Exercise 1. Let $X_1, X_2, \dots$ be i. i. d. $\frac{1}{2}$-Bernoulli random variables. Prove that $Y_n = (X_{n+1}; X_n) \in \{\pm 1\}^2$ is a Markov chain. Compute its transition matrix and its stationary distribution. Bonus question: is $X_{n+1} + X_n$ a Markov chain?

Exercise 2. Given $p \in (0;1)$ and $q \in (0;1)$, denote $a = p(1-q)$, $b = q(1-p)$ and $c = 1 - a - b$. Recall that the queueing process was defined to be the Markov chain with the state space $S = \mathbb{Z}_{\geq 0}$ (number of customers in the queue), with the transition matrix given by

$$P_{xy} = \begin{cases} a, & y = x+1; \\ b, & y = x-1; \\ c, & y = x; \\ 0, & \text{else}, \end{cases} \quad x > 0, \text{ and } P_{0y} = \begin{cases} 1-p, & y = 0 \\ p, & y = 1 \\ 0 & \text{else} \end{cases}$$

(p is the probability that a new customer arrives, and q is the probability that a customer is served, should there be any in the queue). Prove that, when $p \geq q$, the queueing process does not have a stationary distribution, while for $p < q$ there exists a unique stationary distribution. Compute that distribution.

In Exercises 3-7, we consider a simple random walk X_n on the group

$$S = (\mathbb{Z}/2\mathbb{Z})^N = \{(x_1, \dots, x_N) : x_i \in \{0,1\}\},$$

that is, a Markov process with state space S and with transition matrix

$$P_{xy} = \begin{cases} \frac{1}{N}, & \text{if the strings } x \text{ and } y \text{ differ in exactly one bit} \\ 0, & \text{otherwise.} \end{cases}$$

The ultimate goal, achieved in Exercise 8, is to solve the Ehrenfest diffusion model.

Exercise 3. (Fourier-Walsh basis) For every $T \subset \{1, \dots, N\}$, define the function $f_T : S \to \mathbb{R}$ by

$$f_T((x_1, \dots, x_N)) = \prod_{j \in T} e^{\pi i x_j} = \prod_{j \in T} (-1)^{x_j}.$$

Prove that f_T are orthogonal as row vectors, that is,

$$\sum_{x \in S} f_T(x) f_{T'}(x) = \begin{cases} 0, & T \neq T', \\ 2^N, & T = T', \end{cases}$$

Exercise 4. (Fourier-Walsh transform) Prove that every function $\mu : S \to \mathbb{R}$ can be written as

$$\mu = \sum_{T \subset \{1, \dots, N\}} \mu_T f_T,$$

where the coefficients $\mu_T \in \mathbb{R}$ are given by

$$\mu_T = \frac{1}{2^N} \sum_{x \in S} \mu(x) f_T(x).$$

Exercise 5. Prove that for every T, f_T is an eigenvector of P, that is, $f_T P = \lambda_T f_T$. Compute λ_T.

Exercise 6. Given $x, y \in S$, compute $(P^n)_{xy}$ explicitly in terms of $f_T(x)$ and $f_T(y)$, $T \subset \{1, \ldots, N\}$.

Exercise 7. Prove that, for any $x, y \in S$,

$$(P^n)_{xy} = (1 \pm (-1)^n) 2^{-N} + O\left(\alpha_N^n\right) \text{ as } n \to \infty,$$

where "$\pm$" is "$+$" if and only if the total number of 1's in x and y is even, and $\alpha_N = 1 - \frac{2}{N}$.

Exercise 8. Prove that, in the Ehrenfest diffusion model, for any states $k, k' \in \{0, \ldots, N\}$ (representing the number of particles in chamber 1), one has

$$\mathbb{P}_{k'}(X_n = k) = (1 + (-1)^{n+k+k'}) \frac{N!}{k!(N-k)!} 2^{-N} + O(\alpha_N^n), \quad n \to \infty.$$