1. Prove the Hardy inequality

$$\int_0^\infty \frac{1}{x^2}|u(x)|^2 dx \le 4 \int_0^\infty |u'(x)|^2 dx \tag{0.1}$$

say, for real valued $u \in C_0^\infty(]0,\infty[)$. (Instruction: integrate the identity

$$\frac{d}{dx}\Big(\frac{u^2}{x}\Big) = -\frac{u^2}{x^2} + \frac{2u'u}{x}$$

and use Cauchy-Schwartz.)

2. Prove the Hardy inequality

$$\int_{\mathbb{R}^3} \frac{1}{|x|^2}|u(x)|^2 dx \le 4 \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx \tag{0.2}$$

for real valued $u \in C_0^\infty(\mathbb{R}^3)$. Use polar coordinates and Probl. 1.

3. Let $u \in C_0^\infty(\mathbb{R}^3)$ and denote by $\hat{u}$ its Fourier-transform. Then, show that

$$\int_{\mathbb{R}^3} |x|^{-2}|u(x)|^2 dx \le 4 \int_{\mathbb{R}^3} |\xi|^2 |\hat{u}(\xi)|^2 d\xi.$$

4. Let $\Omega = \mathbb{R}^d$, $d \in \mathbb{N}$, and let $A = -\Delta$ with domain $D(A) = C_0^\infty(\Omega)$. What is the domain of the Friedrichs extension of A? (You should take into account that $C_0^\infty(\Omega)$ is dense for example in $H^2(\Omega)$ for this Ω.)