Assume that A is a symmetric operator in the Hilbert space H and that H has on orthonormal basis $(e_n)_{n=1}^{\infty}$ consisting of eigenvectors of A, so that

$$Ae_n = \lambda_n e_n \quad \forall\, n \in \mathbb{N}\ . \tag{0.1}$$

We still assume $\lambda_n > 0$ for all n and that all the vectors e_n belong to $D(A)$. Our task is to show that A is essentially self-adjoint.
Recall that if $x \in H$, then

$$x = \sum_{n=1}^{\infty} x_n e_n \tag{0.2}$$

where $x_n = (x|e_n) \in \mathbb{C}$ and

$$\sum_{n=1}^{\infty} |x_n|^2 = \|x\|^2, \tag{0.3}$$

in particular both sums in (0.2) and (0.3) converge.

1. Show that if $x = \sum_{n=1}^{\infty} x_n e_n \in D(A) \subset H$, then

$$\sum_{n=1}^{\infty} (1 + \lambda_n^2)|x_n|^2 < \infty. \tag{0.4}$$

Let $\mathcal{D} \subset H$ be the set of all $x = \sum_{n=1}^{\infty} x_n e_n$ such that (0.4) holds, and define the operator with domain $\mathcal{D}$,

$$\mathcal{A}x := \sum_{n=1}^{\infty} \lambda_n x_n e_n. \tag{0.5}$$

Verify that $\mathcal{A} : \mathcal{D} \to H$ is an extension of A.

2. Show that $\sigma(\mathcal{A})$ is the closure of the set $\{\lambda_n\ :\ n \in \mathbb{N}\}$ in $\mathbb{C}$.

3. Show that $\mathcal{A}$ is the same as the closure of A. Here, you may assume it is known that $\mathcal{A}$ is closed.

4. Show that $\mathcal{A}$ is self-adjoint.

(We plan to apply these results to the study of the harmonic oscillator of quantum mechanics in the forthcoming exercises.)