Probability theory I - exercise set III

Exercise 1. Let X be a geometric random variable, that is,

$$\mathbb{P}(X = n) = (1 - a)a^n, \quad n = 0, 1, \dots$$

for some $a \in (0, 1)$, and $\mathbb{P}(X \notin \mathbb{N} \cup \{0\}) = 0$. Compute $\mathbb{E}X$, $\mathbb{E}X^2$ and $\mathbb{E}X^3$.

Exercise 2. Let X be an exponential random variable, that is, the density of X is given by

$$F'_X(x) = \begin{cases} \exp(-x) & , \quad x \geq 0 \\ 0 & \quad x < 0 \end{cases}.$$

Compute $\mathbb{E}X^n$ for every $n \in \mathbb{N}$.

Exercise 3. Consider a random variable X such that

$$F_X(x) = \begin{cases} 0, & x < -10 \\ \frac{1}{2}, & -10 \leq x < 0 \\ 1 - \frac{1}{2}e^{-x}, & x \geq 0 \end{cases}$$

Compute $\mathbb{E}X$.

Exercise 4. Given $\alpha > 1$, let X be a random variable with probability density function $F'_X(x) = \frac{1}{2(\alpha-1)}|x|^{-\alpha}$. Compute $\mathbb{E}X$.

Exercise 5. Let X be a random variable such that its distribution μ_X is the uniform measure on the Cantor set, as constructed in Exercise 7 of Set II. Compute $\mathbb{E}X$ and $\mathbb{E}X^2$.

Exercise 6. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, and let $f : \Omega \to \mathbb{R}_{\geq 0}$ be a measurable function. Its subgraph S_f is defined by

$$S_f := \{(\omega, \omega') \in \Omega \times \mathbb{R}_{\geq 0} : \omega' < f(\omega)\}.$$

Prove that S_f is an $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_{\geq 0})$ measurable set, and that

$$(\mu \otimes \lambda)(S_f) = \int_\Omega f d\mu,$$

where λ is the Lebesgue measure.

Exercise 7. Prove that, for a non-negative random variable X, one has

$$\mathbb{E}X = \int_0^\infty (1 - F_X(t))\, dt.$$

Exercise 8. Given $\alpha > 0$, denote

$$f_\alpha(x, y) := \frac{x^2 - y^2}{(x^2 + y^2)^{2\alpha}}$$

Prove that for any $0 < \alpha < 1$, one has

$$\int_{(0,1)} \left(\int_{(0,1)} f_\alpha(x, y) dx \right) dy = \int_{(0,1)} \left(\int_{(0,1)} f_\alpha(x, y) dy \right) dx.$$

Check that, however,

$$\int_{(0,1)} \left(\int_{(0,1)} f_1(x,y)dx \right) dy \neq \int_{(0,1)} \left(\int_{(0,1)} f_1(x,y)dy \right) dx.$$

Hint: For the second part, you might use the identity

$$\frac{\partial}{\partial y} \frac{y}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$