Probability theory I - exercise set V

In this series, given a scalar random variable X, we define, for $\theta \in \mathbb{R}$ and $a \in \mathbb{R}$,

$$\varphi(\theta) := \log \mathbb{E} e^{\theta X} \in \mathbb{R} \cup \{+\infty\}$$

and

$$\gamma(a) := \sup_{\theta \in \mathbb{R}} (\theta a - \varphi(\theta)).$$

In this situation, γ is called the *Legendre transform* of φ.

Exercise 1. Compute the function $\gamma(a)$ for the following random variables:

(1) Standard Gaussian, that is, with density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$;
(2) Exponential with parameter 1, that is, with density $\mathbb{I}_{(0;+\infty)} e^{-x}$;
(3) $\frac{1}{2}$-Bernoully, that is, $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \frac{1}{2}$.

Exercise 2. One tosses a fair coin 1000 times. Give a reasonable upper bound for the probability that the number of heads is at least 600.

Exercise 3. One tosses a fair coin 1000 times. Give a reasonable lower bound for the probability that the number of heads is at least 600.

Exercise 4. Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a convex function, that is,

$$\varphi(\lambda x + (1-\lambda) y) \leq \lambda \varphi(x) + (1-\lambda)\varphi(y)$$

for all $0 \leq \lambda \leq 1$ and all $x, y \in \mathbb{R}$. Let $\gamma(a) := \sup_{\theta \in \mathbb{R}} (a\theta - \varphi(\theta))$ be its Legendre transform. Prove that γ is convex, and that the Legendre transform of γ is φ.

Exercise 5. Let X, Y be independent scalar random variables with densities f_X, f_Y. Prove that $X + Y$ has the density

$$f_{X+Y}(t) = \int_{\mathbb{R}} f_X(t - x) f_Y(x) dx.$$

Exercise 6. A Cauchy random variable with parameter $a > 0$ is a scalar random variable with density

$$f(x) = \frac{1}{\pi} \cdot \frac{a}{x^2 + a^2}.$$

Prove that:

- If X is a Cauchy random variable with parameter 1, then aX is a Cauchy random variable with parameter a;
- If X, Y are independent Cauchy random variables with parameters a, b, then $X + Y$ is a Cauchy random variable with parameter $a + b$.

Couclude that the weak law of large numbers does not hold for Cauchy random variables.

Exercise 7. Let X be a scalar random variable such that $\mathbb{E}X = 0$ and $\mathbb{E}e^{\theta X} < +\infty$ for some $\theta > 0$. Prove that φ' and φ'' are continuous on $[0; \theta)$, and $\mathbb{E}X = \varphi'(0)$, $\mathbb{E}X^2 = \varphi''(0)$ (Note that, compared to the lectures, we do not assume 0 to be the interior point of I).

Exercise 8. Let X be a scalar random variable such that $\mathbb{E}X = 0$, and let $a > 0$. Prove that $\gamma(a) = +\infty$ if and only if $\mathbb{P}(X \geq a) = 0$.