SPECTRAL THEORY / AUTUMN 2016 / EXERCISE 2

1. Let $\varphi : \mathbb{R} \to \mathbb{C}$, $\varphi \neq 0$, be, say, a continuous function such that its support is contained in the interval $[0, 1]$. (In other words, φ vanishes everywhere outside this interval.)
a) Show that if $B \subset L^2(\mathbb{R})$ is a set which contains all functions

$$g_n(t) := \varphi(t - n) \ , \quad n \in \mathbb{Z},$$

then B cannot be compact.
b) Show that the convolution operator $T_\varphi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

$$T_\varphi f(x) = \int_{-\infty}^{\infty} \varphi(x - t) f(t) \, dt$$

is bounded but not compact.

2. Consider the Hilbert space $L^2(0, 1)$, and some $\alpha > 1/2$. Define the operator

$$Af(t) = t^{-\alpha} f(t)$$

with domain

$$D(A) = \{ f \in L^2(0, 1) \ : \ \exists \, \delta > 0 \text{ such that } f(t) = 0 \text{ for almost all } t \in [0, \delta] \}.$$

Prove that A is not bounded nor closed. For the latter, you can for example consider functions $f_n(t)$ defined as t^α for $1/n \leq t \leq 1$ and as 0 for $t < 1/n$, where $n \in \mathbb{N}$.

3. Consider the Hilbert space $L^2(0, 2\pi)$ and let $g : [0, 2\pi] \to \mathbb{C}$ be a C^1-function such that its nth Fourier-coefficient $\hat{g}(n)$ satisfies the bound $|\hat{g}(n)| \leq 1/n$. Show that the (convolution) operator

$$K_g f(t) := \int_0^{2\pi} g(t - s) f(s) ds$$

is compact. (You should use the isomorphism of $L^2(0, 2\pi)$ and ℓ^2 defined by the Fourier series, and Probl. 3 in Exercise 1.)

4. A linear operator $T : H \to H$, where H is a Hilbert space, is called a finite rank operator, if the range $R(T) = T(H)$ is a finite dimensional subspace of H.
Show that a finite rank operator is compact. Moreover, show that if $T \in \mathcal{L}(H)$ can be approximated with respect to the operator norm by finite rank operators T_n, $n \in \mathbb{N}$, i.e. for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\| T - T_n \| \leq \varepsilon,$$

then T is compact.
You can use the result that in a finite dimensional normed space, bounded subsets are precompact.