Probability theory II - exercise set VI

Exercise 1. Let X_n be a simple random walk on $\mathbb{Z}$, and define

$$\tau := \min\{n : X_n = -1\}.$$

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(1) Prove that $e^{\theta X_n}(\cosh\theta)^{-n}$ is a martingale.
(2) Deduce that, for every $0 \le \alpha < 1$, we have $\mathbb{E}\alpha^\tau = \frac{1-\sqrt{1-\alpha^2}}{\alpha}$,that is,

$$\mathbb{P}(\tau = 2k-1) = (-1)^{k+1} \cdot \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-k+1)}{k!}.$$

Exercise 2. Let X_n be a simple random walk on $\mathbb{Z}$, and $a, b \in \mathbb{N}$, and denote $\tau := \min\{n : X_n = -a \text{ or } X_n = b\}$, $Y_n := X_{n\wedge\tau}$. Prove that, conditionally on the event $A = \{X_\tau = a\}$, Y_n is a Markov chain, that is,

$$\mathbb{P}(Y_n = x_n | Y_{n-1} = x_{n-1}, \dots, Y_1 = x_1, A) = \mathbb{P}(Y_n = x_n | Y_{n-1} = x_{n-1}, A).$$

Compute the transition probabilities of that chain.

Exercise 3. Let $X_1, X_2, \dots$ be a sequence of i. i. d., random variables, such that each X_i is uniformly chosen from the 26 letters of the English alphabet. Define τ to be the minimal n such that

$$(X_n, \dots, X_{n+10}) = (A, B, R, A, C, A, D, A, B, R, A).$$

(1) At each time n, a new gambler arrives and bets 1 EUR on the event that $X_n = A$. If he loses, he leaves; if he wins, he receives 26 EUR, all of which he bets on the event that $X_{n+1} = B$. If he loses, he leaves; if he wins, he receives 26^2 EUR, all of which he bets on $X_{n+2} = R$, and so on. Prove that if Y_m is the total gain of all the gamblers by time m, then Y_m is a martingale.
(2) Prove that

$$Y_{\tau+10} = 26^{11} + 26^4 + 26 - \tau - 10.$$

Deduce that

$$\mathbb{E}(\tau) = 26^{11} + 26^4 + 26 - 10.$$

Exercise 4. Alice and Bob are playing the following game: Bob names a string of three letters (e. g., AAB), after which Alice names another string of three letters (e. g. BAB). Then, they sample independent random variables $X_1, X_2, \dots$ with $\mathbb{P}(X_i = A) = \mathbb{P}(X_i = B) = \frac{1}{2}$; Alice wins if her string occurs in the sequence $X_1, X_2, \dots$ before Bob's. We say that Alice's string beats Bob's if the probability of Alice to win is greater than $\frac{1}{2}$.

(1) Assume that the strings chosen by Alice and Bob are given. Explain how to compute $\mathbb{P}(\text{Alice wins})$ in terms of $\mathbb{P}(\tau_x < \tau_y)$, where $\tau_a := \min\{n : Y_n = a\}$ and $Y_0, Y_1, \dots$ is a certain Markov chain with six states.
(2) Prove that AAB beats ABB.
(3) Prove that whatever Bob does, Alice can guarantee a win with probability strictly greater than $\frac{1}{2}$.

Let $\mathcal{D}_k := \{[l2^{-k}, (l+1)2^{-k}), l = 0, \dots, 2^k - 1\}$ denote the set of diadic intervals of rank k in $[0;1)$. For $x \in [0;1)$, let $I_k(x)$ denote $I \in \mathcal{D}_k$ such that $x \in I$. Given an integrable function $f : [0;1) \mapsto \mathbb{R}$, denote by $M_k f$ its diadic averages:

$$(M_k(f))(x) = |I_k(x)|^{-1} \int_{I_k(x)} f(y)dy.$$

Exercise 5. Check that if an integrable function f is viewed as a random variable on the probability space $[0;1)$, equipped with the Borel σ-algebra and the Lebesgue measure, then

$$M_k(f) = \mathbb{E}(f|\sigma(\mathcal{D}_k)).$$

Before doing the next exercises, consult Definition 4.8.9 and the statement of Theorem 4.8.10 in the lecture notes (to be covered in the next lecture).

Exercise 6. Define the *diadic Hardy-Littlewood maximal operator* $Af := \sup_k M_k(|f|)$. Prove that there exists a constant $C > 0$ such that

$$\int_0^1 (Af)^p \leq C \int_0^1 |f|^p.$$

Exercise 7. Prove the weak-type Hardy-Littlewood bound

$$a\lambda(\{\omega \in [0;1] : (Af)(\omega) > a) \leq C \int_0^1 |f|,$$

where $f : [0;1] \mapsto \mathbb{R}$ is an integrable function.

Exercise 8. Prove the *diadic Lebesgue differentiation theorem:* if $f : [0;1) \to \mathbb{R}$ is integrable, then

$$\lim_{k\to\infty} (M_k f)(x) \to f(x)$$

for almost every x.