Exercise 1. Prove the conditional Chebyshev inequality: if X is a random variable such that $X \geq 0$ almost surely and $\mathbb{E}X < \infty$, then

$$\mathbb{P}(X > a|\mathcal{G}) \leq \frac{\mathbb{E}(X|\mathcal{G})}{a} \text{ almost surely.}$$

Exercise 2. Let (X, Y) be a centered Gaussian vector with $\mathbb{E}Y^2 = 1$. Prove that almost surely, the conditional distribution of X given Y is Gaussian with mean $Y\mathbb{E}(XY)$ and variance $\mathbb{E}X^2 - (\mathbb{E}(XY))^2$.

Exercise 3. Let $\xi_1, \xi_2, \dots$ be independent centered random variables with finite variance, and define $S_n = \xi_1 + \dots + \xi_n$. Prove that

$$S_n^2 - \operatorname{Var} S_n$$

is a martingale.

Exercise 4. Let X_n be a submartingale, and let τ be a stopping time such that $\tau \leq n$ almost surely. Prove that

$$\mathbb{E}X_\tau \leq \mathbb{E}X_n.$$

Exercise 5. Let X_n and Y_n be submartingales (adapted to the same filtration). Prove that $\max(X_n; Y_n)$ is a submartingale.

Exercise 6. Let M be a separable metric space such that $d(x, y) < 1$ for all $x, y \in \Omega'$. Let $\{q_1, q_2, \dots\} \subset M$ be a countable dense subset, and define a function $\varphi : M \mapsto [0; 1)^{\mathbb{N}} = \{(x_1, x_2, \dots) : x_i \in [0; 1)\}$ by

$$\varphi(x) = (d(q_1; x), d(q_2; x), \dots).$$

Prove that φ is injective. Viewing $[0; 1)^{\mathbb{N}}$ as a metric space with the metric $d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i| 2^{-i}$, prove that φ is also continuous.

Exercise 7. Consider the map $\psi : [0; 1)^{\mathbb{N}} \mapsto [0; 1)$, defined as follows: if $x = (x_1, x_2, \dots)$ and $x_{ij} \in \{0, 1\}$ are the binary digits of x_i (that is, $x_i = \sum_{j=1}^{\infty} x_{ij} 2^{-j}$, with the convention that infinite tails of 1's are prohibited), define $\psi(x)$ to be the real number whose binary digits are $x_{11}, x_{12}, x_{21}, x_{13}, x_{22}, x_{31}, \dots$, etc. Formally,

$$\psi(x) = \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} x_{n,m-n} 2^{-m(m-1)/2-n}.$$

Prove that $\psi(x)$ is $\mathcal{B}([0; 1)^{\mathbb{N}})$-to-$\mathcal{B}([0; 1))$ measurable and injective, that $A := \psi([0; 1)^{\mathbb{N}})$ is Borel measurable, and that $\psi^{-1} : A \mapsto [0; 1)^{\mathbb{N}}$ is also measurable.

Exercise 8. Let M be a metric space which is a countable union of compacts. Derive from the previous two exercises that $(M; \mathcal{B}(M))$ is isomorphic as a measurable space to a Borel subset B of $\mathbb{R}$, that is, there exists a measurable bijection $\rho : M \mapsto B$ with measurable inverse. Conclude that any random variable with values in M has regular conditional distributions.