SPECTRAL THEORY / AUTUMN 2016 / EXERCISE 1

1. Prove the parallelogram and polarization identities:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$(x|y) = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\big).$$

2. Let ℓ^2 be the Hilbert space of square summable sequences (of complex numbers), endowed with the norm

$$\|x\| = \Big(\sum_{n=1}^{\infty} |x_n|^2\Big)^{1/2}, \quad x = (x_n)_{n=1}^{\infty} \in \ell^2$$

Consider the backward shift operator $B : \ell^2 \to \ell^2$, $B : (x_n)_{n=1}^{\infty} \mapsto (x_{n+1})_{n=1}^{\infty}$. Find some eigenpairs (λ, y) for B, that is, eigenvalues $\lambda \in \mathbb{C}$ and eigenvectors $y \in \ell^2$ satisfying $By = \lambda y$.

3. A set $A \subset H$ is precompact, if for every $\varepsilon > 0$ one can find finitely many balls $B(a_j, \varepsilon) = \{x \in H : \|x - a_j\| < \varepsilon\}$, where $a_j \in H$, $j = 1, \ldots, N$, such that A can be covered by them, i.e.

$$A \subset \cup_{j=1}^{N} B(a_j, \varepsilon) = \cup_{j=1}^{N} \Big(a_j + B(0, \varepsilon)\Big).$$

As H is complete, precompact sets are precisely the same as relatively compacts sets, i.e. sets whose closure is compact.
A linear operator $T : H \to H$, where H is a Hilbert space, is compact, if it maps the unit ball $B_H = \{x \in H : \|x\| < 1\}$ of H into a precompact set. A compact operator is always bounded.
Let us now consider $H = \ell^2$ and a fixed, bounded sequence $\Gamma = (\gamma_n)_{n=1}^{\infty}$, $\gamma_n \in \mathbb{C}$ (not necessarily belonging to ℓ^2) and the corresponding multiplier operator $M_\Gamma : \ell^2 \to \ell^2$,

$$M_\Gamma : (x_n)_{n=1}^{\infty} \mapsto (\gamma_n x_n)_{n=1}^{\infty}$$

Show that $M_\Gamma : \ell^2 \to \ell^2$ is a bounded operator (can you calculate its operator norm?). Show that M_Γ is compact, if and only if the sequence $(\gamma_n)_{n=1}^{\infty}$ converges to 0.

4. Is the backward shift operator of Problem 2 compact? Find some eigenpairs of the multiplier M_Γ of Problem 3. Finally, show that if $S : H \to H$ is a bounded linear operator and the linear operator $T : H \to H$ is compact, then ST and TS are compact.