

## Algebra I

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General examination

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1. Denote  $H = \{5^n \mid n \in \mathbb{Z}\}$ .

a) Show that  $H$  is a subgroup of the group  $(\mathbb{Q} \setminus \{0\}, \cdot)$ .

b) Show that the groups  $(H, \cdot)$  and  $(\mathbb{Z}, +)$  are isomorphic.

2. a) On  $\mathbb{Z}_6$ , the set of residue classes modulo 6, we define a binary operation  $*$  by

$$[a]_6 * [b]_6 = [a + b + 2]_6.$$

The binary operation has an identity. What is the identity?

b) Show that on  $\mathbb{Z}_6$  it is not possible to define a binary operation by

$$[a]_6 * [b]_6 = [|a| - 5b]_6.$$

3. Denote  $R = \mathbb{Z}_3 \times \mathbb{Z}_5$ .

a) Is  $R$  an integer domain?

b) The set  $R^*$  consists of all the units of  $R$ . Show that  $([2]_3, [2]_5) \in R^*$ .

c) The set  $R^*$  of units is a multiplicative group. Determine the subgroup generated by the element  $a = ([1]_3, [2]_5)$ , that is, the subgroup  $\langle a \rangle$ .

4. Consider the group  $S_4$  and its normal subgroup

$$V = \{(1), (12)(34), (13)(24), (14)(23)\}.$$

a) Determine the coset  $(123)V$ .

b) Which of the following cosets coincide? Remember to justify your answer.

$$(123)V, \quad (12)V \cdot (24)V, \quad ((23)V)^{-1}$$

c) Show that the factor group  $S_4/V$  is not cyclic.