Linear algebra and matrices I
University of Helsinki, Department of mathematics and statistics
General examination 18.12.2012

1. (a) Denote $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Show that the inverse matrix of A is B. Deduce that A is invertible.

   (b) Assume that A and B are 4×4-matrices such that $\det(A) = -1$ ja $\det(B) = 2$. Determine $\det(B^{-1}A)$ and $\det(2A)$.

2. Denote $\bar{w}_1 = (1, 2, 0)$, $\bar{w}_2 = (1, 1, -1)$ and $\bar{w}_3 = (1, 4, 2)$. We wish to find out if the vector $\bar{v} = (1, -5, -7)$ is an element of the subspace $\text{span}(\bar{w}_1, \bar{w}_2, \bar{w}_3)$. What kind of an equation should we consider? What is the system of linear equations that the equation corresponds to?

   When the lecturer modified the matrix of this system of linear equations with elementary row operations, she obtained the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -6 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

   On the basis of this matrix, decide whether $\bar{v}$ is in the subspace $\text{span}(\bar{w}_1, \bar{w}_2, \bar{w}_3)$.

3. The vector space $\mathbb{R}^3$ has a basis $\mathcal{B} = ((1, 1, 1), (1, 1, 0), (1, 0, 0))$.

   a) Determine the vector whose coordinates with respect to $\mathcal{B}$ are 1, -5 and 4.

   b) What are the coordinates of the vector $(0, 0, 1)$ with respect to $\mathcal{B}$?

4. a) What is the definition of linear independence of vectors?

   b) Suppose that $\bar{v}_1, \ldots, \bar{v}_k \in \mathbb{R}^n$ and $\bar{w} \in \text{span}(\bar{v}_1, \ldots, \bar{v}_k)$. Show that the vectors $\bar{v}_1, \ldots, \bar{v}_k, \bar{w}$ are linearly dependent.

5. Assume that $\bar{v}, \bar{w} \in \mathbb{R}^n$ and $\bar{w} \neq \bar{0}$.

   a) Which of the following is $\text{proj}_{\bar{w}}(\bar{v})$, that is, the projection of $\bar{v}$ onto the line generated by $\bar{w}$?

$$\frac{\bar{v} \cdot \bar{w}}{\bar{w} \cdot \bar{w}}\bar{w}, \quad \frac{\bar{v} \cdot \bar{w}}{\bar{w} \cdot \bar{w}}\bar{v}, \quad \frac{\bar{v} \cdot \bar{w}}{\bar{w} \cdot \bar{w}}$$

   b) Assume that the vectors $\bar{u}$ and $\bar{w}$ are parallel. Show that $\text{proj}_{\bar{w}}(\bar{v}) = \text{proj}_{\bar{u}}(\bar{v})$.