Linear algebra and matrices I
University of Helsinki, Department of Mathematics and Statistics
General examination 17.6.2015

1. How many solutions does a system of linear equations have if
   (a) its matrix can be modified with elementary row operations to the form
   $$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & -15 & 4 \\ 0 & 0 & 8 & 0 \end{array}\right]$$
   (b) its matrix can be modified with elementary row operations to the form
   $$\left[\begin{array}{ccc|c} 2 & -1 & -1 & -2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
   (c) if its coefficient matrix is
   $$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}?$$
2. You wish to know whether the vectors $(2, 1, 0)$, $(4, 4, 1/2)$ and $(5, -6, 0)$ are linearly independent.
   (a) What kind of an equation should you consider? What kind of a system of linear equations does the equation give?
   (b) When the lecturer modified the system on linear equations with elementary row operations, she obtained the matrix
   $$\left[\begin{array}{ccc|c} 1 & 4 & -6 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right].$$
   Ont the basis of the matrix, what can you say about the linear independence of the vectors?
3. (a) It is known that the matrix A has an eigenvector $\bar{v} = (-4, 1)$. Which of the following vectors could be $A\bar{v}$ and which could not? Justify your answer using the definition of eigenvector.
   $$\bar{a} = (2, -1/2), \qquad \bar{b} = (1, 4), \qquad \bar{c} = (1, 0)$$
   (b) Assume that the matrix B has an eigenvalue λ and a corresponding eigenvalue $\bar{v}$. Show that also $-53\bar{v}$ is an eigenvector corresponding to λ.
4. (a) Draw a picture of the set $W = \{(-3, 4) + (1, 2)t \mid t \in \mathbb{R}\}$. Is W a subspace of $\mathbb{R}^2$?
   (b) We wish to know if vectors $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$ generate $\mathbb{R}^3$. When we try to find out whether the vector $(a_1, a_2, a_3) \in \mathbb{R}^3$ is a linear combination of $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$, we obtain the matrix
   $$\left[\begin{array}{ccc|c} 1 & 4 & 4 & a_3 - 2a_2 \\ 0 & -3 & 0 & 2a_1 \\ 0 & 0 & 0 & a_1 - a_2 - a_3 \end{array}\right].$$
   Do the vectors generate $\mathbb{R}^3$?
5. Assume that $\bar{v}, \bar{u}, \bar{w} \in \mathbb{R}^n$. Assume also that $\bar{u}$ is parallel to $\bar{w}$.
   (a) Draw a picture that illustrates the vectors $\bar{v}$, $\bar{u}$ and $\bar{w}$ and the projections $\text{proj}_{\bar{w}}(\bar{v})$ and $\text{proj}_{\bar{u}}(\bar{v})$.
   (b) Show that $\text{proj}_{\bar{w}}(\bar{v}) = \text{proj}_{\bar{u}}(\bar{v})$.