SPECTRAL THEORY / AUTUMN 2016 / EXERCISE 3

1. Work out some details of Example 3.4., 2°:
- Given $f \in C_0^\infty(\mathbb{R}^d)$, let $\hat{f} = \mathcal{F}f$ denote the Fourier transform of f. Show that the Fourier-transform $\mathcal{F}(-\Delta f)$ of $-\Delta f$ satisfies

$$\mathcal{F}(-\Delta f)(\xi) = C_d \xi^2 \hat{f}(\xi), \qquad (0.1)$$

where C_d is a(n unimportant) positive normalization constant. Notice that $\xi^2 = \sum_{j=1}^d \xi_j^2$ for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.
- Show that the Sobolev-norm of $H^2(\mathbb{R}^d)$,

$$\|f; H^2(\mathbb{R}^d)\| := \Big(\sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le 2}} \int_{\mathbb{R}^d} |D^\alpha f(x)|^2 dx \Big)^{1/2}$$

is equivalent with the norm

$$\Big(\int_{\mathbb{R}^d} \hat{f}(\xi)^2 d\xi + \int_{\mathbb{R}^d} |\xi|^4 \hat{f}(\xi)^2 d\xi \Big)^{1/2}.$$

You may use the well-known fact that $\|f; L^2(\mathbb{R}^d)\| = c_d \|\hat{f}; L^2(\mathbb{R}^d)\|$ for $f \in C_0^\infty(\mathbb{R}^d)$ and some normalization constant c_d.

2. Consider the operators $A : D(A) \to \ell^2$, where

$$D(A) = \{x = (x_n)_{n=1}^\infty \in \ell^2 \,:\, (nx_n)_{n=1}^\infty \in \ell^2\}$$

and $B : \ell^2 \to \ell^2$, where $B : (x_n)_{n=1}^\infty \mapsto (x_n/n)_{n=1}^\infty$.
(i) Show that $A^* = A$.
(ii) Show that $(AB)^* = I$ (the identity operator of ℓ^2) and $B^*A^* = I\big|_{D(A)}$; hence, $(AB)^* \neq B^*A^*$.

3. Let $A : D(A) \to H$, $D(A) \subset H$, and $B : D(B) \to H$, $D(B) \subset H$, be densely defined operators in the Hilbert space H.
(i) Assume that $A + B : D(A + B) \to H$ with $D(A + B) = D(A) \cap D(B)$ is densely defined. Show that $(A + B)^* \supset A^* + B^*$.
(ii) Prove that if $T \in \mathcal{L}(H)$, then $(A + T)^* = A^* + T^*$.

4. Consider the operator $T : D(T) \to L^2(0, 1)$, $Tf(t) = i\frac{df}{dt}$, $t \in [0, 1]$, where

$$D(T) = C_0^\infty(]0, 1[).$$

Is T symmetric, self-adjoint or essentially self-adjoint?