cos\theta, \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}, \\ Y_{20} &= \sqrt{\frac{5}{4\pi}} (\frac{3}{2}\cos^2\theta - \frac{1}{2}), \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\varphi}, \\ Y_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{i2\varphi}. \end{aligned} \tag{112}$$

For example, the following recursive relations can be derived for the associated Legendre functions:

$$\begin{aligned}
& P_l^{m+1} - \frac{2mx}{\sqrt{1-x^2}} P_l^m + [l(l+1) - m(m-1)] P_l^{m-1} = 0 \\
(2l+1) x P_l^m &= (l+m) P_{l-1}^m + (l-m+1) P_{l+1}^m \\
(2l+1)\sqrt{1-x^2} P_l^m &= P_{l+1}^{m+1} - P_{l-1}^{m+1} \\
&= (l+m)(l+m-1) P_{l-1}^{m-1} - (l-m+1)(l-m+2) P_{l+1}^{m-1} \\
\sqrt{1-x^2} P_l^m &= \frac{1}{2} P_l^{m+1} - \frac{1}{2}(l+m)(l-m+1) P_l^{m-1}.
\end{aligned} \tag{113}$$

Let us study the spherical wave expansion of the Green's function corresponding to an outgoing wave. The Green's function fulfils the inhomogeneous wave equation

$$(\nabla^2 + k^2) G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \tag{114}$$

and is of the form

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \tag{115}$$

Let us write

$$G(\mathbf{x}, \mathbf{x}') = \sum_{lm} g_l(r, r') Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \tag{116}$$

and insert this expression into the partial differential equation above. Then, we obtain

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right] g_l = -\frac{1}{r^2}\delta(r - r') \tag{117}$$

with the following wave solution that is finite at the origin and and outgoing wave at infinity,

$$g_l(r, r') = A j_l(kr_<) h_l^{(1)}(kr_>) \tag{118}$$

where $r_> = \max(r, r')$ and $r_< = \min(r, r')$ and $A = ik$, so that the discontinuity of the derivatiuve is correct at $r = r'$. The spherical wave expansion of the Green's function is thus

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik \sum_{l=0}^{\infty} j_l(kr_<) h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \tag{119}$$

(Lecture 8)

In order to solve the vector wave equation, we return one more time to the angular part of the scalar wave equation and introduce useful auxiliary tools. The spherical harmonics are solutions of the following equation:

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_{lm} = l(l+1) Y_{lm},$$

which can be written in the form (cf. quantum mechanics)

$$L^2 Y_{lm} = l(l+1) Y_{lm}$$

where

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

$$\mathbf{L} = \frac{1}{i}(\mathbf{r} \times \nabla)$$

so that $\mathbf{L}$ is $\hbar^{-1}$ times the orbital impulse momentum operator in wave mechanics. $\mathbf{L}$ can be presented conveniently using the operators L_+, L_-, and L_z,

$$\begin{aligned} L_+ &= L_x + iL_y = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \\ L_- &= L_x - iL_y = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \quad (120) \\ L_z &= -i\frac{\partial}{\partial\varphi} \quad (121) \end{aligned}$$

$\mathbf{L}$ only operates on the angular variables and $\mathbf{r} \cdot \mathbf{L} = 0$. For what follows, it is useful to notice that, based on the recursive relations of the spherical harmonics,

$$\begin{aligned} L_+ Y_{lm} &= \sqrt{(l-m)(l+m+1)}Y_{l,m+1} \\ L_- Y_{lm} &= \sqrt{(l+m)(l-m+1)}Y_{l,m-1} \quad (122) \\ L_z Y_{lm} &= mY_{lm} \quad (123) \end{aligned}$$

In addition, $\mathbf{L}$, L^2 and ∇^2 fulfil the following commutation rules:

$$\begin{aligned} L^2\mathbf{L} &= \mathbf{L}L^2 \\ \mathbf{L} \times \mathbf{L} &= i\mathbf{L} \quad (124) \\ L_j\nabla^2 &= \nabla^2 L_j \quad (125) \end{aligned}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r) - \frac{L^2}{r^2}$$

15 Multipole expansions of electromagnetic fields

In free space, Maxwell's equations take the form (time dependence $e^{-i\omega t}$)

$$\begin{aligned} \nabla \times \mathbf{E} &= ik\zeta_0\mathbf{H}, \qquad \nabla \times \mathbf{H} = -ik\mathbf{E}/\zeta_0 \quad (126) \\ \nabla \cdot \mathbf{E} &= 0, \qquad \nabla \cdot \mathbf{H} = 0 \quad (127) \end{aligned}$$

where $k = \omega/c$. If the $\mathbf{E}$-field is eliminated, one obtains

$$(\nabla^2 + k^2)\mathbf{H} = 0, \qquad \nabla \cdot \mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{k\zeta_0}\nabla \times \mathbf{E}$$

Alternatively, eliminating the $\mathbf{H}$-field yields

$$(\nabla^2 + k^2)\mathbf{E} = 0, \qquad \nabla \cdot \mathbf{E} = 0$$

$$\mathbf{E} = \frac{i\zeta_0}{k}\nabla \times \mathbf{H}.$$

Both groups of three equations are equivalent to the original Maxwell's equations. We attempt to find multipole solutions for the vector fields $\mathbf{E}$ and $\mathbf{H}$. It is clear that each Cartesian component of $\mathbf{E}$ and $\mathbf{H}$ fulfil the scalar wave equation so that each component could be developed into series in multipoles of the scalar wave equation. However, the conditions about the sourceless nature of both $\mathbf{E}$ and $\mathbf{H}$ would be difficult to account for and it would be difficult to construct pure multipoles for the vector wave equation.

Instead, we start from the scalar quantity $\mathbf{r}\cdot\mathbf{A}$, where $\mathbf{A}$ is a regularly behaving vector field. First,

$$\nabla^2(\mathbf{r}\cdot\mathbf{A}) = \mathbf{r}\cdot(\nabla^2\mathbf{A}) + 2\nabla\cdot\mathbf{A}$$

so that

$$\nabla^2(\mathbf{r}\cdot\mathbf{E}) = \mathbf{r}\cdot(-k^2\mathbf{E}) \Leftrightarrow (\nabla^2 + k^2)(\mathbf{r}\cdot\mathbf{E}) = 0$$

and, in a corresponding way,

$$(\nabla^2 + k^2)(\mathbf{r}\cdot\mathbf{H}) = 0$$

Therefor, the general solution for $\mathbf{r}\cdot\mathbf{E}$ and $\mathbf{r}\cdot\mathbf{H}$:n can be presented as series of basis functions of the scalar wave equation.

We define the magnetic multipole of order (l,m) by the conditions

$$\begin{aligned}\mathbf{r}\cdot\mathbf{H}_{lm}^{(M)} &= \frac{l(l+1)}{k}g_l(kr)Y_{lm}(\theta,\varphi)\\ \mathbf{r}\cdot\mathbf{E}_{lm}^{(M)} &= 0\end{aligned} \tag{128}$$

where $g_l(kr) = A_l^{(1)}h_l^{(1)}(kr) + A_l^{(2)}h_l^{(2)}(kr)$ (the coefficient $l(l+1)/k$ has been introduced for convenience).

Now

$$\zeta_0 k\mathbf{r}\cdot\mathbf{H} = \frac{1}{i}\mathbf{r}\cdot(\nabla\times\mathbf{E}) = \frac{1}{i}(\mathbf{r}\times\nabla)\cdot\mathbf{E} = \mathbf{L}\cdot\mathbf{E}$$

where $\mathbf{L}$ is the operator showin g up when solving the scalar wave equation. When $\mathbf{r}\cdot\mathbf{H} = \mathbf{r}\cdot H_{lm}^{(M)}$, it must be true that

$$\mathbf{L}\cdot\mathbf{E}_{lm}^{(M)}(r,\theta,\varphi) = l(l+1)\zeta_0 g_l(kr)Y_{lm}(\theta,\varphi)$$

and

$$\mathbf{r}\cdot\mathbf{E}_{lm}^{(M)} = 0$$

Since $\mathbf{L}$ only operates on the angular variables (θ,φ), the r-dependence of $\mathbf{E}_{lm}^{(M)}$ is $g_l(kr)$. In order for $\mathbf{L}\cdot\mathbf{E}_{lm}^{(M)}$ to produce a pure $Y_{lm}(\theta,\varphi)$ angular dependence, $\mathbf{E}_{lm}^{(M)}$ need to be prepared using the L_z, L_+, and L_--operators so that, ultimately,

$$\begin{aligned}\mathbf{E}_{lm}^{(M)} &= \zeta_0 g_l(kr)\mathbf{L}Y_{lm}(\theta,\varphi)\\ \mathbf{H}_{lm}^{(M)} &= -\frac{1}{k\zeta_0}\nabla\times\mathbf{E}_{lm}^{(M)}\end{aligned} \tag{129}$$

This is the definition for the electromagnetic fields of the magnetic multipole of order (l,m). Occasionally, this is also called the transverse electric multipole (TE).

The electromagnetic fields of an electric or transverse magnetic (TM) multipole of order (l,m) follow from the conditions

$$\begin{aligned}\mathbf{r}\cdot\mathbf{E}_{lm}^{(E)} &= -\zeta_0\frac{l(l+1)}{k}f_l(kr)Y_{lm}(\theta,\varphi)\\ \mathbf{r}\cdot\mathbf{H}_{lm}^{(E)} &= 0\end{aligned}$$

and are of the form

$$\begin{aligned}\mathbf{H}_{lm}^{(E)} &= f_l(kr)\mathbf{L}Y_{lm}(\theta,\varphi)\\ \mathbf{E}_{lm}^{(E)} &= \frac{i\zeta_0}{k}\nabla\times\mathbf{H}_{lm}^{(E)}\end{aligned} \tag{130}$$

where the r-dependent part $f_l(kr)$ is again a combination of the spherical Hankel or Bessel and Neumann functions.

It can be shown that the electric and magnetic multipole fields constitute a complete vectorial set of solutions for Maxwell's equations in source-free space. In what follows, the terminology of electric and magnetic multipoles is being used as, physically, the sources are the electric charge density and the magnetic moment density, respectively.

In the consideration of vector spherical harmonics, the vector spherical harmonics functions $\mathbf{L}Y_{lm}$ assume a central role. For convenience, the vector functions are normalized so that the final vector spherical harmonics are

$$\mathbf{X}_{lm}(\theta,\varphi) \equiv \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}(\theta,\varphi)$$

We define $\mathbf{X}_{00} \equiv 0$, since spherically symmetric solutions to Maxwell's equations only exist in source-free space at the static limit $k \to 0$. For $\mathbf{X}_{lm}$, the following orthogonality relations can be ascertained,

$$\int_{(4\pi)} d\Omega \mathbf{X}^*_{l',m'}\cdot\mathbf{X}_{lm} = \delta_{ll'}\delta_{mm'}$$

$$\int_{(4\pi)} d\Omega \mathbf{X}^*_{l',m'}\cdot(\mathbf{r}\times\mathbf{X}_{lm}) = 0$$

The proof is left for an exercise.

The general solution for Maxwell's equations can now be written as an expansion of electric and magnetic multipoles,

$$\mathbf{H} = \sum_{l,m}\left[a_E(l,m)f_l(kr)\mathbf{X}_{lm} - \frac{i}{k}a_M(l,m)\nabla\times g_l(kr)\mathbf{X}_{lm}\right]$$

$$\mathbf{E} = \zeta_0\sum_{l,m}\left[\frac{i}{k}a_E(l,m)\nabla\times f_l(kr)\mathbf{X}_{lm} + a_M(l,m)g_l(kr)\mathbf{X}_{lm}\right]$$

where the coefficients $a_E(l,m)$ and $a_M(l,m)$ give the amount of electric and magnetic multipoles of order (l,m). The functions $f_l(kr)$ and $g_l(kr)$ are linear combinations of $h_l^{(1,2)}$ or j_l and n_l. The coefficients $a_E(l,m)$ and $a_M(l,m)$ are determined by the sources and the boundary conditions. Explicitly, this is seen by the scalar quantitites $\mathbf{r}\cdot\mathbf{H}$ and $\mathbf{r}\cdot\mathbf{E}$ being sufficient to determine the unknown coefficients:

$$a_M(l,m)g_l(kr) = \frac{k}{\sqrt{l(l+1)}}\int_{(4\pi)} d\Omega Y^*_{lm}\mathbf{r}\cdot\mathbf{H}$$

$$\zeta_0 a_E(l,m)f_l(kr) = -\frac{k}{\sqrt{l(l+1)}}\int_{(4\pi)} d\Omega Y^*_{lm}\mathbf{r}\cdot\mathbf{E}$$

When $\mathbf{r}\cdot\mathbf{H}$ and $\mathbf{r}\cdot\mathbf{E}$ are known at two distances differing from one another in the source-free region, the fields can be unambiguously determined, all the way to the mutual proportions of the two parts in the radial dependences f_l and g_l.

16 Energy in multipole fields

Consider multipole fields in the near zone $kr << 1$. Then, the leading contribution derives from the Neumann function so that $f_l \propto n_l$; assume that the coefficient of the multipole in question differs from zero. We obtain

$$\mathbf{H}_{lm}^{(E)} \to -\frac{k}{l}\mathbf{L}\frac{Y_{lm}}{r^{l+1}}$$

where the factor $-k/l$ is introduced for convenience. In order to calculate the electric field, we must compute the curl of the right-hand side of the equation; in doing this, we make use of the result

$$i\nabla \times \mathbf{L} = \mathbf{r}\nabla^2 - \nabla(1 + r\frac{\partial}{\partial r})$$

The electric field is

$$\mathbf{E}_{lm}^{(E)} \to -\frac{i}{l}\zeta_0\nabla \times \mathbf{L}\left(\frac{Y_{lm}}{r^{l+1}}\right)$$

and, since Y_{lm}/r^{l+1} obeys the Laplace equation,

$$\nabla^2\frac{Y_{lm}}{r^{l+1}} = 0$$

and, for the electric field, we obtain

$$\mathbf{E}_{lm}^{(E)} \to -\zeta_0\nabla\frac{Y_{lm}}{r^{l+1}}$$

which is the multipole field of electrostatics. The magnetic field $\mathbf{H}_{lm}^{(E)}$ is smaller than $\mathbf{E}_{lm}^{(E)}/\zeta_0$ by a factor of kr so that, in the near zone, the magnetic field of the electric multipole is considerably smaller than the electric field (cf. earlier treatment for an electric dipole moment).

By exchanging $\mathbf{E}$ and $\mathbf{H}$ in the previous analysis, we can obtain the case of the magnetic multipole,

$$\mathbf{E}^{(E)} \to -\zeta_0\mathbf{H}^{(M)}, \qquad \mathbf{H}^{(E)} \to \mathbf{E}^{(M)}/\zeta_0$$

Let us study the multipole fields in the far zone $kr >> 1$. The fields depend on the boundary conditions set and, as an example, we study outgoing waves that are applicable to the case of radiation by a localized source, too. Now $f_l(kr) \propto h_l^{(1)}(kr)$ and

$$\mathbf{H}_{lm}^{(E)} \to (-i)^{l+1}\frac{e^{ikr}}{kr}\mathbf{L}Y_{lm}$$

and the electric field is of the form

$$\mathbf{E}_{lm}^{(E)} = \zeta_0\frac{(-i)^l}{k^2}\left[\nabla\left(\frac{e^{ikr}}{r}\right) \times \mathbf{L}Y_{lm} + \frac{e^{ikr}}{r}\nabla \times \mathbf{L}Y_{lm}\right]$$

The asymptotic form of $h_l^{(1)}$ is already used in the expression of the electric field so only factors proportional to r^{-1} can be conserved in the expressions. By using, again, the aforedescribed result for $\nabla \times \mathbf{L}$, we obtain

$$\mathbf{E}_{lm}^{(E)} = -\zeta_0(-i)^{l+1}\frac{e^{ikr}}{kr}\left[\mathbf{n} \times \mathbf{L}Y_{lm} - \frac{1}{k}(\mathbf{r}\nabla^2 - \nabla)Y_{lm}\right]$$

where $\mathbf{n} = \mathbf{r}/r$. The second term on the right is of the order of $1/kr$ and can be omitted from the expression in parentheses in the limit $kr >> 1$. We obtain

$$\mathbf{E}_{lm}^{(E)} = \zeta_0 \mathbf{H}_{lm}^{(E)} \times \mathbf{n}$$

where $\mathbf{H}_{lm}^{(E)}$ is asymptotic form given above.

The multipole fields can be utilized in the computation of the energy transported by the radiation. As an example, consider the linear superposition of electric multipoles of order (l, m) with different values of m, when l is kept constant. The fields are of the form

$$\mathbf{H}_l = \sum_m a_E(l, m) \mathbf{X}_{lm} h_l^{(1)}(kr) e^{-i\omega t}$$

$$\mathbf{E}_l = \frac{i}{k} \zeta_0 \nabla \times \mathbf{H}_l$$

The time-averaged energy density of time-harmonic fields is

$$u = \frac{\epsilon_0}{4} (\mathbf{E} \cdot \mathbf{E}^* + \zeta_0^2 \mathbf{H} \cdot \mathbf{H}^*)$$

In the far zone, the two terms of the energy density are equal and, in a spherical shell $r, r + dr$, there is the following amount of energy:

$$dU = \frac{\mu_0 dr}{2k^2} \sum_{m,m'} a_E^*(l, m') a_E(l, m) \int_{(4\pi)} d\Omega \mathbf{X}_{lm'}^* \cdot \mathbf{X}_{lm}$$

and, due to the orthogonality,

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2} \sum_m |a_E(l, m)|^2$$

which is independent of r. In the general case of electric and magnetic multipoles, the summation goes over both l and m and $|a_E|^2 \to |a_E|^2 + |a_M|^2$. In the spherical shell in the radiation zone, the total energy is thus the incoherent sum over all multipoles.

17 Angular dependence of multipole radiation

For an arbitrary localized source distribution, the fields in the radiation zone are obtained as a superposition

$$\mathbf{H} \to \frac{e^{ikr - i\omega t}}{kr} \sum_{lm} (-i)^{l+1} \left[a_E(l, m) \mathbf{X}_{lm} + a_M(l, m) \mathbf{n} \times \mathbf{X}_{lm} \right]$$

$$\mathbf{E} \to \zeta_0 \mathbf{H} \times \mathbf{n}, \qquad \mathbf{n} = \frac{\mathbf{r}}{r}$$

where the coefficients $a_E(l, m)$ and $a_M(l, m)$ are connected to the properties of the source. The time-averaged power as per solid angle is

$$\frac{dP}{d\Omega} = \frac{\zeta_0}{2k^2} | \sum_{l,m} (-i)^{l+1} \left[a_E(l, m) \mathbf{X}_{lm} \times \mathbf{n} + a_M(l, m) \mathbf{X}_{lm} \right] |^2$$

The dimension of the expression inside the || marks is the dimension of the magnetic field. The directions of the vectors determine the polarization of the radiation. The angular dependence of the electric and magnetic multipoles of order (l,m) coincide but the polarizations are perpendicular to one another. It then follows that the order of the multipoles can be determined from the angualr dependence but the electric or magnetic nature can be determined only after a polarization measurement.

The angular dependence of a pure multipole of order (l,m) is

$$\frac{dP(l,m)}{d\Omega} = \frac{\zeta_0}{2k^2}|a(l,m)|^2|\mathbf{X}_{lm}|^2$$

Based on the definition of $\mathbf{X}_{lm}$ and the rules of calculus for L_+ and L_-,

$$\frac{dP(l,m)}{d\Omega} = \frac{\zeta_0|a(l,m)|^2}{2k^2 l(l+1)}\left[\frac{1}{2}(l-m)(l+m+1)|Y_{l,m+1}|^2 + \frac{1}{2}(l+m)(l-m+1)|Y_{l,m-1}|^2 + m^2|Y_{lm}|^2\right]$$

Examples of angular dependences $|\mathbf{X}_{lm}(\theta,\varphi)|^2$ follow:
Dipole: (dipole vibrating in the direction of the z-axis)

$$l=1, m=0 \qquad \frac{3}{8\pi}\sin^2\theta$$

(dipoles vibrating along the x- and y-axes with a phase difference $\frac{\pi}{2}$)

$$l=1, m=\pm 1 \qquad \frac{3}{16\pi}(1+\cos^2\theta)$$

Quadrupole:

$$l=2, m=0 \qquad \frac{15}{8\pi}\sin^2\theta\cos^2\theta$$

$$l=2, m=\pm 1 \qquad \frac{5}{16\pi}(1-3\cos^2\theta+4\cos^4\theta)$$

$$l=2, m=\pm 2 \qquad \frac{5}{16\pi}(1-\cos^4\theta)$$

With the help of the addition rule for spherical harmonics, one can show that

$$\sum_{m=-l}^{l}|\mathbf{X}_{lm}(\theta,\varphi)|^2 = \frac{2l+1}{4\pi}$$

so that the vector spherical harmonics have their own addition rule. This implies that the angular dependence of radiation is isotropic when the source is composed of incoherently radiating multipoles of order l with coefficients $a(l,m)$ independent of m.

The total power radiated by a pure multipole can be obtained via integration and, due to the orthonormality,

$$P(l,m) = \frac{\zeta_0}{2k^2}|a(l,m)|^2$$

For a general source, the angular dependence follows from the coherent that has been shown above. When computing the total power, due to the orthgonality, the interference terms do not contribute, and the total power is the incoherent sum of the contributions from the different multipoles:

$$P = \frac{\zeta_0}{2k^2}\sum_{l,m}\left[|a_E(l,m)|^2 + |a_M(l,m)|^2\right]$$

18 Vector spherical harmonics expansion for a plane wave

(Lecture 9)

In the scattering and absorption problem for localized objects, we need the vector spherical-harmonics expansion of the electromagnetic plane wave.

Let us first derive the spherical-harmonics expansion of the scalar plane wave using the Green's function $e^{ikR}/4\pi R$:

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

In the limit $|\mathbf{x}'| \to \infty$ pätee $|\mathbf{x}-\mathbf{x}'| \approx r' - \frac{\mathbf{x}'}{r'}\cdot\mathbf{x}$ ja $r_> = r'$, $r_< = r$ and $h_l^{(1)}(kr_>) \approx (-i)^{l+1}\frac{e^{ikr_>}}{kr_>}$. Then,

$$\frac{e^{ikr'}}{4\pi r'}e^{-ik\frac{\mathbf{x}'}{r'}\cdot\mathbf{x}} = ik\frac{e^{ikr'}}{kr'}\sum_{lm}(-i)^{l+1}j_l(kr)Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

After reorganizing the terms and taking the complex conjugate,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\sum_{l=0}^{\infty} i^l j_l(kr)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

where $\mathbf{k}$ id the wave vector k,θ',φ'. According to the addition rule for the spherical harmonics,

$$P_l(\cos\gamma) \;=\; \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \tag{131}$$

where γ is the great-circle angle between (θ,φ) and (θ',φ'). With the help of the addition rule,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\gamma)$$

where γ is now the angle between $\mathbf{k}$ and $\mathbf{x}$. Moreover,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{l=0}^{\infty} i^l\sqrt{4\pi(2l+1)}j_l(kr)Y_{l0}(\gamma)$$

In what follows, we develop the corresponding expansion for a circularly polarized vector plane wave

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= (\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)e^{ikz}\\ c\mathbf{B}(\mathbf{x}) &= \hat{\epsilon}_3\times\mathbf{E} = \mp i\mathbf{E}(\mathbf{x}),\end{aligned}$$

where $\hat{\epsilon}_3 = \hat{e}_z$. Since the plane wave is finite everywhere, we write its multipole expansion using the regular radial functions $j_l(kr)$:

$$\mathbf{E}(\mathbf{x}) = \sum_{lm}\left[a_\pm(l,m)j_l(kr)\mathbf{X}_{lm} + \frac{i}{k}b_\pm(l,m)\nabla\times j_l(kr)\mathbf{X}_{lm}\right],$$

$$c\mathbf{B}(\mathbf{x}) = \sum_{lm} \left[\frac{-i}{k} a_\pm(l,m) \nabla \times j_l(kr)\mathbf{X}_{lm} + b_\pm(l,m) j_l(kr)\mathbf{X}_{lm} \right].$$

When deriving the coefficients $a_\pm(l,m)$ and $b_\pm(l,m)$, we make use of the orthogonality properties of the vector spherical-harmonics functions $\mathbf{X}_{lm}$ that we summarize in the following:

$$\int_{4\pi} d\Omega [f_l(r)\mathbf{X}_{l'm'}]^* \cdot [g_l(r)\mathbf{X}_{lm}] = f_l^* g_l \delta_{ll'} \delta_{mm'},$$

$$\int_{4\pi} d\Omega [f_l(r)\mathbf{X}_{l'm'}]^* \cdot [\nabla \times g_l(r)\mathbf{X}_{lm}] = 0,$$

$$\frac{1}{k^2} \int_{4\pi} d\Omega [\nabla \times f_l(r)\mathbf{X}_{l'm'}]^* \cdot [\nabla \times g_l(r)\mathbf{X}_{lm}] = \delta_{ll'}\delta_{mm'} \{ f_l^* g_l + \frac{1}{k^2 r^2} \frac{d}{dr} [r f_l^* \frac{d}{dr}(r g_l)] \}.$$

Above, $f_l(r)$ and $g_l(r)$ are, again, linear combinations of spherical Bessel, Neumann, and Hankel functions. The second and third relation follow from the results

$$i\nabla \times \mathbf{L} = \mathbf{r}\nabla^2 - \nabla(1 + r\frac{\partial}{\partial r}),$$

$$\nabla = \frac{\mathbf{r}}{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\mathbf{r} \times \mathbf{L},$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2} \right] f_l(r) = 0,$$

and the proof is left for an exercise.

The coefficients $a_\pm(l,m)$ and $b_\pm(l,m)$ are determined via the scalar product between $\mathbf{X}^*_{lm}$ and the multipole expansions of the fields and the integration over the angular variables:

$$a_\pm(l,m) j_l(kr) = \int_{(4\pi)} d\Omega \mathbf{X}^*_{lm} \cdot \mathbf{E}(\mathbf{x}),$$

$$b_\pm(l,m) j_l(kr) = \int_{(4\pi)} d\Omega \mathbf{X}^*_{lm} \cdot \mathbf{B}(\mathbf{x}).$$

Explicitly,

$$a_\pm(l,m) j_l(kr) = \int_{(4\pi)} d\Omega (\hat{\epsilon}_1 \pm i\hat{\epsilon}_2) \cdot \frac{\mathbf{L}^* Y^*_{lm}}{\sqrt{l(l+1)}} e^{ikz} = \int_{(4\pi)} d\Omega \frac{(L_\mp Y_{lm})^*}{\sqrt{l(l+1)}} e^{ikz}$$

where the operators $L_\mp$ have been defined earlier. Here we recognize the strength of these operators together with the analyses based on circular polarization, as it follows, in a straightforward way, that

$$a_\pm(l,m) j_l(kr) = \frac{\sqrt{(l \pm m)(l \mp m + 1)}}{\sqrt{l(l+1)}} \int_{(4\pi)} d\Omega Y_{l,m\mp 1} e^{ikz}$$

and, by incorporating the expansion for e^{ikz},

$$\begin{aligned} a_\pm(l,m) &= i^l \sqrt{4\pi(2l+1)} \delta_{m,\pm 1} \\ b_\pm(l,m) &= \mp i a_\pm(l,m) \end{aligned}$$

The multipole expansion of the circularly polarized vector plane wave is thus

$$\mathbf{E}(\mathbf{x}) = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \Big[j_l(kr)\mathbf{X}_{l,\pm 1} \pm \frac{1}{k} \nabla \times j_l(kr)\mathbf{X}_{l,\pm 1} \Big],$$

$$c\mathbf{B}(\mathbf{x}) = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}\Big[-\frac{1}{k}\nabla\times j_l(kr)\mathbf{X}_{l,\pm1} \mp i j_l(kr)\mathbf{X}_{l,\pm1}\Big].$$

The expansions for plane waves linearly polarized in the directions of the vectors $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ can be obtained from the previous results,

$$\hat{\epsilon}_1 e^{ikz} = \sum_{l=1}^{\infty} i^l \sqrt{\pi(2l+1)}\Big[j_l(kr)[\mathbf{X}_{l,1}+\mathbf{X}_{l,-1}] + \frac{1}{k}\nabla\times j_l(kr)[\mathbf{X}_{l,1}-\mathbf{X}_{l,-1}]\Big],$$

$$\hat{\epsilon}_2 e^{ikz} = \sum_{l=1}^{\infty} i^{l-1} \sqrt{\pi(2l+1)}\Big[j_l(kr)[\mathbf{X}_{l,1}-\mathbf{X}_{l,-1}] + \frac{1}{k}\nabla\times j_l(kr)[\mathbf{X}_{l,1}+\mathbf{X}_{l,-1}]\Big].$$

19 Scattering by a spherical particle

Outside the spherical particle, the electromagnetic field is a superposition of the original incident field and the scattered field:

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= \mathbf{E}_i(\mathbf{x}) + \mathbf{E}_s(\mathbf{x}),\\ \mathbf{B}(\mathbf{x}) &= \mathbf{B}_i(\mathbf{x}) + \mathbf{B}_s(\mathbf{x}).\end{aligned}$$

where the plane-wave fields $\mathbf{E}_i, \mathbf{B}_i$ have been given earlier. Since the scattered fields are, asymptotically at the infinity, outgoing waves, they must be of the form

$$\mathbf{E}_s = \frac{1}{2}\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}\Big[\alpha_\pm(l) h_l^{(1)}(kr)\mathbf{X}_{l,\pm1} \pm \frac{\beta_\pm(l)}{k}\nabla\times h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\Big],$$

$$c\mathbf{B}_s = \frac{1}{2}\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}\Big[\frac{-i\alpha_\pm(l)}{k}\nabla\times h_l^{(1)}(kr)\mathbf{X}_{l,\pm1} \mp i\beta_\pm(l) h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\Big],$$

where the coefficients $\alpha_\pm(l)$ and $\beta_\pm(l)$ are determined from the boundary conditions on the surface of the particle. Generally, the expansions include a summation over the order m but, in the case of the spherical symmetry, only the multipoles $m = \pm1$ contribute to the expansion.

With the help of the coefficients $\alpha(l)$ and $\beta(l)$, we obtain the total scattered and absorbed power. The scattered power follows from the integration of the outward directed component of the scattered-field Poynting vector over the spherical surface. The absorbed power follows from the integration of the inward directed Poynting-vector component of the total field. By reorganizing triple scalar products, we obtain

$$\begin{aligned}P_s &= -\frac{a^2}{2\mu_0}\Re\int_{(4\pi)} d\Omega\mathbf{E}_s\cdot(\mathbf{n}\times\mathbf{B}_s)\\ P_a &= \frac{a^2}{2\mu_0}\Re\int_{(4\pi)} d\Omega\mathbf{E}\cdot(\mathbf{n}\times\mathbf{B})\end{aligned}$$

where $\mathbf{n} = \hat{e}_r$. Only the transverse field components contribute to the values of the integrals.

Explicitly,

$$
\begin{aligned}
\mathbf{X}_{lm}(\theta,\varphi) &= \frac{-m}{\sqrt{l(l+1)}\sin\theta}\Bigg[\hat{e}_\theta Y_{lm}(\theta,\varphi) - \\
&\quad i\hat{e}_\varphi\Bigg[\sqrt{\frac{(l^2-m^2)}{(2l-1)(2l+1)}}Y_{l-1,m}(\theta,\varphi) + \sqrt{\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}}Y_{l+1,m}(\theta,\varphi)\Bigg]\Bigg], \\
\frac{1}{k}\nabla\times Z_l(kr)\mathbf{X}_{lm}(\theta,\varphi) &= \frac{i\hat{e}_r\sqrt{l(l+1)}}{kr}Z_l(kr)Y_{lm}(\theta,\varphi) + \\
&\quad \frac{1}{kr}\frac{d}{d(kr)}[krZ_l(kr)]\hat{e}_r\times\mathbf{X}_{lm}(\theta,\varphi), \qquad (132)
\end{aligned}
$$

where we see that $\mathbf{X}_{lm}$ is transverse and that, in the latter term, the transverse component is proportional to $\hat{e}_r \times \mathbf{X}_{lm}$. Upon inserting the multipole expansions of the fields into the expressions for the power, we obtain a double summation over l and l' of relations that are of the form $\mathbf{X}^*_{lm}\cdot\mathbf{X}_{l'm'}$, $\mathbf{X}^*_{lm}\cdot(\hat{e}_r\times\mathbf{X}_{l'm'})$, $(\hat{e}_r\times\mathbf{X}_{lm})\cdot(\hat{e}_r\times\mathbf{X}_{l'm'})$. Integration over the angles removes the other summation. Each remaining term in the sum contains a product of spherical Bessel's functions and/or their derivatives—these products can be eliminated with the help of the Wronskian determinants. The cross sections of scattering and absorption are finally (exercise)

$$
\begin{aligned}
\sigma_s &= \frac{\pi}{2k^2}\sum_l (2l+1)[|\alpha(l)|^2 + |\beta(l)|^2], \\
\sigma_a &= \frac{\pi}{2k^2}\sum_l (2l+1)[2 - |\alpha(l)+1|^2 - |\beta(l)+1|^2],
\end{aligned}
$$

and the exticntion cross section is the sum of the two above,

$$
\sigma_t = -\frac{\pi}{k^2}\sum_l (2l+1)\Re[\alpha(l)+\beta(l)]
$$

The differential scattering cross section is

$$
\frac{d\sigma_s}{d\Omega} = \frac{\pi}{2k^2}|\sum_l \sqrt{2l+1}[\alpha_\pm(l)\mathbf{X}_{l,\pm1} \pm i\beta_\pm(l)\hat{e}_r\times\mathbf{X}_{l,\pm1}]|^2,
$$

for the original polarization state $\hat{\epsilon}_1 \pm i\hat{\epsilon}_2$. Thereby, scattered radiation is in general elliptically polarized.

Let us study the solution for the coefficients $\alpha(l)$ and $\beta(l)$ based on the boundary conditions and start by defining the fields. The original plane-wave field is $\mathbf{E}_i = (\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)e^{ikz}$:

$$
\begin{aligned}
\mathbf{E}_i(\mathbf{x}) &= \sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[j_l(kr)\mathbf{X}_{l,\pm1} \pm \frac{1}{k}\nabla\times j_l(kr)\mathbf{X}_{l,\pm1}\right], \\
\mathbf{H}_i(\mathbf{x}) &= \frac{1}{\mu_0}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[\frac{-i}{k}\nabla\times j_l(kr)\mathbf{X}_{l,\pm1} \mp ij_l(kr)\mathbf{X}_{l,\pm1}\right].
\end{aligned}
$$

The scattered field takes the form

$$
\mathbf{E}_s(\mathbf{x}) = \frac{1}{2}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[\alpha_\pm(l)h_l^{(1)}(kr)\mathbf{X}_{l,\pm1} \pm \frac{\beta_\pm(l)}{k}\nabla\times h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\right],
$$

$$\mathbf{H}_s(\mathbf{x}) = \frac{1}{2\mu_0 c}\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}\left[\frac{-i\alpha_\pm(l)}{k}\nabla\times h_l^{(1)}(kr)\mathbf{X}_{l,\pm1} \mp i\beta_\pm(l)h_l^{(1)}(kr)\mathbf{X}_{l,\pm1}\right].$$

The internal field is

$$\mathbf{E}_t(\mathbf{x}) = \frac{1}{2}\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}\left[\eta_\pm(l)j_l(k_t r)\mathbf{X}_{l,\pm1} \pm \frac{\zeta_\pm(l)}{k_t}\nabla\times j_l(k_t r)\mathbf{X}_{l,\pm1}\right],$$

$$\mathbf{H}_t(\mathbf{x}) = \frac{1}{2\mu c}\sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}\left[\frac{-i\eta_\pm(l)}{k_t}\nabla\times j_l(k_t r)\mathbf{X}_{l,\pm1} \mp i\zeta_\pm(l)j_l(k_t r)\mathbf{X}_{l,\pm1}\right].$$

The boundary conditions on the surface of the spherical particle (radius a) are

$$\begin{aligned}\hat{e}_r\times[\mathbf{E}_i+\mathbf{E}_s-\mathbf{E}_t]|_{r=a} &= 0,\\ \hat{e}_r\times[\mathbf{H}_i+\mathbf{H}_s-\mathbf{H}_t]|_{r=a} &= 0.\end{aligned}$$

As found earlier,

$$\hat{e}_r\times\left[\frac{1}{k}\nabla\times Z_l(kr)\mathbf{X}_{lm}(\theta,\varphi)\right] = -\frac{1}{kr}[krZ_l(kr)]'\mathbf{X}_{lm}(\theta,\varphi),$$

so that, due to the orthogonality of the functions $\mathbf{X}_{lm}$ and $\hat{e}_r\times\mathbf{X}_{lm}$, the boundary conditions simplify into the form,

$$j_l(ka)+\frac{1}{2}\alpha_\pm(l)h_l^{(1)}(ka)-\frac{1}{2}\eta_\pm(l)j_l(k_t a)=0,$$

$$\pm\frac{1}{ka}[kaj_l(ka)]'\mp\frac{1}{2}\beta_\pm(l)\cdot\frac{1}{ka}[kah_l^{(1)}(ka)]'\mp\zeta_\pm(l)\frac{1}{k_t a}[k_t a j_l(k_t a)]'=0,$$

$$\frac{1}{ka}[kaj_l(ka)]'+\frac{1}{2}\alpha_\pm(l)\frac{1}{ka}[kah_l^{(1)}(ka)]'-\frac{\mu_0}{2\mu}\cdot\eta_\pm(l)\frac{1}{k_t a}[k_t a j_l(k_t a)]'=0,$$

$$\mp j_l(ka)\mp\frac{1}{2}\beta_\pm(l)h_l^{(1)}(ka)\pm\frac{1}{2}\frac{\mu_0}{\mu}\zeta_\pm(l)j_l(k_t a)=0.$$

With the help of the Riccati-Bessel functions and by writing $x=ka$, $k_t a = mx$, we obtain

$$\psi_l(x)+\frac{1}{2}\alpha_\pm(l)\xi_l(x)-\frac{1}{2}\eta_\pm(l)\frac{1}{m}\psi_l(mx)=0,$$

$$\mp\psi_l'(x)\mp\frac{1}{2}\beta_\pm(l)\xi_l'(x)\mp\frac{1}{2}\zeta_\pm(l)\frac{1}{m}\psi_l'(mx)=0,$$

$$\psi_l'(x)+\frac{1}{2}\alpha_\pm(l)\xi_l'(x)-\frac{\mu_0}{2\mu}\eta_\pm(l)\frac{1}{m}\psi_l'(mx)=0,$$

$$\mp\psi_l(x)\mp\frac{1}{2}\beta_\pm(l)\xi_l(x)\pm\frac{\mu_0}{2\mu}\zeta_\pm(l)\frac{1}{m}\psi_l(mx)=0,$$

that leads to

$$\alpha_\pm(l) = -2\frac{\frac{\mu_0}{\mu}\psi_l(x)\psi_l'(mx)-\psi_l'(x)\psi_l(mx)}{\frac{\mu_0}{\mu}\xi_l(x)\psi_l'(mx)-\xi_l'(x)\psi_l(mx)},$$

$$\beta_\pm(l) = -2\frac{\mp\frac{\mu_0}{\mu}\psi_l'(x)\psi_l(mx)\mp\psi_l(x)\psi_l'(mx)}{\mp\frac{\mu_0}{\mu}\xi_l'(x)\psi_l(mx)\mp\xi_l(x)\psi_l'(mx)},$$

$$\eta_\pm(l) = 2\frac{\psi_l(x)\xi_l'(x)-\psi_l'(x)\xi_l(x)}{\frac{1}{m}\psi_l(mx)\xi_l'(x)+\frac{1}{m}\frac{\mu_0}{\mu}\psi_l'(mx)\xi_l(x)},$$

$$\zeta_{\pm}(l) = 2\frac{\mp\psi_l'(x)\xi_l(x)pm\psi_l(x)\xi_l'(x)}{\frac{1}{m}\psi_l'(mx)\xi_l(x) \mp \frac{1}{m}\frac{\mu_0}{\mu}\psi_l(mx)\xi_l'(x)}.$$

(Lecture 10)

We have solved for the scattered field completely for two circular polarization states of the original field $\epsilon_1 \pm i\epsilon_2$. We thus know the elements $S_j^{(c)}$ (j=1,2,3,4) of the amplitude scattering matrix that relates the original field to the scattered field in the circular-polarization representation.

$$\begin{pmatrix} E_{s-} \\ E_{s+} \end{pmatrix} = \frac{e^{ik(r-z)}}{-ikr} \begin{pmatrix} S_2^{(c)} & S_3^{(c)} \\ S_4^{(c)} & S_1^{(c)} \end{pmatrix} \begin{pmatrix} E_{i-} \\ E_{i+} \end{pmatrix} \tag{133}$$

The amplitude scattering matrix elements of the circular-polarization representation relate linearly to the commonly used ones of the linear-polarization representation.

$$\begin{pmatrix} S_1^{(c)} \\ S_2^{(c)} \\ S_3^{(c)} \\ S_4^{(c)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & -i & i \\ -1 & 1 & i & i \\ -1 & 1 & -i & -i \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}, \tag{134}$$

where, in the case of Lorenz-Mie scattering ($S_3 = S_4 = 0$),

$$\left.\begin{array}{c} S_1^{(c)} = (S_1 + S_2)/2 \\ S_2^{(c)} = (S_1 + S_2)/2 = S_1^{(c)} \\ S_3^{(c)} = (-S_1 + S_2)/2 \\ S_4^{(c)} = S_3^{(c)} \end{array}\right\} \Rightarrow \left\{ \begin{array}{c} S_1 = S_1^{(c)} - S_3^{(c)} \\ S_2 = S_1^{(c)} + S_3^{(c)} \end{array}\right.$$

The complete scattering matrix follows now from the elements S_1, S_2 in a standard manner. The elements $S_1^{(c)}$, $S_2^{(c)}$, $S_3^{(c)}$, $S_4^{(c)}$ follow from the far-zone expressions for the scattered fields, for which the vector spherical harmonics expansions reduce to the form utlized earlier in the expression for the differential scattering cross section. A more detailed assessment is left for an exercise.

20 Scattering at the short-wavelength limit. Scalar diffraction theory.

Traditionally, diffraction entails those deviations from geometric optics that derive from the finite wavelength of the waves. Thereby, diffraction is connected to objects (e.g., holes, obstacles) that are large compared to the wavelength. The possible geometries are described in the figure below (see Jackson). The sources of the radiation are located in region I and we want to derive the diffracted fields in the diffraction region II. The regions are bounded by the interfaces S_1 and S_2. Kirchhoff was the first one to treat this topic systematically.

For simplicity, we will first study scalar fields, whereafter we will extend the analysis to vector fields. Let $\psi(\mathbf{x}, t)$ be a scalar field, for which we assume a harmonic time dependence $e^{-i\omega t}$. In essence, ψ is one of the components of the $\mathbf{E}$ or $\mathbf{B}$ fields. We assume that ψ fulfils the scalar Helmholtz wave equation

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = 0$$

in the volume V bounded by S_1 and S_2. We introduce the Green's function $G(\mathbf{x}, \mathbf{x}')$,

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$$

and start from Green's theorem

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3\mathbf{x}' = \oint_S [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}]dA'$$

$$\frac{\partial\psi}{\partial n} \equiv \mathbf{n}' \cdot \nabla\psi$$

where $\mathbf{n}'$ is the unit inward normal vector of S. Let us now set $\psi = G$ and $\phi = \psi$ so that, with the help of the wave equations for ψ and G,

$$\psi(\mathbf{x}) = \oint_S dA'[\psi(\mathbf{x}')\mathbf{n}' \cdot \nabla' G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\mathbf{n}' \cdot \nabla'\psi(\mathbf{x}')]$$

Kirchhoff's diffraction integral follows from this relation when G is chosen to be the free-space Green's function describing outgoing waves,

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{ikR}}{4\pi R}, \qquad \mathbf{R} = \mathbf{x} - \mathbf{x}', R = |\mathbf{R}|$$

Then

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S dA'\frac{e^{ikR}}{R}\mathbf{n}' \cdot [\nabla'\psi + ik(1 + \frac{i}{kR})\frac{\mathbf{R}}{R}\psi]$$

The surface S is composed of S_1 and S_2 and the integration can be divided into two parts. In the proximity of S_2, ψ is an outgoing wave and fulfils the so-called radiation condition

$$\psi \to f(\theta, \varphi)\frac{e^{ikr}}{r}, \qquad \frac{1}{\psi}\frac{\partial\psi}{\partial r} \to (ik - \frac{1}{r}).$$

By inserting these results into the integral above, it is possible to show that the integral over S_2 vanishes at least as the inverse of the radius of the sphere when the radius approaches infinity. There remains the integral over S_1, giving the final form of the Kirchhoff integral relation,

$$\psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{S_1} dA'\frac{e^{ikR}}{R}\mathbf{n}' \cdot [\nabla'\psi + ik(1 + \frac{i}{kR})\frac{\mathbf{R}}{R}\psi]$$

In applying the integral relation, it is necessary to know both ψ and $\partial\psi/\partial n$ on the surface S_1. In general, these are not known, at least not precisely. Kirchhoff's approach was based on the idea that ψ and $\partial\psi/\partial n$ are approximated on S_1 for the computation of the diffracted wave. This so-called Kirchhoff's approximation consists of the following assumptions:

1. ψ and $\partial\psi/\partial n$ vanish everywhere else but the holes of S_1
2. ψ and $\partial\psi/\partial n$ in the holes are equal to the original field values when there are no diffracting elements in space.

These assumptions contain a serious mathemtical inconsistency: if ψ and $\partial\psi/\partial n$ are zero on a finite surface, then $\psi = 0$ everywhere. In spite of the inconsistency, the Kirchhoff approximation works in an excellent way in practical problems and constitutes the basis of all diffraction calculus in classical optics.

The mathematical inconsistencies can be removed by a proper choice of the Green's function. In the setup of the figure below (see Jackson), (both P and P' are located several wavelengths away from the hole) we obtain

$$\psi(P) = \frac{k}{2\pi i}\int_{S_1} dA' \frac{e^{ikr}}{r}\frac{e^{ikr'}}{r'}\mathcal{O}(\theta,\theta')$$

$$\mathcal{O}(\theta,\theta') = \begin{cases} \cos\theta, & ; \\ \cos\theta', & ; \\ \frac{1}{2}(\cos\theta + \cos\theta'), & \text{(Kirchhoffin approksimaatio).} \end{cases}$$

The obliquity factor $\mathcal{O}(\theta,\theta')$ assumes less significance than the phase factors, which partly explains the success of the Kirchhoff approximation.

21 Vector Kirchhoff integral relation

The scalar Kirchhoff integral relation is an exact relation between the scalar fields on the surface and at infinity. In a corresponding way, the vector Kirchhoff integral relation is an exact relation between the $\mathbf{E},\mathbf{B}$ fields on the surface S and the diffracted or scattered fields at infinity. Such a relation is interesting in itself and it is a correct guess that the relation carries practical significance, too.

In what follows, we derive the vector relation for the electric field $\mathbf{E}$, starting from the generalization of Green's theorem already appearing in the scalar case for all components of the $\mathbf{E}$-field,

$$\mathbf{E}(\mathbf{x}) = \oint_S dA'[\mathbf{E}(\mathbf{n}'\cdot\nabla'G) - G(\mathbf{n}'\cdot\nabla')\mathbf{E}],$$

when $\mathbf{x}\in V$ and V is the volume bounded by S. Again, $\mathbf{n}'$ is the unit normal vector pointing into the volume V. Since G is singular at $\mathbf{x}' = \mathbf{x}$ and we make use of vector calculus valid for smooth functions, we assume that S is composed of the outer surface S' and an infinitesimally small inner surface S'' so that the point $\mathbf{x}' = \mathbf{x}$ is left out from volume V (but the point is inside S''). In such a case, the left-hand side of the previous equation disappears, but the integration over S'' on the right-hand side returns $-\mathbf{E}(\mathbf{x})$ when the radius of S'' goes to zero.

The vector relation can now be written in the form

$$0 = \oint_S dA'[2\mathbf{E}(\mathbf{n}'\cdot\nabla'G) - \mathbf{n}'\cdot\nabla'(G\mathbf{E})]$$

and, with the help of the divergence theorem ja divergenssiteoreeman

$$\int_V dV'\nabla\cdot A = \oint_S dA'\mathbf{A}\cdot\mathbf{n},$$

the latter term can be transformed to a volume integral

$$0 = \oint_S dA' 2\mathbf{E}(\mathbf{n}'\cdot\nabla'G) + \int_V dV'\nabla'^2(G\mathbf{E})$$

Now

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A})$$

$$\int_V dV\nabla\phi = \oint_S dA\mathbf{n}\phi, \qquad (\mathbf{n}\ \text{ ulkonormaali})$$

$$\int_V dV \nabla \times \mathbf{A} = \oint_S dA(\mathbf{n} \times \mathbf{A})$$

and the volume integral can be returned back to a surface integral

$$0 = \oint_S dA'[2\mathbf{E}(\mathbf{n}' \cdot \nabla' G) - \mathbf{n}'(\nabla' \cdot (G\mathbf{E})) + \mathbf{n}' \times (\nabla' \times (G\mathbf{E}))]$$

When the ∇-operations are carried out for $G\mathbf{E}$ and use is made of Maxwell's equations $\nabla' \cdot \mathbf{E} = 0, \nabla' \times \mathbf{E} = i\omega\mathbf{B}$, one obtains

$$0 = \oint_S dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G]$$

and, furthermore,

$$\mathbf{E}(\mathbf{x}) = \oint_S dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G]$$

where the volume bounded by S now again includes the point $\mathbf{x}$.

As in the case of the scalar relation, we can now derive the vector Kirchhoff integral relation

$$\mathbf{E}(\mathbf{x}) = \oint_{S_1} dA'[i\omega(\mathbf{n}' \cdot \mathbf{B})G + (\mathbf{n}' \times \mathbf{E}) \times \nabla' G + (\mathbf{n}' \cdot \mathbf{E})\nabla' G],$$

where the integration extends over S_1 only.

Finally, we derive a relation between the scattering amplitude and the near fields. For the fields in the vector Kirchhoff integral relation, we choose the scattered fields $\mathbf{E}_s, \mathbf{B}_s$, that is, the total fields $\mathbf{E}, \mathbf{B}$ minus the original fields $\mathbf{E}_i, \mathbf{B}_i$. If the observation point is far away from the scatterer, both the Green's function and the scattered electric field can be given in their asymptotic forms

$$G(\mathbf{x}, \mathbf{x}') = \rightarrow \frac{1}{4\pi} \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

$$\mathbf{E}_s(\mathbf{x}) \rightarrow \frac{e^{ikr}}{r} \mathbf{F}(\mathbf{k}, \mathbf{k}_0)$$

where $\mathbf{k}$ is a wave vector pointing in the direction of the observer, $\mathbf{k}_0$ is the wave vector of the original field, and $\mathbf{F}(\mathbf{k}, \mathbf{k}_0)$ is the vector scattering amplitude. In this limit, $\nabla' G = -i\mathbf{k}G$ and we obtain an integral relation for the scattering amplitude,

$$\mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{i}{4\pi} \oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}[\omega(\mathbf{n}' \cdot \mathbf{B}_s) + \mathbf{k} \times (\mathbf{n}' \times \mathbf{E}_s) - \mathbf{k}(\mathbf{n}' \cdot \mathbf{E}_s)]$$

The relation depends explicitly on the direction of $\mathbf{k}$ and the dependence on $\mathbf{k}_0$ is implicit in $\mathbf{E}_s$ and $\mathbf{B}_s$. Since $\mathbf{k} \cdot \mathbf{F} = 0$, we can reduce the relation to

$$\mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{1}{4\pi i} \mathbf{k} \times \oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}[\frac{c\mathbf{k} \times (\mathbf{n}' \times \mathbf{B}_s)}{k} - \mathbf{n}' \times \mathbf{E}_s]$$

Alternatively, one may want the scattering amplitude in direction $\mathbf{k}$ for a specific polarization state ϵ^*,

$$\epsilon^* \cdot \mathbf{F}(\mathbf{k}, \mathbf{k}_0) = \frac{i}{4\pi} \oint_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}[\omega\epsilon^* \cdot (\mathbf{n}' \times \mathbf{B}_s) + \epsilon^* \cdot (\mathbf{k} \times (\mathbf{n}' \times \mathbf{E}_s))]$$

These integral relations are useful in scattering problems entailing short wavelengths and in the derivation of the optical theorem.

22 Diffraction by a circular aperture

(Lecture 11)

Diffraction is divided into Fraunhofer and Fresnel diffraction depending on the geometry under consideration. There are three length scales involved: the size of the diffracting system d, the distance from the system to the observation point r and the wavelength λ. The diffraction pattern is generated when $r >> d$. In tht case, the slowly changing parts of the vector integral relation can be kept constant. Particular attention needs to be paid to the phase factor e^{ikR}. When $r >> d$, we obtain

$$kR = kr - k\mathbf{n}\cdot\mathbf{x}' + \frac{k}{2r}[r'^2 - (\mathbf{n}\cdot\mathbf{x}')^2] + \ldots, \qquad \mathbf{n} = \frac{\mathbf{x}}{r},$$

where $\mathbf{n}$ is a unit vector pointing in the direction of the observer. The magnitudes of the terms in the expansion are $kr, kd, (kd)^2/kr$. In Fraunhofer diffraction, the terms from the third one (inclusive) onwards are negligible. When the third term becomes significant (e.g., large diffracting systems), we enter the domain of Fresnel diffraction. Far enough from any diffracting system, we end up in the domain of Fraunhofer diffraction.

If the observation point is far away from the diffracting system, Kirchhoff's scalar integral relation assumes the form

$$\Psi(\mathbf{x}) = -\frac{e^{ikr}}{4\pi r}\int_{S_1} dA' e^{-i\mathbf{k}\cdot\mathbf{x}'}\Big[\mathbf{n}\cdot\nabla'\Psi(\mathbf{x}') + i\mathbf{k}\cdot\mathbf{n}\Psi(\mathbf{x}'),$$

where $\mathbf{n}$ now is the unit normal vectoron, $\mathbf{x}'$ denotes the position of the element dA', and $r = |\mathbf{x}|, \mathbf{k} = k(\mathbf{x}/r)$. The so-called Smythe-Kirchhoff integral relation is an improved version of the pure Kirchhoff relation and, in the present limit, takes the form

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}}{2\pi r}\mathbf{k}\times\int_{S_1} dA'\mathbf{n}\times\mathbf{E}(\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

Let us study next what the different diffraction formulae give for a circular hole (radius a) in an infinitesimally thin perfectly conducting slab.

Figure (see Jackson)

In the vector relation,

$$(\mathbf{n}\times\mathbf{E}_i)_{z=0} = E_0\epsilon_2\cos\alpha e^{ik\sin\alpha x'}$$

and, in polar coordinates,

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}E_0\cos\alpha}{2\pi r}(\mathbf{k}\times\epsilon_2)\int_0^a d\zeta\zeta\int_0^{2\pi} d\beta e^{ik\zeta[\sin\alpha\cos\beta - \sin\theta\cos(\varphi-\beta)]}$$

Define

$$\xi \equiv \frac{1}{k}|\mathbf{k}_\perp - \mathbf{k}_{0,\perp}| = \sqrt{\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\varphi},$$

in which case the integral takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta' e^{-ik\zeta\xi\cos\beta'} = J_0(k\zeta\xi)$$

that is, the result is the Bessel function J_0. Hereafter, the integration over the radial part can be calculated analytically, and

$$\mathbf{E}(\mathbf{x}) = \frac{ie^{ikr}}{r}a^2E_0\cos\alpha(\mathbf{k}\times\epsilon_2)\frac{J_1(ka\xi)}{ka\xi}$$

The time-averaged power as per unit solid angle is then

$$\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}(\cos^2\theta + \cos^2\varphi \sin^2\theta)|\frac{2J_1(ka\xi)}{ka\xi}|^2$$

$$P_i = (E_0^2/2z_0)\pi a^2 \cos\alpha,$$

where P_i is the total power normally incident on the hole. If $ka >> 1$, the function $[(2J_1(ka\xi)/ka\xi)^2]$ peaks sharply at 1 with the argument $\xi = 0$ and falls down to zero at $\Delta\xi \approx 1/ka$. The main part of the wave propagates according to geometric optics and only modest diffraction effects show up. If, however, $ka \approx 1$, the Bessel function varies slowly as a function of the angles and the transmitted wave bends into directions considerably deviting from the propagation direction of the incident field. In the extreme limit $ka << 1$, the angular dependence derives from the polarization factor $\mathbf{k} \times \epsilon_2$, but the analysis fails because the field in the hole can no longer be the original undisturbed field as assumed earlier.

let us study the scalar solution assuming that Ψ corresponds the magnitude of the $\mathbf{E}$ field,

$$\begin{aligned} \Psi(\mathbf{x}) &= -ik\frac{e^{ikr}}{r}a^2 E_0 \frac{1}{2}(\cos\alpha + \cos\theta)\frac{J_1(ka\xi)}{ka\xi} \\ \frac{dP}{d\Omega} &\cong P_i \frac{(ka)^2}{4\pi}\cos\alpha(\frac{\cos\alpha + \cos\theta}{2\cos\alpha})|\frac{2J_1(ka\xi)}{ka\xi}|^2 \end{aligned} \tag{135}$$

Both the vector and scalar results include the Bessel part $[(2J_1(ka\xi)/ka\xi)^2]$ and the same wave number dependence. But whereas there is no azimuthal dependence in the scalar result, the vector result is significantly affected by the azimuthal dependence. The dependence derives from the polarization of the vector field. For an original field propagating in the direction of the normal vector, the polarization effects are not important, when additionally $ka >> 1$. Then, all the results reduce into the familiar expression

$$\frac{dP}{d\Omega} \cong P_i \frac{(ka)^2}{\pi}|\frac{J_i(ka\sin\theta)}{ka\sin\theta}|^2$$

However, for oblique directions, there are large deviations and, for very small holes, the analysis fails completely.

23 Scattering in detail

Let us now consider a small particle that is much larger than the wavelength and study what kind of tools the vector Kirchhoff integral relation offers, if the fields close to the surface can be estimated somehow.

For example, the surface of the scatterer is divided into the illuminated and shadowed parts. The boundary between the two parts is sharp only in the limit of geometric optics and, in the transition zone, the breadth of the boundary is of the order of $(2/kR)^{1/3} \cdot R$, where R is a typical radius of curvature on the surface of the particle.

On the shadow side, the scattered field must be equal to the original field but opposite in sign, in which case the total field vanishes. On the illuminated side, the field depends in a detailed way on the properties of the scattering particle. If the curvature radii are large compared to the wavelength, we can make use of Fresnel's coefficients and geometric optics in general. The analysis can be generalized into the case of a transparent particle and the method is known as the physical-optics approximation (or Kirchhoff approximation).

Let us write the scattering amplitude explicitly in two parts,

$$\epsilon^* \cdot \mathbf{F} = \epsilon^* \cdot \mathbf{F}_{sh} + \epsilon^* \cdot \mathbf{F}_{ill}$$

and assume that the incident fields is a plane wave

$$\begin{aligned} \mathbf{E}_i &= E_0 \epsilon_0 e^{i\mathbf{k}_0 \cdot \mathbf{x}} \\ \mathbf{B}_i &= \mathbf{k}_0 \times \mathbf{E}_i / kc \end{aligned}$$

The shadow scattering amplitude is then ($\mathbf{E}_s \approx -\mathbf{E}_i, \mathbf{B}_s \approx -\mathbf{B}_i$)

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{E_0}{4\pi i} \int_{sh} dA' \epsilon^* \cdot [\mathbf{n}' \times (\mathbf{k}_0 \times \epsilon_0) + \mathbf{k} \times (\mathbf{n}' \times \epsilon_0)] \cdot e^{i(\mathbf{k}_0 - \mathbf{k})\cdot \mathbf{x}'}$$

where the integration is over the shadowed region. The amplitude can be rearranged into the form

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{E_0}{4\pi i} \int_{sh} dA' \epsilon^* \cdot [(\mathbf{k} + \mathbf{k}_0) \times (\mathbf{n}' \times \epsilon_0) + (\mathbf{n}' \cdot \epsilon_0)\mathbf{k}_0] \cdot e^{i(\mathbf{k}_0 - \mathbf{k})\cdot \mathbf{x}'}$$

In the short-wavelength limit, $\mathbf{k}_0 \cdot \mathbf{x}'$ and $\mathbf{k} \cdot \mathbf{x}'$ vary across a large regime and the exponential factor fluctuates rapidly and eliminates the integral everywhere else but the forward-scattering direction $\mathbf{k} \approx \mathbf{k}_0$. In that direction ($\theta \lesssim 1/kR$), the second factor is negligible compared to the first one since $(\epsilon^* \cdot \mathbf{k}_0)/k$ is of the order of $\sin\theta << 1, (\epsilon^* \cdot \mathbf{k} = 0, \mathbf{k}_0 \approx \mathbf{k})$. Thus,

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{iE_0}{2\pi} (\epsilon^* \cdot \epsilon_0) \int_{sh} dA' (\mathbf{k}_0 \cdot \mathbf{n}') e^{i(\mathbf{k}_0 - \mathbf{k})\cdot \mathbf{x}'} \tag{136}$$

In this approximation, the integral over the shadow side only depends on the projected area against the propagation direction of the original field. This can be seen from the fact that $\mathbf{k}_0 \cdot \mathbf{n}' dA' = k dx' dy' = k d^2\mathbf{x}'_\perp$ ja $(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}' = k(1 - \cos\theta) z' - \mathbf{k}_\perp \cdot \mathbf{x}'_\perp \approx -\mathbf{k}_\perp \cdot \mathbf{x}'_\perp$. The final form of the shadow scattering amplitude is thus

$$\epsilon^* \cdot \mathbf{F}_{sh} = \frac{ik}{2\pi} E_0 (\epsilon^* \cdot \epsilon_0) \int_{sh} d^2\mathbf{x}'_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{x}'_\perp} \tag{137}$$

In this limit, all scatterers producing the same prohjected area will have the same shadow scattering amplitude. For example, in the case of a circular cylindrical slab (radius a)

$$\int_{sh} d^2\mathbf{x}'_\perp e^{-i\mathbf{k}_\perp \cdot \mathbf{x}'_\perp} = 2\pi a^2 \frac{J_1(ka\sin\theta)}{ka\sin\theta},$$

$$\epsilon^* \cdot \mathbf{F}_{sh} \cong ika^2 E_0 (\epsilon^* \cdot \epsilon_0) \frac{J_1(ka\sin\theta)}{ka\sin\theta}.$$

This explains nicely the forward diffraction pattern in scattering by small particles.

(Lecture 12)

The scattering amplitude due to the illuminated side of the scatterer cannot be calculated without defining the shape and optical properties of the particle. Let us assume in the following example that the illuminated region is perfectly conducting. Then, the tangential components of the fields $\mathbf{E}_s$ and $\mathbf{B}_s$ on S_1 are approximately opposite and similar to those of the original fields, respectively. The scattering amplitude due to the illuminated part is then

$$\epsilon^* \cdot \mathbf{F}_{ill} = \frac{E_0}{4\pi i} \int_{ill} dA' \epsilon^* \cdot [-\mathbf{n}' \times (\mathbf{k}_0 \times \epsilon_0) + \mathbf{k} \times (\mathbf{n}' \times \epsilon_0)] \cdot e^{i(\mathbf{k}_0 - \mathbf{k})\cdot \mathbf{x}'}$$

When this is compared with the shadow amplitude, the only notable difference is the sign in the first term. This sign difference results in a completely different scattering amplitude that can also be written in the form muodossa

$$\epsilon^* \cdot \mathbf{F}_{ill} = \frac{E_0}{4\pi i} \int_{ill} dA' \epsilon^* \cdot [(\mathbf{k} - \mathbf{k}_0) \times (\mathbf{n}' \times \epsilon_0) - (\mathbf{n}' \cdot \epsilon_0)\mathbf{k}_0] \cdot e^{i(\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}'}$$

When again $kR >> 1$, the exponential factor fluctuates rapidly and one would expect a strong contribution in the forward direction; however, the first term goes to zero in the forward direction and no strong contribution can follow. The illuminated region contributes to scattering in the form of a reflected wave.

Assume next that the scattering particle is spherical (radius a). The predominating contribution to the scattering amplitude now derives from a region of integration where the phase of the exponential factor is stationary. If (θ, φ) are the coordinates of $\mathbf{k}$ and (α, β) those of $\mathbf{n}'$ (with respect to $\mathbf{k}_0$), the phase factor is

$$\phi(\alpha, \beta) = (\mathbf{k}_0 - \mathbf{k}) \cdot \mathbf{x}' = ka[(1 - \cos\theta)\cos\alpha - \sin\theta \sin\alpha \cos(\beta - \varphi)]$$

The stationary point can be found at angles α_0, β_0, where $\alpha_0 = \pi/2 + \theta/2$ and $\beta_0 = \varphi$. These angles correspond exactly to the angles of reflection on the surface of the sphere as dictated by geometric optics. At that point, the vector $\mathbf{n}'$ points in the direction of $(\mathbf{k} - \mathbf{k}_0)$. In the proximity of angles $\alpha = \alpha_0$ and $\beta = \beta_0$

$$\phi(\alpha, \beta) = -2ka \sin\frac{\theta}{2}[1 - \frac{1}{2}(x^2 + \cos^2\frac{\theta}{2} y^2) + \ldots]$$

where $x = \alpha - \alpha_0$ and $y = \beta - \beta_0$. The integration can be carried out approximately:

$$\epsilon^* \cdot \mathbf{F}_{ill} \cong ka^2 E_0 \sin\theta e^{-2ika\sin\frac{\theta}{2}} (\epsilon^* \cdot \epsilon_r) \cdot \int dx e^{i[ka\sin\frac{\theta}{2}]x^2} \int dy e^{i[ka\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}]y^2}$$

$$\epsilon_r = -\epsilon_0 + 2(\mathbf{n}_r \cdot \epsilon_0)\mathbf{n}_r, \qquad \mathbf{n}_r = \frac{\mathbf{k} - \mathbf{k}_0}{|\mathbf{k} - \mathbf{k}_0|}$$

When $2ka\sin\frac{\theta}{2} >> 1$, the integrals can be calculated using the result $\int_{-\infty}^{\infty} dx e^{i\alpha x^2} = \sqrt{\pi i/\alpha}$,

$$\epsilon^* \cdot \mathbf{F}_{ill} \cong E_0 \frac{a}{2} e^{-2ika\sin\frac{\theta}{2}} \epsilon^* \cdot \epsilon_r$$

For large $2ka\sin\frac{\theta}{2}$, the intensity of the reflected part of the radiation is constant as a function of the angle, but the part has a rapidly varying phase. When $\theta \to 0$, the intensity vanishes as θ^2 (see the integral above).

Comparison of the amplitudes due to the shadowed and illuminated parts of the surface shows that, in the forward direction, the former amplitude predominates over the latter by a factor $ka >> 1$ whereas, at the scattering angles $2ka\sin\theta >> 1$, the ratio of the amplitudes is of the order of $1/(ka\sin^3\theta)^{1/2}$. The differential scattering cross section (summed over the polarization states of the original and scattered waves) is

$$\frac{d\sigma}{d\Omega} \cong \begin{cases} a^2(ka)^2 |\frac{J_1(ka\sin\theta)}{ka\sin\theta}|^2, & \theta \lesssim \frac{10}{ka} \quad ; \\ \frac{a^2}{4}, \quad \theta >> \frac{1}{ka} & . \end{cases}$$

The total scattering cross section is twice the geometric cross section of the particle.

24 Optical theorem

The optical theorem is a fundamental relation that connects the exticntion cross section to the imaginary part of the forward-scattering amplitude. Consider a plane wave with a wave vector $\mathbf{k}_0$ and field components $\mathbf{E}_i, \mathbf{B}_i$. The plane wave is incident on a finite-sized scatterer inside the surface S_1. The scattered field $\mathbf{E}_s, \mathbf{B}_s$ propagates away from the scatterer and is observed in the far zone in the direction $\mathbf{k}$. The total field outside the surface S_1 is, by definition,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_i + \mathbf{E}_s \\ \mathbf{B} &= \mathbf{B}_i + \mathbf{B}_s. \end{aligned}$$

In the general case, the scatterer absorbs energy from the original field. The absorbed power can be calculated by integrating the inward-directed Poynting-vector component of the total field over the surface S_1:

$$P_{abs} = -\frac{1}{2\mu_0} \oint_{S_1} dA' \mathrm{Re}(\mathbf{E} \times \mathbf{B}^*) \cdot \mathbf{n}'$$

The scattered power is computed in the usual way from the asymptotic form of the Poynting vector for the scattered fields in the regime, where the fields are simple transverse spherical waves that attenuate as $1/r$. But since there are no sources between S_1 and infinity, the scattered power can as well be calculated as an integral of the outward-directed component of the Poynting vector for the scattered field over S_1:

$$P_{sca} = \frac{1}{2\mu_0} \oint_{S_1} dA' \mathrm{Re}(\mathbf{E}_s \times \mathbf{B}_s^*) \cdot \mathbf{n}'$$

The total power is the sum of the absorbed and scattered power so that, after rearranging,

$$P = P_{abs} + P_{sca} = -\frac{1}{2\mu_0} \oint_{S_1} dA' \mathrm{Re}(\mathbf{E}_s \times \mathbf{B}_i^* + \mathbf{E}_i^* \times \mathbf{B}_s) \cdot \mathbf{n}'$$

When the original field in written explicitly in the form

$$\begin{aligned} \mathbf{E}_i &= E_0 \epsilon_0 e^{i\mathbf{k}_0 \cdot \mathbf{x}} \\ c\mathbf{B}_i &= \frac{1}{k} \mathbf{k}_0 \times \mathbf{E}_i \end{aligned}$$

the total power can be transformed to the form

$$P = \frac{1}{2\mu_0} Re E_0^* \oint_{S_1} dA' e^{-i\mathbf{k}_0 \cdot \mathbf{x}} [\epsilon_0^* \cdot (\mathbf{n}' \times \mathbf{B}_s) + \epsilon_0^* \cdot \frac{\mathbf{k}_0 \times (\mathbf{n}' \times \mathbf{E}_s)}{kc}]$$

By comparing this with the scattering amplitude $\mathbf{F}(\mathbf{k}, \mathbf{k}_0)$ derived earlier, we can recognize that the total power is proportional to the value of $\mathbf{F}$ in the forward-scattering direction $\mathbf{k} = \mathbf{k}_0$ in the polarization state coinciding with that of the original field:

$$P = \frac{2\pi}{kZ_0} \mathrm{Im}[E_0^* \epsilon_0^* \cdot \mathbf{F}(\mathbf{k} = \mathbf{k}_0)],$$

which is the basic form of the optical theorem.

The total or extinction cross section σ_e is defined as the ratio of the total and original flux densities ($|E_0|^2/2Z_0$, power as per unit surface area).

In a corresponding way, one can define a normalized scattering amplitude **f** (against the original field value at origin)

$$f(\mathbf{k} = \mathbf{k}_0) = \frac{\mathbf{F}(\mathbf{k}, \mathbf{k}_0)}{E_0}$$

The final form of the optical theorem is then

$$\sigma_e = \frac{4\pi}{k}\mathrm{Im}[\epsilon_0^* \cdot \mathbf{f}(\mathbf{k} = \mathbf{k}_0)].$$

25 Scattering by nonspherical particles

(Lecture 13)

Perfectly spherical particles constitute, practically, an exception in nature and even in industrial applications. In the recent past, numerical methods have been actively developed for light scattering by nonspherical particles. In practice, the methods require extensive computational capacity including supercomputers.

In what follows, one possible modeling of a nonspherical particle geometry is presented: the Gaussian random sphere. Thereafter, computation of scattering by Gaussian particles is discussed in various approxiamtions, whereafter a summary is given on essentially exact numerical methods and possibilities to apply these methods to scattering by Gaussian particles.

26 Gaussian random particle

Statistical modeling of nonspherical particle shapes seems reasonable, since nonspherical shapes usually show up as a wide spectrum of different-looking shapes. In the Gaussian-random-sphere model, the particle is assumed to be mathematically star-like so that there is an origin with respect to which the shape can be expressed as a function of the spherical coordinates. In the spherical geometry, the so-called lognormal statistics are being used so that the radial distance of the particle varies within $]0, \infty[$. The shape is unambiguously defined by the mean of the radial distance a and the covariance function of the logarithm of the radial distance Σ_s. Explicitly,

$$r(\theta, \varphi) = ae^{s(\theta,\varphi)-\frac{1}{2}\beta^2},$$

where s is the logarithmic radial distance and $\beta^2 = \Sigma_s(0)$ is the variance of s. Now

$$s(\theta, \varphi) = \sum_{lm} s_{lm} Y_{lm}(\theta, \varphi)$$

and, due to s being real-valued,

$$s_{l,-m} = (-1)^m s_{lm}^* \begin{cases} l = 0, 1, 2, \ldots, & ; \\ m = -l, \ldots, -1, 0, 1, \ldots, l, & \end{cases},$$

$$Im(s_{l0}) = 0.$$

The spherical harmonics coefficients of the logarithmic radial distance $s_{lm}, m \geq 0$ are independent Gaussian random variables with zero means and with variances (l and m as above)

$$\begin{aligned} \mathrm{Var}[\Re(s_{lm})] &= (1 + \delta_{m0})\tfrac{2\pi}{2l+1}c_l \\ \mathrm{Var}[\Im(s_{lm})] &= (1 - \delta_{m0})\tfrac{2\pi}{2l+1}c_l \end{aligned}$$

The coefficients $c_l \geq 0, l = 0, \ldots, \infty$ are the coefficients of the Legendre expansion for the covariance function Σ_s:

$$\Sigma_s(\gamma) = \beta^2 C_s(\gamma) = \sum_{l=0}^{\infty} c_l P_l(\cos\gamma), \qquad \sum_{l=0}^{\infty} c_l = \beta^2,$$

where γ is the angualr distance between two directions (θ_1, φ_1) and (θ_2, φ_2).

The two slopes on the Gaussian random particle (subscripts refering to partial derivatives)

$$s_\theta = \frac{r_\theta}{r}, \qquad \frac{1}{\sin\theta} s_\varphi = \frac{r_\varphi}{r\sin\theta}$$

are, again, independent Gaussian random variables with zero means and with standard deviations

$$\rho = \sqrt{-\Sigma_s^{(2)}(0)},$$

where $\Sigma_s^{(2)}$ is the second derivative of the covariance function with respect to γ. The correlation length l_c and correlation angle Γ_c are

$$l_c = 2\sin\frac{1}{2}\Gamma_c = \frac{1}{\sqrt{-c_s^{(2)}(0)}}.$$

Natural random shapes often exhibit covariance functions, for which the coefficients c_l follow the exponent form $c_l \propto l^{-\nu}, l \geq 2$. For $\nu = 4$, one obtains random shapes applicable, in the first place, to modeling Saharan sand particles, asteroids, as well as the shapes of terrestrial planets.

In the limiting case, the Gaussian random shape thus depends on a single free parameter insofar as the shape is concerned: the variance β^2 of the logarithmic radial distance. β^2 relates to the relative variance of the radius σ^2 via the simple relation

$$\sigma^2 = e^{\beta^2} - 1.$$

Increasing σ results in shapes, where the radial fluctuations are enhanced.

If, additionally, ν is treated as a free parameter, one obtains shorter correlation lengths with smaller values of ν (when the expansions are always truncated at a certain degree l_{max}) and thereby lrger numbers of hills and valleys as per unit solid angle.

For $\nu \geq 4$, non-fractal smooth shapes are obtained whereas, for $\nu < 4$, fractal shapes follow, in which case infinite expansions would yield non-differentiable surfaces rendering the discussion of slopes meaningless.

27 Scattering by Gaussian particles in different approximations

Light scattering by Gaussian random particles has been studied in the ray-optics, Rayleigh-volume, Rayleigh-Gans, anomalous-diffraction and perturbation-series approximations, as well as in the Rayleigh-ellipsoid approximation.

In the Rayleigh-volume approximation, the scattering by a small particle follows from its volume. In the case of the Gaussian particle, the (ensemble-averaged) absorption cross section is proportional to the mean of the volume, whereas the scattering cross section is proportional to the mean of the squared volume. The angular characteristics of the scattering matrix are the same as in the Rayleigh approximation for spherical particles. The results are largely analytical.

In the Rayleigh-ellipsoid approximation, an ellipsoid is fitted to each realization of the Gaussian particle, the ellipsoid volume being equal to the volume of the realization. Scattering is then approximated with the existing electrostatics approximation for ellipsoidal scatterers. The most significant challenge in the Rayleigh-ellipsoid approximation is the numerical computation of the best-fit ellipsoid, whereafter the results follow in a straightforward way.

In the Rayleigh-Gans approximation (or the first Born approximation), the numerical computation of the form factor can be aided by analytical intermediate results. In practice, some numerical integrattion remains, preventing the treatment of arbitrarily large particles.

In anomalous diffraction, path lengths of rays inside the Gaussian sample particles are numerically computed in cases where the refractive index is close to unity. The absorption follows directly from the exponential attenuation and extinction is computed from the optical theorem. The angular dependence of scattering is obtained by averaging the square of the scattering amplitude. The most demanding task is the computation of the path lengths inside the particle, which is difficult for extremely nonspherical shapes.

In the second-order perturbation-series approach for the boundary conditions, analytical results follow for the cross sections and scattering matrices and the most challenging numerical part is the computation of the so-called $3j$-symbols. The unknown accuracy of the results is a problem. In practice, the perturbation-series method is applicable to wavelength-scale scatterers only, if the deviations from the spherical shape are small compared to the wavelength.

Approximations can be taken to be "the spice" that makes the scattering research "delicious", since, in practice, all so-called exact methods are based on approximation in some part. One can make the provocative statement that only approximations allow the computation of light scattering by realistic small particles. The applicability of the exact methods is usually limited to a narrow range of simple shapes. By the rapid development of computers and by the development of new analytical methods, the applicability of certain exact methods grows slowly but steadily.

28 Exact methods and their applicability to Gaussian particles

The numerical methods in light scattering can be divided into differential-equation and integral-equation methods. The traditional computational method is the separation-of-varaibles method that has been successful in the solution of the following scattering problems:

1. isotropic, homogeneous sphere
2. coated sphere consisting of the interior and coating (with common origin)
3. layered sphere that consists of several layers defined by concentric spherical cells
4. radially inhomogeneous sphere
5. optically active (chiral) sphere
6. homogeneous, istropic infinite circular cylinder
7. optically active infinite circular cylinder
8. isotropic infinite elliptic cylinder
9. isotropic, homogeneous spheroid

10. coated spheroid that consists of the interior and coating (with common origin)
11. optically active spheroid

The separation-of-variables method is not applicable to scattering by Gaussian particles.

The FEM-method (finite-element method) is a differenttial-equation method, where the scatterer is placed in a finite computational volume that is discretized into numerous small computational cells. Typically, there are 10-20 cells per wavelength and the electromagnetic field is solved for in the nodal points of the cells. The resulting linear group of equations consists of a sparse matrix. In the boundaries of the computational volume, an artifical absorbing boundary condition is invoked. Although FEM allows for the computations for arbitrary, even inhomogeneous particles, it has not yet been applied to Gaussian particles.

The FDTD-method (finite-difference time-domain method) is a differential-equation method that solves for the time dependence of the electromagnetic fields based on Maxwell's curl equations. Both time and spatial derivatives are expressed with finite differences and time elapses in finite steps. The scattering particle is again palced in a finite computational volume and an absorbing boundary condition is required in the boundary of the computational volume. The density of the discretization is as in the FEM-method. In FDTD, there is no need to solve a large group of equations. Recently, the method has yielded promising results in light scattering by Gaussian particles.

(Lecture 14)

In the PM-method (point matching), the boundary conditions of the electromagnetic fields are required in a finite number of points on the surface of the particle. In the original method, there were as many points as unknown coefficients in the vector spherical harmonics expansion. It was concluded that the method was numerically instable. There is, however, nothing that prevents us from expanding the number of points and computing the coefficients using the least-squares method. This version of the method has been noticed to be stable and is one of the most popular numerical methods. The regime of application can be improved by expanding the fields with a number of suitably chosen origins within the particle. PM is promising also for scattering by Gaussian particles. It is intriguing to ponder whether "an educated guess" can help speed up the solution of the coefficients.

The integral-equation methods are divided into a wide spectrum of different methods. In the VIEM method (volume-integral-equation), one considers the integral equation

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_i(\mathbf{r}) + k^2 \int_V d^3\mathbf{r}'[\mathbf{1} + \frac{1}{k^2}\nabla\nabla]\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \cdot [m^2(\mathbf{r}') - 1]\mathbf{E}(\mathbf{r}').$$

By discretizing the integral on the right-hand side, one obtains a group of linear equations for the field values at the discretization points within the volume of the particle. Solving the equations results in the field inside the particle. Typically, again, 10-20 discretization points are required as per wavelength so that, after a straightforward calculation, it is clear that a group of equations with thousands of unknowns easily follows. In practice with current computers, up to 200 million unknowns can be treated (as of December 12, 2008). Various versions of the VIEM method have been successfully applied to Gaussian-particle scattering (foremost DDA, discrete-dipole approximation).

In the case of VIEM, the matrix of the group of linear equations is full, which makes the solution more difficult. When the internal field has been solved for, the same integral relation gives the scattered field outside the particle via straightforward integration (subtracting the original field).

DDA (discrete-dipole approximation) is a certain version of solution methods for the integral equation. DDA can be visualized in the following: the particle can be thought to be composed of dipole scatterers interacting with each other. In practice, the VIEM methods differ from one another in how they treat the singular self-term inside the integral, which is essential for the accuracy of the method.

The surface-integral-equation methods (SIEM) make use of two-dimensional integral equations that seem like a reasonable starting point, in particular, for homogeneous particles. However, the SIEM-methods are less stable than the VIEM-methods and usually require additional regularization.

The integral equation shown above in connection to the VIEM-method is Fredholm-type and the kernel has a singularity at $\mathbf{r} = \mathbf{r}'$. Via Fourier-transformation, handling of the singularity can be improved and the integral equation can be solved numerically in the wavenumber (or frequency) space. Surprisingly, the disadvantage of the method is the considerable analytical work needed for each different particle. These so-called FIEM-methods have not been very popular.

In the TMM method (transition matrix method), the analysis proceeds with the help of vector spherical harmonics functions and the word "transition" refers to the linear matrix relation between the original field and the scattered field. Compared to the direct vector spherical harmonics treatment of the boundary conditions, TMM has the advantage that a linear relation is obtained purely between the internal and original fields, reducing the number of unknowns in the group of linear equations. After solving the group of equations, the scattered is computed from the vector Kirchhoff integral relation. The TMM method is an efficient method, in particular, for axially symmetric particles and useful results have been obtained, e.g., for spheroids to compare with the implications of the SVM method. However, TMM suffers from unpredictable convergence and instability problems and have not yet been extensively applied to scattering by Gaussian particles. As a tool the actual T-matrix is quite useful and, for a single particle, needs to be computed only once (independently of the orientation). Recently, an analytical version of the T-matrix method has been developed—this version is highly promising for studying scattering by Gaussian random particles.

In the superposition method for spheres and spheroids, scattering by particle clusters is computed using the translation and addition rules of vector spherical harmonics functions. The field scattered by the cluster is expressed as a superposition of the fields scattered by each constituent particle. The partial fields depend on each other due to the mutual electromagnetic interactions of the constituent particles. The scattering problem again manifests itself in a solution of a group of linear equations. Currently, precise solutions can be computed for clusters with several dozens of constituent particles, when constituent-particle size approaches the wavelength.

29 Applications of electromagnetic scattering

In his book, van de Hulst has presented an excellent review of the applications of light scattering in various fields of science. This is recommended reading bearing in mind, in particular, modern computational methods for nonspherical particles. Bohren and Huffman offer additional material on the applications, as well as Mishchenko et al. Finally, the publications from the meeting series entitled *Electromagnetic and Light Scattering by Nonspherical Particles: Theory, Measurements, and Applications* offer up-to-date information about the advances in light scattering by small particles.