Probability theory II - exercise set I

In exercises $1-7$, $X_1, X_2, \ldots$ denote independent $\frac{1}{2}$-Bernoulli random variables ($\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$), so $S_n = \sum_{i=1}^n X_i$ is a simple random walk on $\mathbb{Z}$.

Exercise 1. Check that $\frac{1}{2}S_2, \frac{1}{2}S_4, \frac{1}{2}S_6 \ldots$ is a random walk on $\mathbb{Z}$ that satisfies the conditions of the Local Central Limit Theorem. Deduce that, as $n \to \infty$,

$$\mathbb{P}(S_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}}.$$

Exercise 2. (Method of reflections) Given $a \in \mathbb{Z}$, denote, as in the lecture notes, $\tau_a := \min\{n > 0 : S_n = a\}$. Let

$$X_i' := \begin{cases} X_i, & i \le \tau_a \\ -X_i, & i > \tau_a. \end{cases}$$

Prove that $X_1', X_2', \ldots$ are independent $\frac{1}{2}$-Bernoulli random variables, and hence $S_n' = \sum_{i=1}^n X_i'$ is again a simple random walk.

Exercise 3. Prove that, for every $a, m \in \mathbb{Z}$ such that $0 < a < m$, and every $n \in \mathbb{N}$, one has

$$\mathbb{P}(S_n = m) = \mathbb{P}(S_n = 2a - m \text{ and } \tau_a \le n)$$

Exercise 4. Prove that

$$\mathbb{P}(S_{2n} = 2m) = \frac{(2n)!}{(n-m)!(n+m)!} \cdot 2^{-2n}.$$

Deduce from this and from the previous exercise that, for every $n \in \mathbb{N}$,

$$\mathbb{P}(\tau_0 = 2n) = C_{n-1} \cdot 2^{-2n+1}.$$

where $C_n = \frac{(2n)!}{n!(n+1)!}$ is the n-th Catalan number.

Exercise 5. Prove that $\mathbb{E}(\tau_0) = +\infty$. Generalize this result to arbitrary random walks with centered steps on $\mathbb{Z}^d$ satisfying the conditions of the Local Central Limit theorem.

Exercise 6. Let $a, b \in \mathbb{Z}$ be such that $a < 0$ and $b > 0$. Prove that

$$\mathbb{E}(\min(\tau_a, \tau_b)) < \infty.$$

Exercise 7. Compute $\mathbb{E}(\min(\tau_a, \tau_b))$ for $a < 0$, $b > 0$.

Exercise 8. (Discrete Dirichlet problem via random walks) A function $h : \mathbb{Z}^d \to \mathbb{R}$ is called *discrete harmonic* at $z \in \mathbb{Z}^d$ if $h(z) = \frac{1}{2d}\sum_{z' \sim z} h(z')$, where the sum is over the $2d$ lattice neighbors $z \pm e_1, \ldots, z \pm e_d$ of z. Given a nonempty set $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, let $\tau_A^x := \min\{n \ge 0 : x + S_n \in A\}$, where S_n is a simple random walk on $\mathbb{Z}^d$. Prove that, for any $f : A \cup \infty \to \mathbb{R}$, the function

$$h(x) := \mathbb{E}(f(x + S_{\tau_A^x}))$$

is bounded, harmonic at all points of $\mathbb{Z}^d \setminus A$, and satisfies $h(x) = f(x)$ for all $x \in A$. Here we define $x + S_{\tau_A^x} = \infty$ if $\tau_A^x = \infty$.