Linear algebra and matrices I
Department of mathematics and statistics
General examination
15.11.2012

1. Denote $\bar{a} = (1, 0, -2)$ and $\bar{b} = (-1, 4, 3)$.
   (a) Determine the dot product $\bar{a} \cdot \bar{b}$ and norm $\|\bar{a}\|$.
   (b) Assume that for the vector $\bar{v} \in \mathbb{R}^n$ it holds $\|\bar{v}\| = 4$ and $\bar{a} \cdot \bar{v} = -2$. Determine $\|\bar{a} + \bar{v}\|$.

2. The vector space $\mathbb{R}^3$ has a basis $\mathcal{B} = ((1, 1, 1), (1, 1, 0), (1, 0, 0))$.
   a) Determine the vector whose coordinates with respect to $\mathcal{B}$ are 1, -5 and 4.
   b) What are the coordinates of the vector $(0, 0, 1)$ with respect to $\mathcal{B}$?

3. Denote $\bar{w}_1 = (1, 2, 0)$, $\bar{w}_2 = (1, 1, -1)$ and $\bar{w}_3 = (1, 4, 2)$. We wish to find out if the vector $\bar{v} = (1, 1, 0)$ is an element of the subspace $\text{span}(\bar{w}_1, \bar{w}_2, \bar{w}_3)$ generated by the vectors $\bar{w}_1$, $\bar{w}_2$ and $\bar{w}_3$. What kind of an equation should we consider? What is the system of linear equations that the equation corresponds to?

   When the lecturer modified the matrix of this system of linear equations with elementary row operations, she obtained the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

   On the basis of this matrix, decide whether $\bar{v}$ is an element of the subspace $\text{span}(\bar{w}_1, \bar{w}_2, \bar{w}_3)$.

4. a) What is the definition of linear independence of vectors?
   b) Suppose that $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in \mathbb{R}^n$ and $\bar{v}_4 \in \text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. Show that the vectors $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ and $\bar{v}_4$ are linearly dependent.

5. a) Denote

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}.$$

   Calculate the products AB and AC. Write down the intermediate steps of your calculations.

   b) On the basis of part a), decide whether A is invertible.