Linear algebra and matrices I
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1. (a) Write the definition of the set $\text{span}((1,0,-3),(1,-1,1))$.
   (b) Give three vectors, distinct from $(1,0,-3)$ and $(1,-1,1)$, that are elements of the set $\text{span}((1,0,-3),(1,-1,1))$.
   (c) List the different kind of subspaces of $\mathbb{R}^3$. (You can answer in your own words. Jusfication is not needed.)

2. In each of the following cases, determine how many solutions the system of linear equations has. Justify your answer.

   (a) The matrix corresponding to the system of linear equations can be transformed to the following form using elementary row operations.

$$\text{i.}\ \left[\begin{array}{ccc|c} 1 & -6 & 0 & 7/5 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{ii.}\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

   (b) The determinant of the coefficient matrix is 0, and the system of linear equations has at least one solution.

3. (a) Denote $\bar{v}_1 = (1,0,0)$, $\bar{v}_2 = (1,1,0)$ and $\bar{v}_2 = (1,-1,-1)$.
      i. Determine, using the definition on linear independence, whether the vectors $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$ are linearly independent.
      ii. One can show that $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ is a basis for the vector space $\mathbb{R}^3$. Determine the vector whose coordinates with respect to this basis are -2, 0 and -1.
   (b) Give an example of a subspace of $\mathbb{R}^4$ whose dimension is two. Justify your answer using the definition of dimension.

4. (a) It is known that the matrix A has an eigenvector $\bar{v} = (-4,1)$. Which of the following vectors could be $A\bar{v}$ and which could not? Justify your answer.

$$\bar{a} = (2,\ -1/2), \qquad \bar{b} = (1,4), \qquad \bar{c} = (1,0)$$

   (b) It is known that the matrix B has an eigenvalue -5 and eigenvectors $\bar{v}_1 = (-3,2,1,-2)$ and $\bar{v}_2 = (1,1,1,1)$ that correspond to this eigenvalue. Find an eigenvector corresponding to the eigenvalue -5, such that it is not parallel to $\bar{v}_1$ or $\bar{v}_2$. Justify your answer using the definition of eigenvector.

5. Assume that $\bar{v}, \bar{w} \in \mathbb{R}^n$ and $\bar{w} \neq \bar{0}$. The projection of $\bar{v}$ onto the line generated by $\bar{w}$ can be calculated using the following formula:

$$\text{proj}_{\bar{w}}(\bar{v}) = \frac{\bar{v} \cdot \bar{w}}{\bar{w} \cdot \bar{w}} \bar{w}.$$

   (a) Denote $\bar{v} = (-1,4)$, $\bar{w} = (0,2)$ and $\bar{u} = (0,4)$. Determine $\text{proj}_{\bar{w}}(\bar{v})$ and $\text{proj}_{\bar{u}}(\bar{v})$.
   (b) Explain in your own words why in part a) it holds $\text{proj}_{\bar{w}}(\bar{v}) = \text{proj}_{\bar{u}}(\bar{v})$. You can elaborate your explanation by drawing a picture if you wish.
   (c) Assume that $\bar{v}, \bar{w}, \bar{u} \in \mathbb{R}^n$ and $\bar{w} \neq \bar{0}$. Suppose that the vectors $\bar{u}$ and $\bar{w}$ are parallel to each other. Show that $\text{proj}_{\bar{w}}(\bar{v}) = \text{proj}_{\bar{u}}(\bar{v})$.