SPECTRAL THEORY / AUTUMN 2016 / EXERCISE 9

1. Show that for all $\varepsilon > 0$ there exists a C^∞-function (a "cut-off-function) $\chi_\varepsilon : \mathbb{R} \to [0,1]$ with the following properties:

$$\chi_\varepsilon(x) = 0 \quad \text{for all } x \leq 0$$
$$\chi_\varepsilon(x) = 1 \quad \text{for all } x \geq \varepsilon$$
$$\Big| \frac{\partial \chi_\varepsilon}{\partial x}(x) \Big| \leq \frac{10}{\varepsilon} \quad \text{for all } x \in \mathbb{R}$$
$$\Big| \frac{\partial^2 \chi_\varepsilon}{\partial x^2}(x) \Big| \leq \frac{100}{\varepsilon^2} \quad \text{for all } x \in \mathbb{R}.$$

Instruction. Suitable piecewise linear function & convolution with a mollifier.

2. Generalize the result of problem 1 to higher dimensions, with $B(0,\varepsilon)$ replacing the interval $]-\infty, 0]$, $\mathbb{R}^d \setminus \overline{B}(0, 2\varepsilon)$ replacing the interval $[\varepsilon, \infty[$, gradient replacing the first derivative and the Laplacian replacing the second derivative.

3. Prove the "if " statement of Theorem 6.18.

4. Prove the "only if " statement of Theorem 6.18.