SPECTRAL THEORY / AUTUMN 2016 / EXERCISE 6

1. Let A be a bounded self-adjoint operator in a Hilbert space H. If $K \subset \mathbb{C}$ is a non-empty compact set, then we define the map

$$\Phi : \mathcal{P} \to \mathcal{L}(H) \quad , \quad \Phi : P \mapsto P(A),$$

where $P(z) = \sum_{n=0}^{N} a_n z^n$ is a polynomial, $\mathcal{P}$ is the subspace of $C(K)$ consisting of polynomials P and

$$P(A) = \sum_{n=0}^{N} a_n A^n.$$

Show that Φ is an algebra homomorphism and that it satisfies properties 1°–2° of Theorem 6.11 for $f \in \mathcal{P}$.

2. Prove Lemma 6.6. of the lecture notes (for example, consider Riemann sums.)

3. Prove the first identity in Lemma 6.7.

4. Prove that the two examples after Definition 6.3 are spectral families.