## Probability theory II - exercise set IV

**Exercise 1.** There is a bus stop in Springfield; the buses arrive to the stop in a Poisson process with intensity 1 (the service started long time ago). Homer comes to the stop and wants to figure out the expected time to wait for the bus. He argues as follows: "I know that the intervals between buses are exponentially distributed with parameter 1, therefore, they have expected duration 1. On average, a person who arrives at the stop waits for a half of that interval. Therefore, my expected waiting time is $\frac{1}{2}$". Is Homer right? If not, what's wrong with his reasoning?

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We say that a stochastic process $X_t$ has finite-dimensional marginals of the Poisson process if $X_0 = 0$ and for any $0 = a_0 \leq a_1 \leq \cdots \leq a_n$, the random variables $X_{a_i} - X_{a_{i-1}}$ are independent and Poisson distributed:

$$\mathbb{P}(X_{a_i} - X_{a_{i-1}} = m) = \frac{\varepsilon_i^m}{m!} e^{-\varepsilon_i}, \quad m = 0, 1, \ldots.$$

where $\varepsilon_i = a_i - a_{i-1}$. In Exercise 5, it will be proven that if such a process is almost surely increasing and right-continuous, then it is a Poisson process. You may use this fact in Exercises 2-3.

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**Exercise 2.** Let $X_t$ and $Y_t$ be two independent Poisson processes with intensities $\lambda_x$ and $\lambda_y$, respectively. Prove that $X_t + Y_t$ is a Poisson process with intensity $\lambda_x + \lambda_y$.

**Exercise 3.** Let $X_t$ be a Poisson process with intensity $\lambda$, and let $\zeta_1, \zeta_2, \ldots$ be $p$-Bernoulli random variables ($\mathbb{P}(\zeta_i = 1) = 1 - \mathbb{P}(\zeta_1 = 0) = p$), independent of $X_t$ and among themselves. Define

$$Y_t := \zeta_1 + \cdots + \zeta_{X_t}.$$

(Every time $X_t$ jumps, we flip a coin and make $Y_t$ jump if the coin comes up heads.) Prove that $Y_t$ is a Poisson process with intensity $\lambda p$.

**Exercise 4.** Show by an example that a stochastic process with finite-dimensional marginals of the Poisson process can be discontinuous at every point almost surely.

**Exercise 5.** Let $X_t$ be a stochastic process with finite-dimensional marginals of the Poisson process such that the random function $t \mapsto X_t$ is almost surely increasing and right-continuous. Prove that there exist exponentially distributed i. i. d. random variables $\xi_1, \xi_2, \ldots$ such that almost surely, $X_t = \max\{n : \xi_1 + \cdots + \xi_n \leq t\}$ for all $t \geq 0$. (In this situation, one says that $X_t$ is *indistinguishable* from a Poisson process.)

**Exercise 6.** Let $X_t$ be a stochastic process with finite-dimensional marginals of the Poisson process. Define, for $t \geq 0$, $\tilde{X}_t = \inf\{X_q : q \in \mathbb{Q}, q > t\}$.

(1) Prove that $\tilde{X}_t$ is a stochastic process, that is, $\tilde{X}_t$ is measurable for all $t$.
(2) Prove that almost surely, $\tilde{X}_t$ is increasing, right-continuous, and $X_t \in \mathbb{Z}_{\geq 0}$ for all $t$.

(3) Prove that for every $t \geq 0$, $X_t = \tilde{X}_t$ almost surely (in this situation, $\tilde{X}_t$ is called *a modification* of $X_t$. Note that "for every $t \geq 0$, $X_t = \tilde{X}_t$ almost surely" is not the same as "almost surely, $X_t = \tilde{X}_t$ for every $t \geq 0$".)

**Exercise 7.** Let $\mathcal{G}_1 \subset \mathcal{G}_2$ be two $\sigma$-algebras. Prove that

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(X|\mathcal{G}_1).$$

**Exercise 8.** Let $(X, Y)$ be a centered Gaussian vector with covariance operator $\Sigma$. Compute $\mathbb{E}(X|Y)$ and $\mathbb{E}(X^2|Y)$.