Exercise 1. Consider a Markov chain with $S = \{1, 2, 3, 4, 5, 6\}$ and the transition matrix

$$P = \begin{pmatrix} 0 & 2/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

Determine whether this Markov chain is irreducible and whether it is aperiodic. Does it have a unique stationary distribution?

Exercise 2. Consider the following method of card shuffling: take the card from the top of the deck and insert it into a random position in the deck, chosen uniformly among all possibilities. Iterate; this is a Markov chain whose state space consists of all possible permutations of the deck. Let $X_1, X_2, \dots$ denote the states of the chain at time $1, 2, \dots$, and let N denote the size of the deck.

(1) Let A be the card that was initially at the *bottom* of the deck. Convince yourself and the tutor that at any given moment of time, the cards *below* A are perfectly shuffled (i. e., all possible orders have equal probabilities).
(2) Let τ be the (random) time at which A is picked from the top of the deck and put somewhere in the deck. Prove that, for any permutation π, $\mathbb{P}(X_n = \pi | \tau \le n) = \frac{1}{N!}$, where N is the number of cards in the deck. In other words, after the time τ the deck is perfectly shuffled.
(3) Convince yourself and the tutor that τ has the same distributinon as
$$\xi_1 + \dots + \xi_N,$$
where ξ_i are independent, and ξ_i is geometric with parameter $\alpha_i := \frac{N-i}{N}$, that is $\mathbb{P}(\xi_i = m) = (1 - \alpha_i)\alpha_i^{m-1}$ for all $m \in \mathbb{N}$.
(4) Prove that $\approx N \log N$ iterations are enough to shuffle the deck well[1].
(5) Define τ' to be the first time the card A reaches the upper half of the deck. Prove that the probability of τ' being much smaller than $N \log N$ is small[2]. Conclude that $\approx N \log N$ iterations are necessary to shuffle the deck well[3].

Exercise 3. Consider an irreducible Markov chain that has a stationary distribution μ. Prove that $\mu(y) > 0$ for any state y.

Exercise 4. Given a Markov chain X_t and a state x, define inductively

$$\tau_x^{(k)} := \min\{t > \tau_x^{(k-1)} : X_t = x\}$$

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[1] A suggested precise formulation: if $n > N \log(N+1) + \frac{10\pi}{\sqrt{6}}N$, then $d_{TV}(\mu^{(n)}; \mu) \le \frac{1}{50}$, where, as usuall, $\mu^{(n)}$ denotes the distribution of the chain at time n, and μ denotes the uniform distribution on the set of all permutations.

[2] A suggested estimate: $\mathbb{P}(\tau' < N \log(N/2 - 1)/2 - \frac{\pi}{\sqrt{3}}N) < \frac{1}{2}$

[3] A suggested estimate: if $n < N \log(N/2 - 1)/2 - \frac{\pi}{\sqrt{3}}N$, then $d_{TV}(\mu^{(n)}; \mu) \ge \frac{1}{4}$

whe time of k-th visit to x, with $\tau_x^{(0)} := 0$. Prove that $\tau_x^{(2)} - \tau_x^{(1)}, \tau_x^{(3)} - \tau_x^{(2)}, \ldots$ are i. i. d. random variables.

Exercise 5. Let $X_1, X_2, \ldots$ be i. i. d. random variables with values in $\mathbb{N}$ (we use the convention that $0 \notin \mathbb{N}$), denote $S_n = \sum_{i=1}^n X_i$, and define

$$N(t) := \max\{n : S_n \le t\}.$$

(1) Prove that if $\mathbb{E}X_1 < \infty$, then $\frac{N(t)}{t} \to \frac{1}{\mathbb{E}X_1}$ in probability as $t \to \infty$.
(2) Deduce that if $\mathbb{E}X_1 = \infty$, then $\frac{N(t)}{t} \to 0$ in probability as $t \to \infty$, and hence $\mathbb{E}(N(t)/t) \to 0$.

Exercise 6. Prove that, if a Markov chain has a stationary distribution μ, then any state x satisfying $\mu(x) > 0$ is positive recurrent.