PROBABILITY THEORY I - EXERCISE SET IV

Exercise 1. Let A, B be two events. Prove that the following are equivalent:

- A, B are independent;
- A^c, B are independent;
- $\mathbb{I}_A, \mathbb{I}_B$ are independent random variables.

Exercise 2. Let $X_1, \ldots X_{n-1}$ be independent $\frac{1}{2}$-Bernoulli random variables, that is, $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 0) = \frac{1}{2}$, and let

$$X_n = \sum_{i=1}^{n-1} X_i \mod 2 = \begin{cases} 1, & \sum_{i=1}^{n-1} X_i \text{ is odd} \\ 0, & \text{otherwise} \end{cases}.$$

Prove that $X_1, \ldots, X_n$ are not independent, but every $(n-1)$-element subcollection $X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n$ of them is independent.

Exercise 3. Let X, Y be two independent random variables with probability distribution functions F_X, F_Y. Find the probability distribution functions of the random variables $\max(X, Y)$ and $\min(X, Y)$.

Exercise 4. Let X_1, X_2 be the outcomes of rolling two dice. Find all $a \in \{2, \ldots, 12\}$ such that the random variable $\mathbb{I}_{X_1+X_2=a}$ is independent of X_1.

Exercise 5. Let X and Y be two independent exponential random variables with parameter 1, that is, with density

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find the distribution of the random variable $X/(X+Y)$.

Exercise 6. Let X, Y be two independent exponential random variables with parameter 1. Prove that $\min(X, Y)$ is independent of $\max(X, Y) - \min(X, Y)$.

Exercise 7. Let X be a random variable with values in $\mathbb{N}$ such that

$$\mathbb{P}(X = n) = \frac{n^{-s}}{\zeta(s)},$$

where $s > 1$ and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Given a prime number p, let E_p be the event that X is divisible by p. Prove that the events E_p are independent.

Exercise 8. (Maxwell's theorem) Let X_1, X_2 be two independent scalar random variables such that the distribution of the random vector $(X_1; X_2)$ is invariant under rotations of the plane around the origin. Prove that X_1 and X_2 have normal distribution (that is, have a density of the form Ce^{-ax^2}, where $C > 0$ and $a > 0$). You may assume, if necessary, that (X_1, X_2) has a continuous, or even smooth, density with respect to the 2-dimensional Lebesgue measure.