1. Let $a < b$ and let $z \in \mathbb{C}$ be such that $z \neq \pi^2 n^2 (b-a)^{-2}$ for all $n \in \mathbb{N}$. Let us define the operator $R_z : L^2(a,b) \to L^2(a,b)$,

$$R_z f(t) = \frac{1}{\sqrt{z}\sin(\sqrt{z}(b-a))} \Big(\sin(\sqrt{z}(b-t)) \int_a^t \sin(\sqrt{z}(s-a)) f(s) ds + \sin(\sqrt{z}(t-a)) \int_t^b \sin(\sqrt{z}(b-s)) f(s) ds \Big).$$

Prove that $R_z f \in H_0^1(]a,b[)$ and $\frac{d}{dt} R_z f \in H^1(]a,b[)$ and

$$-\frac{d^2}{dt^2} R_z f - z R_z f = f$$

for all $f \in L^2(a,b)$. (Weak derivatives!)

2. Continuation of Problem 1. Show that R_z is the resolvent operator of the 1-dimensional Dirichlet-Laplacian $Af := -\frac{d^2}{dt^2} f$ with domain

$$D(A) := \big\{ f \in H_0^1(]a,b[) \, : \, \frac{d}{dt} f \in H^1(]a,b[) \big\}.$$

Show that the spectrum of A is $\sigma(A) = \{\pi^2 n^2 (b-a)^{-2} \, : \, n \in \mathbb{N}\}$.

3. Let $B \subset \mathbb{C}$ be an arbitrary compact subset. Find a bounded operator T in some separable Hilbert space H such that $\sigma(T) = B$. (Instruction. Take a dense sequence $(\alpha_n)_{n=1}^\infty$ in B. Consider the sequence space multiplier $(x_n)_{n=1}^\infty \mapsto (\alpha_n x_n)_{n=1}^\infty$ in the space ℓ^2).

4. Let $g : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the unit interval, $g(x) = 1$ for $x \in [0,1]$, $g(x) = 0$ for $x \notin [0,1]$. Calculate the integral

$$\int_{-1}^{5} e^{-x} dg(x). \tag{0.1}$$