Exercise 1. Let $\mathcal{R}$ be a semi-ring, and let $\mathcal{R}' := \{\sqcup_{i=1}^{N} A_i : A_i \in \mathbb{R}\}$, $\mathcal{R}'' := \{\sqcup_{i=1}^{\infty} A_i : A_i \in \mathbb{R}\}$. Prove that R' is closed under finite unions and intersections, and R'' is closed under countable unions and finite intersections. Is R'' necessarily closed under countable intersections?

Exercise 2. (Patch to the proof of Caratheodory's theorem) Assume that μ is a pre-measure on a semi-ring $\mathcal{R}$, and $A_1, A_2, \dots \in \mathcal{R}$, $A_1', A_2', \dots \in \mathcal{R}$ are such that $\cup_{i=1}^{\infty} A_i = \sqcup_{i=1}^{\infty} A_i'$. Prove that $\sum_{i=1}^{\infty} \mu(A_i) \geq \sum_{i=1}^{\infty} \mu(A_i')$. Conclude that

$$\mu^*(A) = \inf_{\substack{A \subset \sqcup_{i=1}^{\infty} A_i \\ A_i \in \mathcal{R}}} \sum_{i=1}^{\infty} \mu(A_i).$$

Exercise 3. Let $\mathcal{R}$ be a semi-ring, and let $\mu : \mathcal{R} \to \mathbb{R}_{\geq 0}$ be a finitely additive function (that is, $\mu(A_1 \sqcup \dots \sqcup A_n) = \mu(A_1) + \dots + \mu(A_n)$ whenever $A_i \in \mathcal{R}$ and $\sqcup_{i=1}^{n} A_i \in \mathcal{R}$.) We say that μ is upper semi-continuous if for every sequence $E_1 \supset E_2 \supset \dots$, such that $\cap_{i=1}^{\infty} E_i = \emptyset$ and each E_i is a finite union of sets in $\mathcal{R}$, one has $\lim_{i\to\infty} \mu(E_i) = 0$. Prove that a finitely additive function is a pre-measure if and only if it is upper semi-continuous.

Exercise 4. Let $\Omega = \mathbb{Q}$ (the set of rational numbers), $\mathcal{R} := \{[a;b) \cap \mathbb{Q} : a, b \in \mathbb{Q}\}$, and define $\mu : \mathcal{R} \to \mathbb{R}_{\geq 0}$ by $\mu([a;b)) := b - a$. Prove that μ is not a pre-measure.

Exercise 5. Let $\Omega := [0;1]$ and $\mathcal{R} := \{A \subset \Omega : A \text{ finite}\}$. Check that $\mathcal{R}$ is a semi-ring. Find two different measures on $\sigma(\mathcal{R})$ that agree on $\mathcal{R}$.

Exercise 6. (Unifrom measure on self-similar sets) Let $K_0 \subset \mathbb{R}^N$ be a compact set, $0 < \lambda < 1$, and let $f_1, \dots, f_m : \mathbb{R}^N \to \mathbb{R}^N$ be maps of the form

$$f_i(x) = \lambda \cdot x + a_i.$$

where $a_i \in \mathbb{R}^N$. Assume that $f_i(K_0) \subset K_0$ for all i, and $f_i(K_0) \cap f_j(K_0) = \emptyset$ for $i \neq j$. Define, inductively, $K_n := \cup_{i=1}^{m} f_i(K_{n-1})$, $n = 1, 2, \dots$, and $K := \cap_{n=1}^{\infty} K_n$.

- Prove that for every n, K_n is compact and $K_n \subset K_{n-1}$. Conclude that $K \neq \emptyset$.
- Show that, when $N = 1$, one can choose K_0 and f_i so that K is the Cantor set in the real line.
- Let $\mathcal{R}_n := \{f_{i_1} \circ \dots \circ f_{i_n}(K) : 1 \leq i_1 \leq m, \dots, 1 \leq i_n \leq m\}$, and $\mathcal{R} := (\cup_{n=1}^{\infty} \mathcal{R}_n) \cup \{\emptyset\}$. Prove that $\mathcal{R}$ is a semi-ring on K, and that μ defined by $\mu(I) := m^{-n}$ when $I \in \mathcal{R}_n$, is a pre-measure on $\mathcal{R}$.
- Prove that $\sigma(\mathcal{R}) = \mathcal{B}(K)$. Conclude that there is a Borel measure on K that coincides with μ on $\mathcal{R}_n$.

Exercise 7. (Completion of measures) Let μ be a measure on a σ-algebra $\mathcal{F}$. We say that $E \in 2^{\Omega}$ is a *null-set* if there is a set $E' \in \mathcal{F}$ such that $E \subset E'$ and $\mu(E') = 0$. Denote the set of all null-sets by $\mathcal{N}$.

- Prove that the set $\overline{\mathcal{F}} := \{E \cup E' : E \in \mathcal{F}, E' \in \mathcal{N}\}$ is a σ-algebra.
- Prove that if $E_1 \cup E_1' = E_2 \cup E_2'$, where $E_{1,2} \in \mathcal{F}$ and $E_{1,2}' \in \mathcal{N}$, then $\mu(E_1) = \mu(E_2)$. Conclude that there is a unique measure $\overline{\mu}$ on $\overline{\mathcal{F}}$ such that $\overline{\mu}(E) = \mu(E)$ for all $E \in \mathcal{F}$.