Linear algebra and matrices I
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General examination 7.4.2016

1. Consider vectors $\bar{v}_1 = (1, -3, 1)$, $\bar{v}_2 = (0, -4, 0)$ and $\bar{v}_3 = (1, 1, 1)$.
   (a) Write the definition of the set $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$.
   (b) Give three examples of vectors of the set $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. Justify your answer.
   (c) What is the dimension of the space $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$? Justify your answer using the definition of dimension.

2. (a) How is linear independence defined?
   (b) Denote $\bar{w}_1 = (-10, -17, -4, -13)$, $\bar{w}_2 = (1, 1, -1, 2)$ ja $\bar{w}_3 = (2, 3, 0, 3)$. We wish to find out whether $\bar{w}_1$,$\bar{w}_2$ and $\bar{w}_3$ are linearly independent.
      i. What kind of equation needs to be considered? What should be shown concerning its solutions?
      ii. Explain carefully what kind of system of linear equations can be obtained from the equation.
      iii. When the matrix of the system of linear equations was modified with elementary row operations, the following matrix was obtained:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/7 & 0 \\ 0 & 1 & 4/7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

      Based on the matrix, what can be deduced concerning the linear independence of $\bar{w}_1$,$\bar{w}_2$ and $\bar{w}_3$?

3. (a) It is known that the matrix A has an eigenvalue $\bar{v} = (-4, 1)$. Which of the following vectors could be $A\bar{v}$ and which could not? Justify your answer using the definition of eigenvalue.

$$\bar{a} = (2, -1/2), \qquad \bar{b} = (1, 4), \qquad \bar{c} = (1, 0)$$

   (b) It is known that the matrix B has an eigenvalue 3 with corresponding eigenvectors $\bar{v}_1 = (-2, 4, 1, -1)$ and $\bar{v}_2 = (0, 1, 0, 1)$. Find an eigenvector corresponding to 3 such that it is not parallel to $\bar{v}_1$ or $\bar{v}_2$. Justify your answer using the definition of eigenvalue.

4. (a) The matrix of a system of linear equations has been modified with elementary row operations to the form

$$\left[\begin{array}{ccc|c} 4 & 1 & b & 0 \\ 0 & -2 & a & -a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

   where a and b are some real numbers. How many solutions does the system of linear equations have?
   (b) Assume that $A \in \mathbb{R}^{3\times 3}$. Suppose in addition that the equation $A\bar{x} = \bar{0}$ has a solution $\bar{x} = (1, 0, -1)$. Is A invertible?

5. Assume that $\bar{a}, \bar{b} \in \mathbb{R}^n$. Show that if $\|\bar{a} + \bar{b}\|^2 = \|\bar{a}\|^2 + \|\bar{b}\|^2$, then $\bar{a}$ and $\bar{b}$ are perpendicular to each other.