SPECTRAL THEORY / AUTUMN 2016 / EXERCISE 8

1. Given a spectral family $\{E_\lambda : \lambda \in \mathbb{R}\}$ in a separable infinite dimensional Hilbert space H, show that the integral

$$\int_0^\infty e^{-\lambda} d(E_\lambda u|v)$$

converges for all $u, v \in H$. (See the instruction below (6.88) in the lecture note.)

2. We denote for all $n \in \mathbb{N}_0$,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

where $x \in \mathbb{R}$ is a variable. Prove the recursion relation

$$H_{n+1} = 2xH_n(x) - H_n'(x) \quad , \quad n \in \mathbb{N},$$

where the prime denotes the derivative. Conclude that H_n is a polynomial of degree n (called the nth Hermite polynomial) and that the highest degree term equals $2^n x^n$. Also conclude that H_n is even, if n is even, and H_n is odd for odd n.

3. We denote

$$G_n(x) = \frac{1}{\sqrt{2^{n+1} n!}} H_n(x) e^{-x^2/2} \quad , \quad x \in \mathbb{R},$$

and define the second order differential operator

$$Su = -\frac{1}{2}u'' + \frac{\omega^2 x^2}{2} u,$$

the harmonic oscillator Hamiltonian; here $\omega > 0$ is a parameter which we in the following set $\omega = 1$ for simplicity.
By assuming that H_n satisfies the differential equation

$$f''(x) - 2xf'(x) + 2nf(x) = 0, \tag{0.1}$$

prove that G_n is an eigenfunction of S with eigenvalue $\lambda_n = n + 1/2$. (The proof of (0.1) is quite a long story ... . Also, the functions G_n, with a proper normalization, form an orthonormal basis of $L^2(\mathbb{R})$, which implies that the operator S is essentially self-adjoint in the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$. Just to let you know, you do not need to prove this.)

4. Prove that the solution of the Laplace-Dirichlet problem

$$-\Delta u(x) = f(x) \quad , \quad x \in \Omega$$
$$u(x) = 0 \quad , \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is the strip $\mathbb{R} \times]0,1[$ and $\partial\Omega = \big(\mathbb{R} \times \{0\}\big) \cup \big(\mathbb{R} \times \{1\}\big)$, is not unique, when f is given and, say $f \in \mathcal{S}(\Omega)$. Instruction. It is enough to find non-zero solutions to the corresponding homogeneous problem $f = 0$. Try this by separating the variables, assuming $u(x) = v(x_1)w(x_2)$ for some functions v, w of one variable.