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You may use the MAOL mathematical tables in the exam

1. The probability density function of the random variable X is

$$f(x) = k\,x, \quad \text{kun } 0 < x < 1$$

(and 0 otherwise).

a) Find the value of the constant k.

b) Find the cumulative distribution function (cdf) as well as the quantile function of X.

c) Find the probability $P(\frac{1}{X} < 2)$.

2. Let $X > 0$ and $Y > 0$ be independent positive random variables such that the expectations EX and $E(1/Y)$ are finite. It follows from these assumptions that one or more of the following properties a, b, c are necessarily true and the others are not. (One or more of the properties may hold for a particular probability distribution but not for every probability distribution satisfying the assumptions.) For each of the properties, state whether it is necessarily true or is not necessarily true. If you answer that the property is not necessarily true, then you should give an example of a distribution (e.g., a discrete distribution) which satisfies the assumptions but which fails to satisfy the property. If you answer that the property is necessarily true, then you should justify briefly your answer.

a) $E\left(\dfrac{X}{Y}\right) = \dfrac{EX}{EY}$, b) $E\left(\dfrac{X}{Y}\right) \leq \dfrac{EX}{EY}$, c) $E\left(\dfrac{X}{Y}\right) \geq \dfrac{EX}{EY}$.

3. Let U and V be independent random variables both of which are uniformly distributed on the interval $(0, 1)$. Define random variables X and Y as follows,

$$X = V, \qquad Y = U/V.$$

Find the joint probability density function of X and Y. In addition, find the marginal probability density function of Y.

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4.

a) Explain what one means when one says that a symmetric matrix $\mathbf{C}$ is *positive semidefinite*. (2 points)

b) Let $\mathbf{C}$ be the covariance matrix of the random vector $\mathbf{X}$, i.e., $\mathbf{C} = \operatorname{Cov} \mathbf{X}$. Give a definition for $\mathbf{C}$ as an expected value. Show that the matrix $\mathbf{C}$ is symmetric and positive semidefinite. (4 points)

5. Let $\mathbf{M}$ be a fixed $n \times p$ matrix such that the matrix $\mathbf{M}^T\mathbf{M}$ is invertible. Let $\boldsymbol{\beta} \in \mathbb{R}^p$ be a fixed vector of coefficients. Define the random variables $Y_1, \dots, Y_n$ as

$$Y_i = \mathbf{m}_i^T \boldsymbol{\beta} + \epsilon_i, \qquad i = 1, \dots, n,$$

where $\epsilon_1, \dots, \epsilon_n$ are independent random variables, which follow the distribution $N(0, \sigma^2)$. The row vector $\mathbf{m}_i^T$ is the ith row vector of matrix $\mathbf{M}$. The error variance $\sigma^2 > 0$ is a fixed positive number.

In this linear model, we estimate the parameter $\boldsymbol{\beta}$ using the estimator

$$\mathbf{B} = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{Y},$$

where $\mathbf{Y} = (Y_1, \dots, Y_n)$.

a) What is the distribution of the random vector $\mathbf{Y}$? (State the name of the distribution as well as its parameters.)

b) Find $E\mathbf{B}$ and $\operatorname{Cov} \mathbf{B}$ in terms of the quantities $\mathbf{M}$, $\boldsymbol{\beta}$ and σ^2.

c) What is the distribution of the random vector $\mathbf{B}$? (State the name of the distribution as well as its parameters.)