Statistical Inference
Exam 22.10.2012

The only materials you are allowed to have at your desk during the exam are writing instruments and approved calculators.

1. Let Y_1, Y_2, Y_3 be three independent observations from the $\text{Exp}(1/\mu)$-distribution. Define
$$S = \frac{1}{4}(2Y_1 + Y_2 + Y_3), \quad T = \frac{1}{3}(Y_1 + Y_2 + Y_3).$$
Verify that S and T are unbiased estimators of μ. Why is T better? Prove that T is, in fact, the best unbiased estimator of μ.

2. Let $Y_1, \ldots, Y_n \sim N(\mu_0, \sigma^2)$ $\perp\!\!\!\perp$ (independently and identically normally distributed). The mean (μ_0) is known in advance but the standard deviation ($\sigma > 0$) is unknown.

   a) What is the joint density function of the observations (n of them)?

   b) Derive the ML estimator $\hat{\sigma}$ for the standard deviation σ. Note: The ML estimator for the standard deviation σ instead of the variance σ^2 is asked for. Check that you have found the (local) maximum of the log-likelihood.

   c) Derive expected (Fisher) information $i(\sigma)$ for the standard deviation σ. (Hint: Calculate $i(\sigma)$ by means of the formula which is easier in general. If you have time, check your result with the other formula which will yield a much more laborous derivation.)

   d) What is the approximate distribution of ML estimator $\hat{\sigma}$ for large samples?

3. a) Let $Y_1, \ldots, Y_n$ be a random sample from a distribution whose probability density function depends on a real valued parameter θ. What is the definition of an interval estimator of θ with confidence coefficient $1 - \alpha$ (also known as a $100(1 - \alpha)\%$ confidence interval)?

   b) Let $Y_1, \ldots, Y_n$ be iid random variables from $N(\mu, \sigma_0^2)$ with *known* variance σ_0^2. Derive a $100(1 - \alpha)\%$ confidence interval for the parameter μ.

4. Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables from $Exp(1/\mu)$-distribution. The null hypothesis is $H_0 : \mu = \mu_0$, where $\mu_0 > 0$.

   a) Derive the formulae for the likelihood ratio, Wald and Rao test statistics.

   b) The sample size is $n = 50$ and the sample mean is $\bar{y} = 800$. Test with the above mentioned test statistics, if the null hypothesis $H_0 : \mu = 1000$ is rejected or not at significance level 0.05 (two-sided test).

You may use the following fact:

- If the random variable Y has an $Exp(\lambda)$ distribution, its pdf is

$$f(y;\lambda) = \lambda e^{-\lambda y}, \quad y > 0,\ \lambda > 0,$$

  and $E(Y) = 1/\lambda$, $\text{Var}(Y) = 1/\lambda^2$.

Appendix 1: Table of the χ^2-distribution.

	Level of significance									
df	0.995	0.990	0.975	0.950	0.10	0.05	0.025	0.01	0.005	0.001
1	0.000	0.000	0.001	0.004	2.706	3.841	5.024	6.635	7.879	10.828
2	0.010	0.020	0.051	0.103	4.605	5.991	7.378	9.210	10.597	13.816
3	0.072	0.115	0.216	0.352	6.251	7.815	9.348	11.345	12.838	16.266
4	0.207	0.297	0.484	0.711	7.779	9.488	11.143	13.277	14.860	18.467
5	0.412	0.554	0.831	1.145	9.236	11.070	12.833	15.086	16.750	20.515
6	0.676	0.872	1.237	1.635	10.645	12.592	14.449	16.812	18.548	22.458
7	0.989	1.239	1.690	2.167	12.017	14.067	16.013	18.475	20.278	24.322
8	1.344	1.646	2.180	2.733	13.362	15.507	17.535	20.090	21.955	26.124
9	1.735	2.088	2.700	3.325	14.684	16.919	19.023	21.666	23.589	27.877
10	2.156	2.558	3.247	3.940	15.987	18.307	20.483	23.209	25.188	29.588
11	2.603	3.053	3.816	4.575	17.275	19.675	21.920	24.725	26.757	31.264
12	3.074	3.571	4.404	5.226	18.549	21.026	23.337	26.217	28.300	32.909
13	3.565	4.107	5.009	5.892	19.812	22.362	24.736	27.688	29.819	34.528
14	4.075	4.660	5.629	6.571	21.064	23.685	26.119	29.141	31.319	36.123
15	4.601	5.229	6.262	7.261	22.307	24.996	27.488	30.578	32.801	37.697
16	5.142	5.812	6.908	7.962	23.542	26.296	28.845	32.000	34.267	39.252
17	5.697	6.408	7.564	8.672	24.769	27.587	30.191	33.409	35.718	40.790
18	6.265	7.015	8.231	9.390	25.989	28.869	31.526	34.805	37.156	42.312
19	6.844	7.633	8.907	10.117	27.204	30.144	32.852	36.191	38.582	43.820
20	7.434	8.260	9.591	10.851	28.412	31.410	34.170	37.566	39.997	45.315
21	8.034	8.897	10.283	11.591	29.615	32.671	35.479	38.932	41.401	46.797
22	8.643	9.542	10.982	12.338	30.813	33.924	36.781	40.289	42.796	48.268
23	9.260	10.196	11.689	13.091	32.007	35.172	38.076	41.638	44.181	49.728
24	9.886	10.856	12.401	13.848	33.196	36.415	39.364	42.980	45.559	51.179
25	10.520	11.524	13.120	14.611	34.382	37.652	40.646	44.314	46.928	52.620
26	11.160	12.198	13.844	15.379	35.563	38.885	41.923	45.642	48.290	54.052
27	11.808	12.879	14.573	16.151	36.741	40.113	43.195	46.963	49.645	55.476
28	12.461	13.565	15.308	16.928	37.916	41.337	44.461	48.278	50.993	56.892
29	13.121	14.256	16.047	17.708	39.087	42.557	45.722	49.588	52.336	58.301
30	13.787	14.953	16.791	18.493	40.256	43.773	46.979	50.892	53.672	59.703
40	20.707	22.164	24.433	26.509	51.805	55.758	59.342	63.691	66.766	73.402
50	27.991	29.707	32.357	34.764	63.167	67.505	71.420	76.154	79.490	86.661
60	35.534	37.485	40.482	43.188	74.397	79.082	83.298	88.379	91.952	99.607
70	43.275	45.442	48.758	51.739	85.527	90.531	95.023	100.425	104.215	112.317
80	51.172	53.540	57.153	60.391	96.578	101.879	106.629	112.329	116.321	124.839
90	59.196	61.754	65.647	69.126	107.565	113.145	118.136	124.116	128.299	137.208
100	67.328	70.065	74.222	77.929	118.498	124.342	129.561	135.807	140.169	149.449

For example
$P(\chi^2(8) \geq 15.507) = 0.05$