In the following we consider the elements $g$ of the Sobolev-space $H^1(]0,1[) = H^1(0,1)$ as continuous functions so that $g(t)$ is well defined for all $t \in [0,1]$. So, $H_0^1(]0,1[) = H_0^1(0,1)$ is the closed subspace of $H^1(0,1)$, consisting of functions which vanish at 0 and 1. Equivalently, $H_0^1(0,1)$ is the closure of the subspace $C_0^\infty(0,1)$ in $H^1(0,1)$.

1. Consider the operator $A : f \mapsto if'$ (weak derivative) with

$$D(A) = \{f \in L^2(0,1) \, : \, f \in H_0^1(0,1)\}.$$

Determine $D(A^*)$ and $A^*$. Is $A$ symmetric or self-adjoint?

2. The same question for the operator $A_\theta : f \mapsto if'$, when it is defined in the domain

$$D(A_\theta) = \{f \in L^2(0,1) \, : \, f \in H^1(0,1), f(0) = e^{i\theta} f(1)\},$$

where $\theta \in [0, 2\pi]$ is a given fixed number. Can you say something about self-adjoint extensions of the operator $A$ of Problem 1?

3. Let $\Omega$ be for example the unit cube $\Omega = [0,1] \times [0,1] \times [0,1] \subset \mathbb{R}^3$ and let $\varphi \in L^1(\Omega)$ be fixed. Consider the multiplication operator

$$M_\varphi f = \varphi f,$$

which is defined as pointwise multiplication in the domain

$$D(M_\varphi) = \{f \in L^2(\Omega) \, : \, \varphi f \in L^2(\Omega)\}.$$

Show that $M_\varphi$ is densely defined and closed. Give examples of bounded and unbounded operators $M_\varphi$.

4. Show that $M_\varphi$ of the Probl. 3 satisfies $(M_\varphi)^* = M_{\overline{\varphi}}$ and thus it is self-adjoint, if the values of $\varphi$ are real numbers.